CHAPTER 0: CONSTRUCTIONS AND EXAMPLES

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Abstract. This is Chapter 0 of a book on 3-Manifolds. This Chapter introduces several standard techniques for building 3-manifolds, together with a range of illustrative examples.

Contents

1. Building 3-manifolds

In this chapter we introduce techniques for constructing 3-manifolds and give a range of examples. The tone is informal, and several concepts are introduced for which a precise technical definition must wait until a later chapter. A manifold is said to be *closed* if it is compact without boundary. Although our main focus and interest is 3-manifolds, along the way we are naturally led to consider 2-manifolds, more usually called *surfaces*, and 1-manifolds which (if closed and embedded in a 3-manifold) are called *knots* or *links*.

1.1. **Euler characteristic.** One of the simplest and most useful invariants of a (sufficiently nice) topological space X is the *Euler characteristic* $\chi(X)$. If X is a finite cell complex (e.g. a simplicial complex) then $\chi(X)$ is equal to the alternating sum of the number of cells in each dimension; i.e.

$$\chi(X) = \sum_i (-1)^i \cdot \text{number of cells of dimension } i$$

One may also compute $\chi(X)$ directly from the singular homology of X, via the formula $\chi(X) = \sum(-1)^i \cdot \text{rank}(H_i(X))$.

Example 1.1 (Surfaces). If M_1, M_2 are manifolds of the same dimension, the *connect sum* of M_1 and M_2 (denoted $M_1 \# M_2$) is obtained from the disjoint union of the M_i by removing an

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open ball from each of the M_i and gluing the resulting boundary spheres by an orientation-reversing homeomorphism. If S_1 and S_2 are surfaces, and if $S = S_1 \# S_2$ then $\chi(S) = \chi(S_1) + \chi(S_2) - 2$.

(1) Every closed connected oriented surface may be obtained from S^2 by repeatedly connect summing with tori T^2. Since $\chi(T^2) = 0$, the result of connect summing S^2 with g tori satisfies $\chi = 2 - 2g$ and is called the *closed surface of genus g*.
(2) Every closed connected nonorientable surface may be obtained from S^2 by repeatedly connect summing with projective planes $\mathbb{RP}^2$. Since $\chi(\mathbb{RP}^2) = 1$ the result of connect summing S^2 with m projective planes satisfies $\chi = 2 - m$. Notice that the 2-sphere uniquely realizes the maximum of χ among all closed connected surfaces.
(3) Every compact surface may be obtained from a closed surface by removing finitely many open disks. If S is a genus g surface minus p disks, then $\chi(S) = 2 - 2g - p$.

If M is a closed connected oriented 3-manifold, Poincaré duality and the universal coefficient theorem together imply that $\text{rank}(H_i(M)) = \text{rank}(H_{3-i}(M))$ for all i. Thus $\chi(M) = 0$. Since Euler characteristic is multiplicative under finite covers and additive under disjoint union, it follows that $\chi(M) = 0$ for *any* closed 3-manifold, orientable or not.

Example 1.2 (Nonorientable 3-manifolds). Let M be a closed connected non-orientable 3-manifold. Then $\chi(M) = 0$. Since $H_0(M) = \mathbb{Z}$ and $H_3(M) = 0$, this implies that $\text{rank}(H_1(M)) > 0$. In particular, the abelianization of $\pi_1(M)$ is infinite, and surjects onto $\mathbb{Z}$.

There is a straightforward relationship between the Euler characteristic of a compact 3-manifold and the Euler characteristic of its boundary:

Lemma 1.3. *Suppose M is a compact 3-manifold with boundary ∂M. Then $\chi(\partial M) = 2\chi(M)$.*

Proof. The *double of M*, denoted DM, is the 3-manifold obtained by taking two disjoint copies of M and gluing them by the identity along ∂M. This is a closed 3-manifold, and therefore satisfies $\chi(DM) = 0$. On the other hand, by e.g. Meyer-Vietoris,

$$\chi(DM) = 2\chi(M) - \chi(\partial M)$$

□

A closed (not necessarily connected) surface S bounds a 3-manifold if and only if $\chi(S)$ is even. Necessity follows from Lemma 1.3; sufficiency follows by an explicit construction.

1.2. **Triangulations.**

Every 3-manifold may be *triangulated* — i.e. obtained from a disjoint collection of tetrahedra by gluing faces in pairs. Compact 3-manifolds may be obtained from finite collections of tetrahedra; non-compact ones require infinitely many.

Every simplicial 2-complex obtained by edge pairing a finite collection of triangles is a closed 2-manifold (not necessarily connected or orientable), but not every simplicial 3-complex obtained by face pairing a finite collection of tetrahedra results in a 3-manifold.

Let's fix notation. Let T be a finite disjoint collection of tetrahedra, and suppose we are given a gluing map that pairs each face of each tetrahedron homeomorphically with some other face. Let X denote the result of gluing, thought of as a quotient space of T, and let's insist that X is a simplicial complex, or perhaps what is more generally called a Δ-complex: i.e. the interior of each simplex of T embeds in the quotient space X. This condition just amounts to the requirement that the different identifications of edges that result by composing the various face gluings are compatible, and do not result (for example) in an edge being 'folded in half'.

Proposition 1.4. *Let X be a simplicial (or Δ) complex obtained from a finite disjoint collection of tetrahedra T by face gluing. Then X is a 3-manifold if and only if $\chi(X) = 0$.*

Proof. A finite simplicial 3-complex is always a manifold away from the vertices, and is a manifold near each vertex if and only if the vertex link is a 2-sphere. We shall see that this holds at every vertex if and only if $\chi(X) = 0$.

Let U be the disjoint union of the triangular vertex links of the tetrahedra of T. The face pairing of T induces an edge pairing of U, and the image of U in X is therefore a triangulated surface L. Evidently the components of this surface are the vertex links of X.

Let's denote the vertices of X by v_i and the corresponding vertex links by L_i. Since each L_i is a closed surface, $\chi(L_i) \le 2$ with equality if and only if L_i is a sphere. Let N be the compact 3-manifold with boundary obtained by removing the vertices from X and replacing each vertex v_i by a copy of L_i. By Lemma 1.3 we have $2\chi(N) = \sum \chi(L_i)$. On the other hand, X may be obtained from N by collapsing each L_i to a single point, and therefore

$$\chi(X) = \chi(N) + \sum_i (1 - \chi(L_i)) = \sum_i (1 - \chi(L_i)/2)$$

It follows that $\chi(X) \ge 0$ with equality if and only if every vertex link is a 2-sphere. □

Example 1.5 (600 cell). Five regular Euclidean tetrahedra very nearly fit together around an edge. The dihedral angles are all $\cos^{-1}(1/3) \sim 70.53°$ which is very close to $72°$. If you take five physical tetrahedra (for example, 4-sided Dungeons and Dragons dice) and squeeze them together, you could imagine that they would close up under enough pressure.

The dihedral angles in a regular spherical tetrahedron are larger than those in a Euclidean one, so five regular spherical tetrahedra all with edge length $\cos^{-1}(1/2) = \pi/3$ will fit together perfectly (and rigidly) around an edge. One may extend this configuration by packing in more regular spherical tetrahedra, five around every edge, until one tessellates S^3 completely with exactly 600 of them.

