HOMEWORK 1 — GROUPS

DANNY CALEGARI

Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due September 29th. Problems marked "Hard" are extra credit; in other words, doing all the ordinary problems correctly will earn full credit. But by doing hard problems, it is possible to make mistakes on, or fail to complete, all the ordinary problems, and still earn full credit.

Problem 1. Verify that each of the following examples are groups under the proposed operation, and calculate the identity and the inverse of a general element.

(1) The integers $\mathbb{Z}$ under $+$
(2) The set of 2×2 matrices with real entries and nonzero determinant under matrix multiplication. This is called the *general linear group in dimension* 2 and denoted $GL_2(\mathbb{R})$
(3) The set of 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ where a is nonzero and b is arbitrary under matrix multiplication.
(4) The group of *integers modulo* n under $+$. These are the sets of equivalence classes of integers, where $a \sim b$ if and only if $a - b$ is divisible by n. This group is denoted $\mathbb{Z}/n\mathbb{Z}$.
(5) The set of 2×2 matrices with entries in $\mathbb{Z}/2\mathbb{Z}$ (i.e. "even" and "odd") with the usual rules of multiplication and addition for even and odd numbers, with odd determinant, under matrix multiplication. This group is denoted $GL_2(\mathbb{Z}/2\mathbb{Z})$.
(6) The group of permutations of n objects (which might as well be the set $\{1, \dots, n\}$) under composition. This group is called the *symmetric group* S_n.
(7) The group of symmetries of a regular n–gon under composition. This group is called the *dihedral group* D_n.
(8) For an arbitrary set X, the group of 1–1 and invertible maps $X \to X$ under composition is a group. This group is called the *symmetric group of* X and denoted S_X.

Problem 2. How many elements are in D_n? How many are in S_n?

Problem 3. List all the homomorphisms from $\mathbb{Z}, +$ to $\mathbb{Z}, +$. List all the homomorphisms from $\mathbb{Z}/2\mathbb{Z}, +$ to $\mathbb{Z}, +$.

Problem 4. Let S be a set and G be a group. Show that the set of maps $f : S \to G$ is a group $\text{Map}(S, G)$, under the operation $c(f, g)(s) = f(s)g(s)$ where the product on the right hand side is taken in the group G. Suppose S has two elements. Show that $\text{Map}(S, G) \cong G \times G$.

Problem 5. For X a subgroup of Y, recall that the index of X in Y is the number of equivalence classes of cosets of X in Y. Denote this by $(Y : X)$. If L is a subgroup of H which is a subgroup of G, show that

$$(G : H)(H : L) = (G : L)$$

Problem 6. Let H be a subgroup of G. Show that the action of G on the set of equivalence classes of cosets G/H defines a homomorphism $\phi : G \to S_{G/H}$. Show that the kernel K of ϕ is contained in H. If H is *normal*, then show $K = H$. Using ϕ show that if the index of H in G is finite, then the index of K in G is finite. (Recall that H is *normal* if for any $g \in G$, $gHg^{-1} = H$ as subsets of G)

Problem 7. Show that

$$\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/mn\mathbb{Z}$$

if and only if n and m are coprime.

Problem 8. Show that $S_2 \cong D_1 \cong \mathbb{Z}/2\mathbb{Z}$. Also, show that $S_3 \cong D_3 \cong GL_2(\mathbb{Z}/2\mathbb{Z})$.

Problem 9. Suppose A is an Abelian group (i.e. $gh = hg$ for all $g, h \in A$). Fix an integer n and let $\phi_n : A \to A$ be defined by $\phi_n(g) = g^n$. Show ϕ_n is a homomorphism. Suppose A is finite and the order of A is coprime with n. Show ϕ_n is an isomorphism.

Problem 10. An *automorphism* of G is an isomorphism from G to itself. Show that the collection of automorphisms of G is itself a group, denoted $\text{Aut}(G)$.

Problem 11 (Hard). If $g \in G$, the map $\phi_g : G \to G$ defined by $\phi_g(h) = ghg^{-1}$ defines an automorphism of G (check!). Such automorphisms are called *inner automorphisms*. Show that the map $G \to \text{Aut}(G)$ defined by $g \to \phi_g$ is a homomorphism, and therefore that the set of inner automorphisms is a group. Show in fact that it is a *normal* subgroup of $\text{Aut}(G)$. The quotient group $\text{Aut}(G)/\text{Inn}(G)$ is called the group of *outer automorphisms of* G.

DEPARTMENT OF MATHEMATICS, HARVARD, CAMBRIDGE, MA 02138
E-mail address: dannyc@math.harvard.edu