HOMEWORK 4 — HYPERBOLIC GEOMETRY

DANNY CALEGARI

This homework is due November 3rd in Kathy Paur's mailbox. There will be no class that day, since I'll be talking at a conference at Columbia. Recall that D usually denotes the Poincaré disk model of hyperbolic space, and that ∂D denotes the *ideal boundary* — otherwise known as the *circle at infinity*.

Problem 1. Let l_1, l_2 be two intersecting lines, in the Klein disk model, and let S be the boundary circle. Let t_1, t_2 be the two (Euclidean) tangents to S at the endpoints of l_1. Show that l_1, l_2 are perpendicular if and only if the three lines l_2, t_1, t_2 are either parallel, or intersect in a single point. (Here we think of l_2 as the Eudlidean straight line extending the hyperolic straight line also called l_2).

Deduce the theorem in Euclidean geometry that t_1, t_2, l_2 are parallel or coincident if and only if s_1, s_2, l_1 are parallel or coincident, where s_1, s_2 are the tangents to S at the endpoints of l_2.

Hint: to find the hyperbolic angle between l_1, l_2, move over to the Poincaré disk model, where apparent angles are equal to hyperbolic angles.

Problem 2. For four distinct points $v_1, v_2, v_3, v_4 \in \mathbb{R} \cup \infty$, define the *cross–ratio* to be

$$c(v_1, v_2, v_3, v_4) = \frac{(v_4 - v_2)(v_3 - v_1)}{(v_2 - v_1)(v_4 - v_3)}$$

If $\alpha \in PSL(2, \mathbb{R})$ acting on $\mathbb{R} \cup \infty$ in the usual way, then show

$$c(v_1, v_2, v_3, v_3) = c(\alpha(v_1), \alpha(v_2), \alpha(v_3), \alpha(v_4))$$

Problem 3. With notation as in the previous problem, if v_1, v_2, v_4 are in anticlockwise circular order, show there is a *unique* choice of $\alpha \in PSL(2, \mathbb{R})$ with

$$\alpha : \{v_1, v_2, v_4\} \to \{0, 1, \infty\}$$

(the bijection induced by α should be the order–preserving one). Show that

$$c(v_1, v_2, v_3, v_4) = \alpha(v_3)$$

Problem 4. In the upper half–space model, show that the set of points P at constant *hyperbolic* distance t away from the hyperbolic straight line $x = 0$ consists of a pair of *Euclidean* straight lines. What is the relationship between the distance t and the angle between the lines in P and the line $x = 0$?

Problem 5. Let K be the group of matrices of the form

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and A the group of matrices of the form

$$\begin{bmatrix} \cosh(\gamma) & 0 & \sinh(\gamma) \\ 0 & 1 & 0 \\ \sinh(\gamma) & 0 & \cosh(\gamma) \end{bmatrix}$$

Show that every element of $SO(2,1)$ can be expressed as $k_1 a k_2$ for some $k_1, k_2 \in K$ and $a \in A$; that is, we can write $SO(2,1) = KAK$. How unique is such an expression?

Problem 6. Let K' be the subgroup of $PSL(2,\mathbb{R})$ consisting of matrices of the form $\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$ and A' the subgroup of matrices of the form $\begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}$. Find an isomorphism from $SO(2,1)$ to $PSL(2,\mathbb{R})$ taking K to K' and A to A'. (Careful! The isomorphism $K \to K'$ might not be the one you first think of . . .)

Problem 7. Recall the subgroups K' and A' defined in the previous question. Let N denote the group of matrices of the form $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. Show that every element of $PSL(2,\mathbb{R})$ can be expressed as kan for some $k \in K'$, $a \in A'$ and $n \in N$. How unique is this expression? This is an example of what is known as the KAN or *Iwasawa decomposition*.

Problem 8. For a group G, the *commutator subgroup* G_1 is the group generated by elements of the form $[g,h]$ for $g,h \in G$ (remember $[g,h] = ghg^{-1}h^{-1}$). We write $G_1 = [G,G]$. Define G_i inductively as the group generated by elements of the form $[g, g_i]$ for $g \in G$ and $g_i \in G_{i-1}$. We write $G_i = [G, G_{i-1}]$. A group G is *nilpotent* (of order n) if G_n is trivial, for some n.

Show N as in the previous problem is nilpotent (this motivates the standard notation).

Problem 9. We inherit notation from the previous problems. In the Poincaré disk model, show that the orbits of points under the group K are circles contained in the interior of D. Show that the orbits of points under the group A are arcs of circles (or straight lines) which intersect the boundary of H transversely. Show that the orbits of points under the group N are circles which intersect ∂D tangentially. Use this to show that no element of one of these subgroups is a conjugate of an element of another one of these subgroups unless they are the identity elements.

These third kind of "circles" are called *horocircles* or sometimes *horocycles*.

Problem 10 (Hard). Let γ be a smooth curve embedded in $\mathbb{R}^2$, bounding a region E. Define a "metric" on E as follows: Let f be a smooth nowhere zero function on E which is equal to $\frac{1}{\text{dist}(p,\gamma)}$ for all p sufficiently close to γ. Let the metric on E at a point p be equal to the Euclidean metric at that point times $f(p)$. That is, the "length elements" on E are given by $(f dx, f dy)$, where (dx, dy) are the usual Euclidean length elements. Show that there is a continuous 1–1 map $\phi : E \to D$ which distorts the lengths of curves by a bounded amount. Here D is the Poincaré disk model of hyperbolic space with the hyperbolic metric. That is, there is a constant $K > 0$ such that for any curve α in E,

$$\frac{1}{K}\text{length}_D(\phi(\alpha)) \le \text{length}_E(\alpha) \le K\text{length}_D(\phi(\alpha))$$

Remark. One form of the Riemann mapping theorem says that there is a unique conformal (i.e. angle preserving) map $\phi : D \to E$ which takes any three distinct points in ∂D to any three distinct points in ∂E. If we define a metric in E by declaring that the length of a segment γ in E is equal to the hyperbolic length of $\phi^{-1}(\gamma)$, then this metric is asymptotically equal to the metric defined in the previous problem, for points close to the boundary.

DEPARTMENT OF MATHEMATICS, HARVARD, CAMBRIDGE, MA 02138
E-mail address: dannyc@math.harvard.edu