The symmetries of this tessellation form the Coxeter group H_4 with diagram

; this acts transitively on the set of tetrahedra, and the stabilizer of each tetrahedron is the full permutation group S_4. Thus the order of the group is 14400.

Example 1.6 (Ideal triangulations). If X is a simplicial 3-complex with $\chi(X) > 0$ then some vertex links are not spheres. Removing these vertices produces a noncompact 3-manifold, that may be compactified by adding back the vertex links as boundary components. An *ideal* tetrahedron is a tetrahedron with the vertices removed; consequently we may speak of *ideal triangulations*.

If M is a closed 3-manifold and $L \subset M$ is a knot or link in M, then $M - L$ is a noncompact 3-manifold that may be compactified by adding back a torus for each component of L (equivalently, one may take an open neighborhood $N(L)$ of L, and then consider the

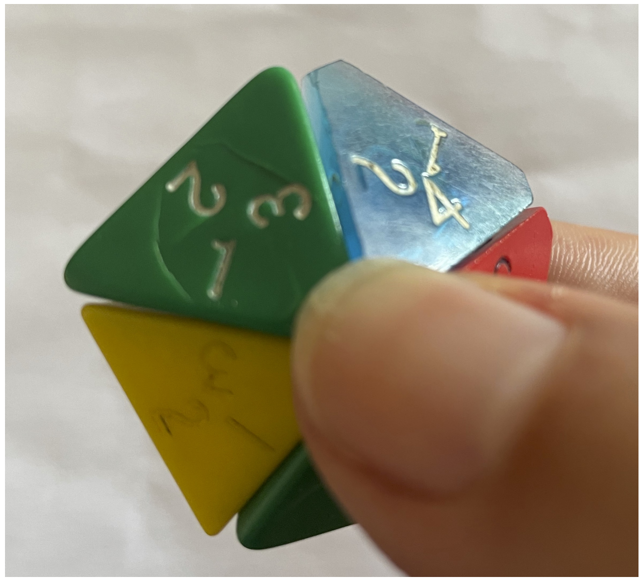

FIGURE 1. Five regular Euclidean tetrahedra very nearly fit together around an edge.

compact 3-manifold $M - N(L)$). Such manifolds may often be efficiently represented by ideal triangulations. See Example 2.2 for some examples.

Example 1.7 (Gieseking manifold). The *Gieseking manifold* is obtained from a single ideal tetrahedron by a suitable face pairing. Figure 2 indicates a choice of orientation on the edges of a tetrahedron; there is a combinatorially unique orientation-preserving face pairing that respects the orientations on edges.

The quotient simplicial complex X has one tetrahedron, two triangles, one edge and one vertex; thus $\chi(X) = 1$ and the vertex link has $\chi = 0$. Since the face gluings are orientable, the manifold obtained from X by removing the vertex is non-orientable, and may be compactified to a closed manifold by adding a Klein bottle boundary.

All six edges of the tetrahedron fit together around the unique edge of X. The dihedral angles in a regular hyperbolic tetrahedron are smaller than those in a Euclidean one; as the common edge lengths get longer the dihedral angle shrinks until as the edges get infinitely long, the dihedral angle becomes exactly $60°$. Thus if one takes a regular ideal hyperbolic tetrahedron one may perform the face gluings for the Gieseking manifold isometrically in such a way that the resulting non-compact manifold inherits a complete hyperbolic structure.

Example 1.8 ($\mathrm{PGL}(2, \mathbb{F}_7)$). Simplicial complexes arise in many ways in mathematics. It is not so common for such complexes to be manifolds, but there are some interesting examples.

Let P be the projective line over $\mathbb{F}_7$, the field with 7 elements. As a set, P consists of 8 points that we may identify with $\mathbb{F}_7 \cup \infty$. The group $G := \mathrm{PGL}(2, \mathbb{F}_7)$ acts on P; the element represented by the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts on $z \in P$ by $z \to (az + b)/(cz + d)$.

G acts simply transitively on ordered distinct triples of points in P. The stabilizer of any specific triple is therefore the full permutation group S_3. For example, the stabilizer

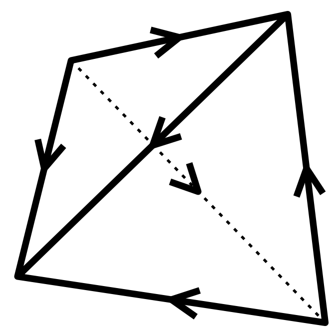

FIGURE 2. Edge orientations determining the face pairing that results in the Gieseking manifold.

of $0, 1, \infty$ is generated by $z \to 1/(1-z)$ and $z \to 1-z$ or as matrices, $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The first matrix fixes 3 and 5 while the second matrix permutes them. The 'meaning' of 3 and 5 is that these are the primitive 6th roots of unity in $\mathbb{F}_7$.

Let X be the simplicial complex whose 'simplices' are the G orbits of $0, 1, \infty, 3$ in the space of distinct 4-tuples of elements of P and whose 'faces' are glued if they agree as subsets of P. By our previous observation, every triangle (for instance $0, 1, \infty$) lies in exactly two tetrahedra, so the resulting complex is a 3-manifold except possibly at the eight vertices. Let M be equal to X minus its vertices. This is a noncompact 3-manifold. One may check that there are exactly six tetrahedra around each edge; for example, the edge $0, \infty$ lies in the tetrahedra $0, 1, \infty, 3$ and $0, 1, \infty, 5$ and their images under powers of the order 3 element $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 5 \end{smallmatrix}\right)$. Consequently, as in Example 1.7 one may realize a complete hyperbolic structure on M by building X out of regular ideal hyperbolic tetrahedra.

The simplicial complex X is made up of 42 tetrahedra, 84 triangles, 42 edges and 8 vertices, so $\chi(X) = 8$. Every face gluing is orientation-reversing, so M is orientable. The group G acts transitively on the vertices; it follows that every vertex link is a torus, and M may be compactified by adding 8 boundary tori, one for each vertex of X.

The full symmetry group of the tessellation of hyperbolic 3-space by regular ideal hyperbolic tetrahedra is $\mathrm{PGL}(2, \mathcal{O}_3)$ where $\mathcal{O}_3$ is the ring of 'Eisenstein integers' in the field $\mathbb{Q}(\sqrt{-3})$. If we denote the primitive sixth root of unity $(1+\sqrt{-3})/2$ by ζ, then $\mathcal{O}_3 = \mathbb{Z}[\zeta]$ and the reduction $\mathcal{O}_3 \to \mathbb{F}_7$ given by sending ζ to 3 induces a surjective homomorphism $\mathrm{PGL}(2, \mathcal{O}_3) \to \mathrm{PGL}(2, \mathbb{F}_7)$ whose kernel is $\pi_1(M)$. Remarkably, M is homeomorphic to the complement of the 8-component link in S^3 in Figure 3 as discovered by Thurston [7].

1.3. **Heegaard splittings.**

Definition 1.9 (Handlebody)**.** Let M be a 3-manifold, and let Γ be a finite connected graph embedded in the interior of M. The closure H of a regular neighborhood of Γ is a compact 3-manifold with boundary called a *handlebody*.

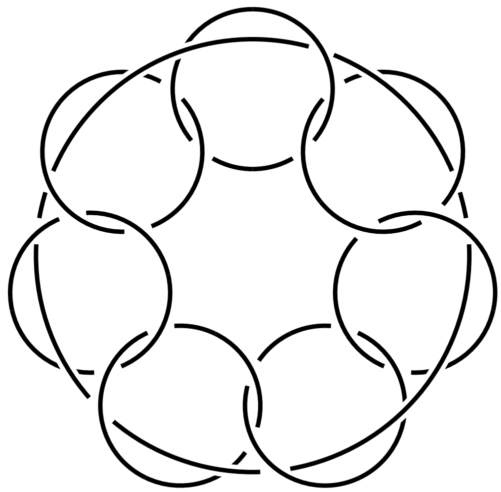

FIGURE 3. The complement of this 8-component link in S^3 is homeomorphic to M.

The Euler characteristic of a connected finite graph Γ is ≤ 1 with equality if and only if Γ is a tree, and in general is $1-g$ if Γ is homotopic to a wedge of g circles (such a homotopy may be obtained by collapsing a maximal subtree to a point).

By Lemma 1.3 we have $\chi(\partial H) = 2-2g$. The surface ∂H is orientable if and only if H is (of course, this is automatic if M is orientable). If H is orientable and $\chi(H) = 1-g$ we say H is a *handlebody of genus* g. Note that H deformation retracts to Γ, so that $\pi_1(H)$ is free on g generators.

Definition 1.10. Let M be a closed 3-manifold. A *Heegaard splitting* for M is a decomposition into two handlebodies $M = H_1 \cup H_2$ glued by a homeomorphism along their common boundaries. The *genus* of the Heegaard splitting is the common genus of H_1 and H_2; this is also equal to the genus of the *splitting surface* $\partial H_1 = \partial H_2$

Example 1.11 (Lens spaces)*.* The only 3-manifold with Heegaard genus 0 is the 3-sphere, obtained by identifying the boundaries of two 3-balls.

In $\mathbb{C}^2$ with standard complex coordinates z_1, z_2 the 3-sphere is the level set $|z_1|^2 + |z_2|^2 = 1$. The *Clifford torus* $|z_1|^2 = |z_2|^2 = 1/2$ is the splitting surface of a genus 1 splitting of S^3, bounding genus 1 handlebodies H_1 where $|z_1| > |z_2|$ and H_2 where $|z_2| > |z_1|$. The core of H_1 is the circle where $|z_1| = 1$ and $z_2 = 0$, and the core of H_2 is the circle where $z_1 = 0$ and $|z_2| = 1$. Note that a genus 1 handlebody is topologically a product $D^2 \times S^1$

For p and q coprime there is a free action of $\mathbb{Z}/p\mathbb{Z}$ on S^3 generated by $(z_1, z_2) \to (e^{2\pi i/p}z_1, e^{2\pi i q/p}z_2)$. The quotient 3-manifold is the Lens space $L(p,q)$. Note that the result only depends on the residue of q mod p, so usually we normalize q by requiring $0 < q < p$. Each of the handlebodies H_1, H_2 is preserved by the group action, and the quotients are also genus 1 handlebodies; thus each Lens space has a Heegaard splitting of genus 1.

Conversely, any 3-manifold with a genus 1 Heegaard splitting is either a Lens space, or $S^2 \times S^1$ obtained by doubling $D^2 \times S^1$.

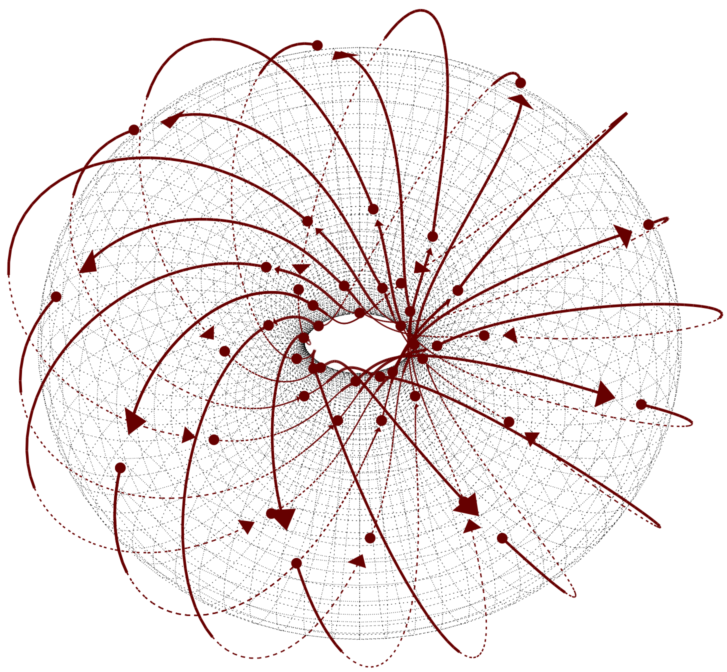

FIGURE 4. An order 5 symmetry of the Clifford torus in S^3 whose quotient is the splitting surface for a genus 1 Heegaard splitting of $L(5, 2)$.

Figure 4 shows the projection of the Clifford torus to $\mathbb{R}^3$ under stereographic projection, together with some arrows indicating the generator of the $\mathbb{Z}/p\mathbb{Z}$ action described above with $p = 5$ and $q = 2$.

Example 1.12 (Triangulations). Let τ be a triangulation of a closed 3-manifold M. Let Γ denote the 1-skeleton of τ, and let Γ' denote the dual 1-skeleton: i.e. Γ has one vertex in the center of each tetrahedron of τ, and one edge intersecting each face and running between the vertices in the center of the adjacent tetrahedra. Evidently M admits a Heegaard splitting where H_1 and H_2 are closed regular neighborhoods of Γ and Γ' respectively.

Example 1.13 (Morse functions). Let M be a closed 3-manifold and let $f : M \to \mathbb{R}$ be a Morse function; i.e. the critical points of f are isolated and nondegenerate. By general position we may assume that f is self-indexing; i.e. that the critical points of index i are all contained in the level set $f^{-1}(i)$ for each of $i = 0, 1, 2, 3$.

A Riemannian metric on M determines the vector field $\text{grad}(f)$, which is everywhere perpendicular to the level sets of f. The integral curves of this vector field are the *gradient flowlines*. At each index 1 critical point there are two gradient flowlines asymptotic to the given point in forward time; in backward time they are asymptotic to index 0 critical points. The closure of the union of these flowlines, over all index 1 critical points, is a graph Γ. Likewise, at each index 2 critical point there are two gradient flowlines asymptotic to the given point in *backward* time; in forward time they are asymptotic to index 3 critical points, and the closure of the union of these flowlines is another graph Γ'. The level surface

$S := f^{-1}(3/2)$ separates M into two handlebodies $H_1 := f^{-1}[0, 3/2]$ and $H_2 := f^{-1}[3/2, 3]$ which deformation retract onto Γ and Γ' respectively, and thereby gives rise to a Heegaard splitting of M.

Example 1.14 (Handle structure). The Heegaard splitting coming from a Morse function inherits some extra combinatorial data in the form of a *handle structure*, which makes M into a kind of thickened CW complex. In a neighborhood of a critical point of index i, the descending resp. ascending gradient flowlines from the critical point fill out disks of dimension i and $3-i$ respectively, and these disks form the *core* $D^i \times 0$ and *cocore* $0 \times D^{3-i}$ of an *i-handle* $D^i \times D^{3-i}$.

In the example above, H_1 is the union of the 0-and 1-handles, and H_2 is the union of the 2 and 3-handles. The cocores of the 1-handles resp. the cores of the 2-handles form a system of disks in H_1 resp. in H_2 that chop it up into 3-balls, each of which contains a single index 0 resp. index 3 critical point. The boundary circles of these cocores resp. cores form a system of circles $\{\alpha_i\}$ resp. $\{\beta_j\}$ on the splitting surface S.

If S has genus g, then by canceling critical points of adjacent index if necessary, we may always choose a self-indexing Morse function with $S := f^{-1}(3/2)$ which has exactly one critical point of index 0 and 3, and exactly g critical points of index 1 and 2. Thus we obtain two systems of g embedded circles $\{\alpha_i\}, \{\beta_j\}$ on S.

The complement $H_1 - \cup$(cocores of 1-handles) is a ball, and the boundary of this ball may be compactified by adding $S - \cup\alpha_i$ together with $2g$ disks, one on either side of each 1-handle cocore. In particular, $S - \cup\alpha_i$ is a planar surface, homeomorphic to S^2 minus $2g$ disks (and the same is true for the β_j).

Conversely, if S is any genus g surface and $\{\alpha_i\}, \{\beta_j\}$ are two systems of g embedded circles for which $S - \cup\alpha_i$ and $S - \cup\beta_j$ are spheres minus $2g$ disks, then this data determines a Heegaard splitting of a 3-manifold. For, we can glue g thickened disks to one side of S along the α_i curves, g thickened disks to the other side of S along the β_j curves, and fill in the resulting S^2 boundaries with a 3-ball on either side.

Example 1.15 (Handle slides). Let H be a genus g handlebody, and let $\{\alpha_i\}$ be a collection of g circles on ∂H bounding the core disks $\{D_i\}$ of a system of 2-handles for H. One may modify this system by isotopy in H, together with a combinatorial operation called a *handle slide*.

Suppose that γ is an oriented arc in ∂H properly embedded in $\partial H - \cup\alpha_i$ that runs from a point on α_i to a point on α_j. A neighborhood of γ on ∂H is a rectangle R, and the union of this rectangle with the disks D_i and D_j is itself a topological disk E. We may push E off itself by a proper isotopy to a new disk E' properly embedded in H and with boundary $\alpha' = \partial E'$ an embedded circle in ∂H disjoint from the α_i.

The disk E' is said to be obtained by a *handle slide of D_i over D_j*. Replacing D_i by E produces a new system of 2-handles for H, and a theorem of Whitehead says that any two systems of 2-handles are related to each other by a finite sequence of handle slides plus isotopy.

Example 1.16 (Stabilization and destabilization). If f is a Morse function on a manifold, we may perturb f locally by introducing a canceling pair of critical points p, p' of respective index i and $i+1$ that are joined by a single gradient flowline; conversely a pair of critical points joined by a single gradient flowline may be perturbed away. The first operation is called *stabilization* and the second *destabilization*. If f is a Morse function on a 3-manifold stabilization usually refers to the introduction of a pair of critical points of index 1 and 2. In the corresponding handle structure the boundary circle of the cocore of the 1-handle intersects the boundary circle of the core of the 2-handle in a single (transverse) point, corresponding to the unique gradient flowline connecting the two critical points. Conversely, if M has a genus g Heegaard splitting with a handle structure for which there is a pair of circles α_i, β_j that meet transversely in a single point, one may 'cancel' the handles to produce a Heegaard splitting of genus $g - 1$.

A Heegaard splitting with a handle structure gives rise to a presentation for $\pi_1(M)$:

Proposition 1.17 (Balanced Presentation)**.** *A handle structure on M with one 0-cell and one 3-cell gives rise to a* balanced presentation *for $\pi_1(M)$; i.e. a presentation with the same number of generators as relations.*

Proof. The Heegaard splitting presents M as the union of two handlebodies H_1 and H_2. As remarked earlier, $\pi_1(H_1)$ is free on g generators if the splitting has genus g; we may take the free generators to be the cores of the 1-handles. By Seifert van-Kampen attaching each 2-handle along the boundary of its cocore imposes a relation on π_1, which may be taken to be the free homotopy class of the attaching loop β_j in H_1. Attaching the 3-handle does not affect π_1. □

Suppose M has a handle structure of genus g. Let

$$\pi_1(M) = \langle a_1, \cdots, a_g \mid r_1, \cdots, r_g \rangle$$

be a balanced presentation obtained as above, where a_i is the core of the 1-handle with cocore α_i, and r_j is the relation associated to the 2-handle with attaching loop β_j. If we isotop the curves α_i and β_j in S to be transverse and to intersect the minimal number of times (one says the α_i and the β_j meet *efficiently*) then each r_j is a cyclically reduced word of length equal to the cardinality of $\beta_j \cap \cup_i \alpha_i$, with one letter a_i or a_i^{-1} (in cyclic order) for each transverse positive or negative intersection of β_j with α_i.

Example 1.18 (Poincaré Homology Sphere). A *Heegaard diagram* is a diagram of a genus g surface together with two families of curves that are the attaching circles for a genus g handle structure on a 3-manifold. In a 1904 paper ([4]) on the nascent field of topology (dubbed 'analysis situs'), Poincaré gave an example of a closed 3-manifold with the homology of the 3-sphere but with nontrivial fundamental group; this manifold is now known as the *Poincaré Homology Sphere*. In Poincaré's paper this example is presented via what is essentially a Heegaard diagram.

Drawing the α and β curves legibly on a genus 2 surface is challenging. What is easier is to draw the result of cutting the surface along the α curves, thereby giving rise to a planar surface — a sphere with four holes — on which the cut up β curves reduce to a system of proper arcs. Figure 5 shows such a diagram for the Poincaré Homology Sphere; it is essentially equivalent to the figure in Poincaré's paper.

From the diagram one can read off a balanced presentation for $\pi_1(M)$:

$$\pi_1(M) = \langle a, b \mid a^4ba^{-1}b, b^{-2}a^{-1}ba^{-1} \rangle$$

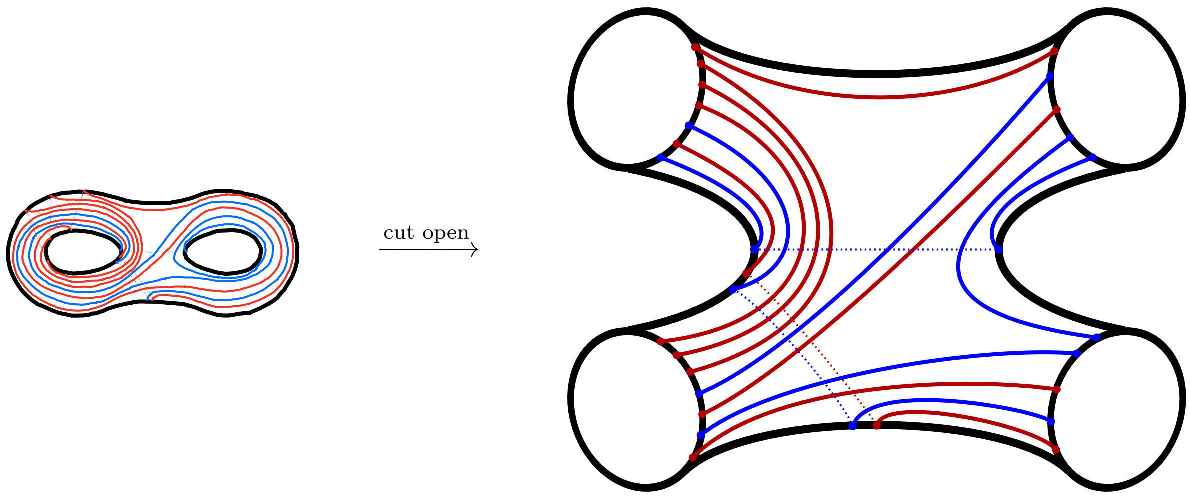

FIGURE 5. A Heegaard diagram for the Poincaré Homology Sphere cut open along the α circles (not pictured) onto a planar surface.

The abelianization of this group is the quotient of the group $\mathbb{Z}^2$ by the subgroup generated by the vectors $\left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}-2\\-1\end{smallmatrix}\right)$; since $\det\left(\begin{smallmatrix}3 & -2\\2 & -1\end{smallmatrix}\right) = 1$ this quotient is trivial, so that $H_1(M) = 0$. The universal coefficient theorem and Poincaré duality imply that $H_2(M) = 0$, and since M is closed and orientable, $H_3(M) = \mathbb{Z}$. Thus $H_*(M) \cong H_*(S^3)$; i.e. M is a homology sphere (as its name implies).

To see that $\pi_1(M)$ is not trivial, we may obtain a homomorphism ϕ from $\pi_1(M)$ to the group of orientation-preserving symmetries of a regular dodecahedron. Let F be a face of the dodecahedron and V a vertex of F. Let $\phi(a)$ be a rotation through angle $-2\pi/5$ with axis through the center of F, and let $\phi(b)$ be a rotation through angle $2\pi/3$ with axis through V. If G denotes the image $\phi(\pi_1(M))$ then $\phi(a)$ has order 5 in G and $\phi(b)$ has order 3, so that the defining relations of $\pi_1(M)$ both say $\phi(a^{-1}b)^2 = 1$ which is true, since $\phi(a^{-1}b)$ is a rotation through angle π with axis through the center of an edge of F containing V. The homomorphism ϕ is easily seen to be surjective to the full group of orientation-preserving symmetries of the dodecahedron which has order 60, and the kernel turns out to be of order 2 generated by the central element $a^5 = ab^{-1}ab^{-1} = b^3$. Thus the order of $\pi_1(M)$ is 120.

In the same paper Poincaré asked the question whether it was possible for a closed 3-manifold with trivial fundamental group to fail to be homeomorphic to the 3-sphere; the negation of this possibility became known as the *Poincaré Conjecture*, which dominated topology for a hundred years until its final resolution by Perelman using methods from Ricci Flow.

Example 1.19 (Compression body). Let M be a compact 3-manifold with boundary. We can construct a self-indexing Morse function without local maxima $f : M \to [0, 5/2]$ for which $\partial M = f^{-1}(5/2)$ is a nonsingular level set. The level surface S separates M into a handlebody $H_1 := f^{-1}[0, 3/2]$ and a *compression body* $H_2 := f^{-1}[3/2, 5/2]$ which is obtained from a collar neighborhood of S by attaching 2-handles.

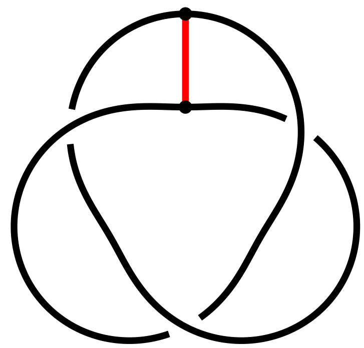

FIGURE 6. The right-handed trefoil knot K is depicted in black. The red arc α is an *unknotting arc*; i.e. the complement of an open neighborhood of $\Gamma := K \cup \alpha$ is a (genus 2) handlebody.

For example, let $K \subset S^3$ be the (right-handed) trefoil knot, and let $M = S^3 - N(K)$ where $N(K)$ is a solid torus open neighborhood of K; Figure 6 depicts the knot K in black. The red arc α in the figure runs from K to itself, and the union $\Gamma := K \cup \alpha$ is what is called a *theta graph* because it is homeomorphic to the letter θ.

The manifold M is not a handlebody; equivalently, the knot K is 'genuinely' knotted, i.e. it is not isotopic to a round circle. On the other hand, the theta graph Γ *is* unknotted in the sense that $H := S^3 - N(\Gamma)$ is a (genus 2) handlebody. Thus M has the structure of a compression body obtained from H by attaching a single 2-handle whose cocore is the arc α.

The arc α in Example 1.19 is called an *unknotting tunnel* for the knot K. More generally, for a knot K (or link L) in S^3 a system of proper embedded arcs $\alpha_1, \cdots, \alpha_g$ in $S^3 - K$ is a *system of unknotting tunnels* if the graph $\Gamma := K \cup_j \alpha_j$ has the property that $S^3 - N(\Gamma)$ is a handlebody (of genus g). The *unknotting number* of a knot K is the least g for which a system of g unknotting tunnels exists; it is always finite.

1.4. **Surgery.** Individual 3-manifolds can be studied on their own, but it is also important to understand the relations between different 3-manifolds. Distinct 3-manifolds may be related by *surgery*.

The idea of surgery is simple: start with a closed 3-manifold M and a compact submanifold $\bar{A} \subset M$. Choose a homeomorphism $\phi : \partial\bar{A} \to \partial\bar{A}$. If A denotes the interior of $\bar{A}$ we can form a compact 3-manifold $M - A$ with $\partial(M - A) = \partial\bar{A}$ and then close it back up by gluing back $\bar{A}$ by attaching its boundary to $\partial(M - A)$ by ϕ. The result is a new closed 3-manifold M'. If M and A are oriented and ϕ is orientation-preserving then M' will be oriented too.

Example 1.20 (Mapping class group). Suppose $\partial\bar{A} = S_g$, the closed oriented surface of genus g. Now $\phi : \partial\bar{A} \to \partial\bar{A}$ is a self-homeomorphism of S_g. The set of orientation-preserving self-homeomorphisms of S_g form a topological group in the compact-open topology, and the path components are precisely the *isotopy classes* of homeomorphisms. This set of path components is itself a group, called the *mapping class group* of genus g, usually denoted Mod_g or Γ_g.

Some self-homeomorphisms of $\partial\bar{A}$ might extend to self-homeomorphisms of $\bar{A}$; this property depends only on the isotopy class of ϕ in Mod_g, and those classes that extend form a subgroup E_A. Evidently if M' is obtained from M by surgery the homeomorphism type of M' depends only on the coset of ϕ in Mod_g/E_A.

Example 1.21 (Heegaard splittings). Any two closed 3-manifolds with Heegaard splittings of genus g are related by surgery where $\bar{A} = H_g$, a handlebody of genus g. By stabilization, any two closed oriented 3-manifolds admit Heegaard splittings of some common genus g.

Example 1.22 (Alexander trick). Any (orientation-preserving) homeomorphism $\phi : S^2 \to S^2$ extends to a self-homeomorphism of B^3 by coning ϕ radially to the center. This extension is not typically smooth even if ϕ is, and in higher dimensions sometimes a smooth extension does not exist, but Smale showed that any orientation-preserving diffeomorphism $\phi : S^2 \to S^2$ may be smoothly isotoped to the identity. Thus there is only one way to (smoothly) attach a 3-ball to a compact 3-manifold with S^2 boundary.

An extremely important special case of surgery is that $\bar{A}$ is a solid torus, i.e. a handlebody of genus 1. This case of surgery was first studied by Dehn, and is known as *Dehn surgery*. We shall return to it in § 2.4.

2. Knots and Links

A circle smoothly embedded in a 3-manifold is called a *knot*, and a disjoint union of smoothly embedded circles is a *link*. Knots or links are *equivalent* if they are smoothly isotopic in their ambient manifolds. Knots and links are interesting in their own right, and additionally they give a powerful method to represent 3-manifolds (and relations between them) via *Dehn surgery*.

2.1. **Diagrams and Reidemeister moves.** Let $L \subset S^3$ be a (knot or) link. We may obtain a *projection* of L by removing a point from $S^3 - L$ to produce $\mathbb{R}^3$, and then projecting $\mathbb{R}^3$ orthogonally to the horizontal plane $\mathbb{R}^2$ by $\pi : \mathbb{R}^3 \to \mathbb{R}^2$. If we wiggle L slightly by an isotopy to put it in general position, then $\pi : L \to \mathbb{R}^2$ will be an immersion on each component, and the image will contain at most finitely many transverse double points (called *crossings*) and no other singularities. Suppose $p \in \pi(L)$ is a crossing, i.e. the preimage $\pi^{-1}(p)$ is a line in $\mathbb{R}^3$ that intersects L transversely in two points. If we orient this line (equivalently, if we coorient $\mathbb{R}^2$ in $\mathbb{R}^3$) then one point of $\pi^{-1}(p) \cap L$ lies 'above' the other, and this is generally indicated in the projection by omitting the image under π of a small segment of L near the point which is 'underneath'. Knot and link projections are the pictures of knots and links one encounters in practice; Figure 3 and Figure 6 are examples.

If L and L' are isotopic links in S^3, the projections of L and L' are related by a finite sequence of combinatorial moves which alternate between isotopy of the projection, and one of the following three local modifications of a diagram, called *Reidemeister moves* (see Figure 7).

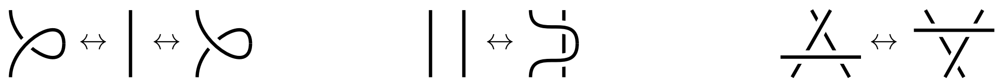

FIGURE 7. Any two projections of isotopic links are related by planar isotopy and a finite number of Reidemeister moves.

One way to see this is to think about singularities of the vertical projections in a generic smooth isotopy of L to L'. A dimension count in spaces of real algebraic curves shows that three kinds of singularities will occur at isolated times during the isotopy:

(1) the tangent to a knot becomes vertical at a point, and the knot projects this tangency to an ordinary cusp;
(2) two arcs of the projection become tangent to first order at a point (this is called a *tacnode*); or
(3) three arcs intersect at an ordinary triple point.

A transition through one of these singularities gives rise to one of the three Reidemeister moves.

Example 2.1 (An amphichiral knot). Reidemeister moves are not always the easiest way to indicate an isotopy between two different projections of isotopic knots. A knot K in S^3 is *chiral* if it is not isotopic to its image under some (equivalently: any) orientation-reversing homeomorphism of S^3 (e.g. a reflection of the round S^3 in $\mathbb{R}^4$ in a hyperplane through the origin), otherwise it is *amphichiral*. The trefoil is chiral; it comes in left and right-handed versions (Figure 6 depicts a right-handed trefoil). The figure 8 knot is amphichiral; see Figure 8.

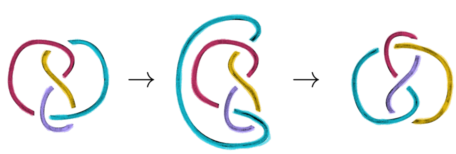

FIGURE 8. The figure 8 knot is amphichiral.

An isotopy between the figure 8 knot and its mirror image is indicated in the figure. First, pull the rightmost strand to the left over the rest of the knot. Then the diagram may be isotoped into the desired form.

It is not hard to visualize how to achieve the first move by an isotopy, but in the course of the isotopy the projection will change by multiple Reidemeister 2 and 3 moves, together with a pair of Reidemeister 1 moves at the end.

Example 2.2 (Alternating link complements). It is a demanding exercise in visualization to translate a knot or link projection into a combinatorial ideal triangulation of the complement, but there is a systematic method which works well for alternating links.

A link projection is *alternating* if, as we move along any component of the link in any orientation, the crossings alternate between overcrossings and undercrossings. A link is alternating if it admits some alternating projection.

Suppose L is a link projection. We can embed L in a graph Γ by adding one "vertical" edge for each crossing, which joins the overcrossing point to the undercrossing point. Complementary regions to the projection are polygons, whose edges are arcs of L joining adjacent crossings. A complementary n-gon P to the projection determines a $2n$-gon $\bar{P}$ obtained by inserting a vertical edge at each vertex of P; see Figure 9. Then we can obtain

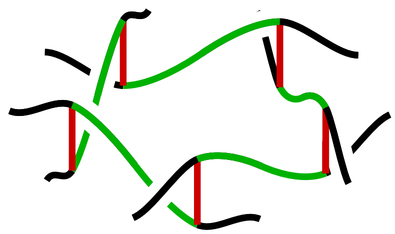

FIGURE 9. Insert red vertical edges at each crossing of the projection to turn a complementary 5-gon P into a 10-gon $\bar{P}$ (edges in red and green). Then collapse the green edges to ideal points to obtain an ideal 5-gon P'.

an ideal n-gon P' from $\bar{P}$ by collapsing the original edges of P (which lie on L) from $\bar{P}$, replacing them by ideal vertices. Thus: the *edges* of P' correspond to the *crossings* on the boundary of the region P and the (ideal) *vertices* of P' correspond to *segments* of the link L running between adjacent crossings.

Now let's suppose L is alternating. The complement $S^3 - L$ is obtained by gluing two (combinatorial) ideal polyhedra $B^\pm$ defined as follows. Each of $\partial B^\pm$ has one copy of each ideal polygon P' as a face, and all faces arise this way. We glue B^+ to B^- along their boundaries by gluing each P' in ∂B^+ to the P' in ∂B^- by the "identity" map. It remains to describe how the copies of P' fit together combinatorially in ∂B^+ and in ∂B^-.

Suppose P, Q are complementary polygons to L which share an edge $e \subset L$ oriented to run from an undercrossing e^- to an overcrossing e^+. Note that the *crossings* $e^\pm$ will correspond to pairs of *edges* of P^+ and Q^+ in ∂B^+ and in ∂B^-. Suppose with respect to the orientation on e that P is on the left and Q is on the right. Then the copies of P' and Q' share one edge in ∂B^+ and one edge in ∂B^- as follows:

- in ∂B^+, P' and Q' meet along e^-; and
- in ∂B^-, P' and Q' meet along e^+.

This determines the way the different P' meet in ∂B^+ and in ∂B^-, and thus the combinatorics of the gluing. In practice we may obtain the gluing by thinking of the projection as the 1-skeleton of (the boundary of) a polyhedral ball E, checkerboard coloring the complementary polygons (i.e. the faces of E) and gluing two copies of E together along these faces, rotating the white faces clockwise and the dark faces anticlockwise through one unit.

If some complementary regions to L are bigons, they give rise to a pair of bigons in ∂B^+ and ∂B^- which may be collapsed to edges without changing the topology of the quotient. This simplification is useful in practice.

The Figure 8 knot K has a projection with 6 complementary regions consisting of 4 triangles and 2 bigons. After collapsing bigons, we obtain $S^3 - K$ by gluing two ideal tetrahedra as in Figure 10. This is an important example; an explicit picture of the face gluing may be found in Figure 7 in Chapter 2.

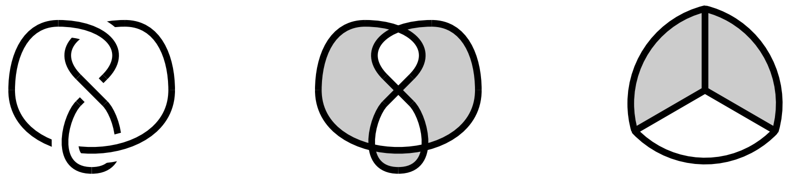

FIGURE 10. Take two copies of the colored tetrahedron (i.e. the rightmost figure) and glue faces to faces, twisting white faces clockwise and dark faces anticlockwise. This exhibits $S^3 -$ figure 8 knot as the union of two ideal tetrahedra.

The Borromean rings L in its standard projection has 8 complementary triangle regions. Both $B^{\pm}$ in this case are ideal octahedra, and $S^3 - L$ can be realized by gluing two ideal octahedra according to the pattern in Figure 11.

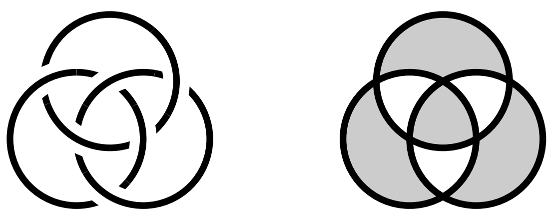

FIGURE 11. Take two copies of the checkerboard colored octahedron (i.e. the rightmost figure) and glue faces to faces, twisting white faces clockwise and dark faces anticlockwise. This exhibits $S^3 -$ Borromean rings as the union of two ideal octahedra.

2.2. **Connect sum.** Suppose $K \subset S^3$ and $K' \subset S^3$ are two oriented knots. We may choose a closed arc $\alpha \subset K$ and $\alpha' \subset K'$ and balls $B_K, B_{K'} \subset S^3$ so that $B_K \cap K = \alpha$ and $B_{K'} \cap K' = \alpha'$. Then $S^3 - B_K$ and $S^3 - B_{K'}$ are both balls containing oriented knotted arcs in their interiors, and we can glue them together along their boundaries to produce S^3 in such a way as to glue the knotted arcs together, forming an (oriented) knot in S^3 called the *connect sum* of K and K', and denoted $K\#K'$. The common sphere along which the balls B_K and $B_{K'}$ are glued is called a *decomposing sphere* for $K\#K'$.

The balls $B_K, B_{K'}$ may be thought of as little beads strung on the knots K and K'. These beads may be slid around the knot between any two places; thus the result of connect sum does not depend (up to isotopy) on the choice of arcs or balls, and one may check that it defines an associative operation on oriented isotopy classes of knots in S^3. Furthermore, this operation is commutative, since it does not depend on an ordering of the knots.

Connect sum with an unknot (i.e. a round circle in S^3) is the identity operation. Apart from associativity and commutativity, there are essentially no other relations between connect sum, so that in the end it is not a very interesting operation. As a special case, the connect sum of two nontrivial knots is never the unknot:

Proposition 2.3. *Let K and K' be oriented knots in S^3. If $K\#K'$ is the unknot then both K and K' are unknots.*

Proof. We give a rather startling proof due to Barry Mazur, and known as the *Mazur swindle*. Suppose $K\#K'$ is the unknot. Then the same is true of $K\#K'\#K\#K'$ and $K\#K'\#K\#K'\#K\#K'$ and so on. We may tentatively form the infinite connect sum $K_\infty := \#^{\mathbb{N}}(K\#K')$ by summing on smaller and smaller copies of $K\#K'$ accumulating only at a single point; see Figure 12.

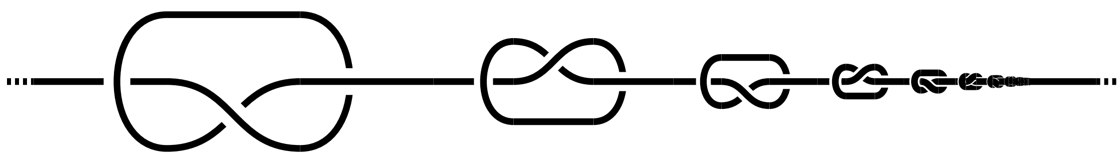

FIGURE 12. The infinite connect sum $K_\infty := \#^{\mathbb{N}}(K\#K')$

A priori this is not a (tame) knot, because it appears to have a wild point where the summands accumulate. However this is deceiving: since by hypothesis $K\#K'$ is unknotted, the apparently knotted segments are really just inefficient projections of unknotted arcs, and this infinite sum K_∞ is an ordinary unknot, albeit drawn with a wild projection. On the other hand, by bracketing the terms differently we have the identity

$$K_\infty = K\#^{\mathbb{N}}(K'\#K) = K\#^{\mathbb{N}}(K\#K') = K\#K_\infty$$

Since K_∞ is an unknot, K is an unknot too, and similarly for K'. □

Remark 2.4. Up to now we have been implicitly working with smooth knots and isotopies; the argument in the proof of Proposition 2.3 implicitly moves us to the world of topological isotopies and homeomorphisms, and technically one ought to show that this does not introduce new equivalences between isotopy classes of smooth knots that cannot be achieved by smooth isotopy. This can be done, but we don't do it here.

2.3. **Linking number.** Suppose K and K' are two disjoint knots in S^3. Let's orient K and K' (arbitrarily). Since S^3 is simply-connected, the knot K' can be shrunk down to a point. The track of this homotopy is an immersed oriented disk D bounding K', and if we wiggle D to be in general position, we can arrange for it to meet K transversely in finitely many points. If D' is any other disk bounding K' in general position (but with the opposite orientation), the union $D \cup D'$ is an oriented 2-sphere S. The signed intersection number of S with K depends only on the homology classes $[S] \in H_2(S^3)$ and $[K] \in H_1(S^3)$ which are both trivial; thus the signed intersection number of D with K is equal to that of D' with K, and depends only on K'. This number is called the *linking number* of K and K', and denoted $\text{link}(K, K')$. From the definition one sees that it only depends on the isotopy class of the pair. Notice that changing the orientation of either knot changes the sign of $\text{link}(K, K')$.

We may compute this number directly from a link projection. Let's suppose K and K' have been projected to the horizontal $\mathbb{R}^2$ in $\mathbb{R}^3$. We may contract K' to a point by first pushing it vertically into some level set of z far below the knot K, and then contracting it radially in that level set. The only intersections of the track of this homotopy with K are on the vertical lines that intersect both K and K' — i.e. at the overcrossings of K' over K. Such an overcrossing contributes 1 or -1 to the linking number depending on the relative orientations of K and K'; see Figure 13.

FIGURE 13. We may compute $\text{link}(K, K')$ by counting crossings of K' over K in any link projection.

If we rotate $\mathbb{R}^3$ about the x-axis through $180°$, overcrossings of K' over K in the original diagram become overcrossings of K over K' in the rotated diagram and vice versa, and therefore $\text{link}(K, K') = \text{link}(K', K)$.

2.4. **Dehn Surgery.** Dehn surgery modifies a closed 3-manifold M by removing an open solid torus and gluing it back in by an automorphism of its boundary. The homeomorphism type of the result depends on the mapping class of the boundary automorphism, modulo those mapping classes that extend over a solid torus.

Let's fix an identification of a torus T with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$. We may think of $\mathbb{Z}^2$ here as either $\pi_1(T)$ or $H_1(T)$ acting on the universal cover $\mathbb{R}^2$ by (integer) translations. Any (orientation-preserving) self-homeomorphism induces a linear automorphism of $\mathbb{Z}^2$; conversely any matrix in $\text{SL}(2, \mathbb{Z})$ acts linearly on $\mathbb{R}^2$ permuting $\mathbb{Z}^2$ and descends to a homeomorphism of the quotient torus. Thus there is a surjective map $\text{Mod}(T) \to \text{SL}(2, \mathbb{Z})$, and this map turns out to be an isomorphism.

If we think of T as the boundary of a solid torus H we may choose a basis for $H_1(T)$ consisting of a pair of oriented curves μ, λ such that

(1) μ is the boundary of a disk cross-section D of H (we say μ is a *meridian* for H); and
(2) μ and λ intersect transversely in one point.

The curve λ is called a *longitude*. There is no canonical choice for λ in general.

We choose coordinates on $H_1(T) = \mathbb{Z}^2$ for which $\mu = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\lambda = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Any mapping class that extends over H must preserve the kernel of the inclusion map $\mathbb{Z}^2 = H_1(T) \to H_1(H) = \mathbb{Z}$. This kernel is $\mathbb{Z}$, generated by μ. Conversely, any mapping class that fixes or reverses μ extends over the disk D that it bounds. The complement of this disk in H is an open ball, and the mapping class extends over this ball by the Alexander trick. Thus the subgroup of $\text{Mod}(T)$ that extends over H is precisely the set of matrices of the form $\pm \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$.

Now let's let K be a knot in S^3. Let $N(K)$ be an open solid torus neighborhood of K and let T be the torus $T := \partial(S^3 - N(K))$. If we orient K, we may choose a meridian μ for T which has linking number 1 with K. Now there *is* a canonical choice of (oriented) longitude, namely the unique oriented curve λ which has linking number 0 with K and for which the intersection number of μ with λ in T is 1.

If p/q is a rational number in reduced form, we define the result of *p/q-surgery on K* to be the 3-manifold obtained by removing $N(K)$ from S^3 and gluing a solid torus back by any automorphism represented in terms of the basis $[\mu], [\lambda]$ by a matrix in $\mathrm{SL}(2, \mathbb{Z})$ of the form $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$. By the discussion above, the resulting closed 3-manifold — denoted $K_{p/q}$ — depends only on the rational number p/q. By abuse of notation we let $K_{1/0}$ denote S^3.

Example 2.5 (Lens spaces). Let $K \subset S^3$ be an oriented unknot, i.e. the isotopy class of any round knot (in the spherical metric) and let $N(K)$ be an open solid torus neighborhood. The complement $S^3 - N(K)$ is a closed solid torus whose core is another unknot K'. Let $T = \partial(S^3 - N(K))$ be the boundary torus with meridian μ and longitude λ. Notice that λ is the meridian of K'; in particular, the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ m & 1 \end{smallmatrix}\right)$ of T (in μ, λ coordinates) extends to an automorphism of $S^3 - N(K)$. This automorphism fixes λ and takes μ to a curve representing $\mu + m\lambda$. It follows that the result of p/q surgery on K and $p/(q + mp)$ surgery on K gives rise to homeomorphic 3-manifolds.

In fact, the manifold $K_{p/q}$ is homeomorphic to the Lens space $L(p, q)$ that we encountered in Example 1.11 whose homeomorphism type (as we have already noted) depends only on q mod p, and under this identification the torus $\partial S^3 - N(K)$ becomes the quotient of the Clifford torus by the $\mathbb{Z}/p\mathbb{Z}$ action.

Example 2.6 (± 1 surgery on the unknot). Even the case of ± 1 surgery on the unknot K is interesting. As we have already noted, the automorphism $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of $T = \partial S^3 - N(K)$ extends to an automorphism of $S^3 - N(K)$ that we denote h. The homeomorphism h is obtained by cutting $S^3 - N(K)$ along D, twisting through $360°$, and regluing. If K' is an oriented knot in $S^3 - K$ then $h(K')$ will be a possibly *different* knot; see Figure 14:

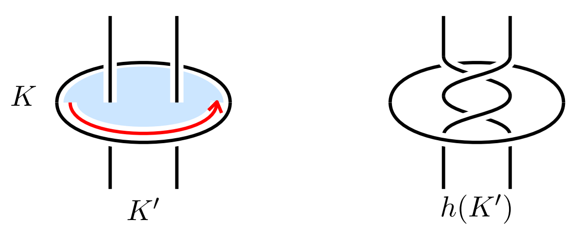

FIGURE 14. The automorphism h of $S^3 - N(K)$ is obtained by cutting along the horizontal disk D (in blue) and rotating through $360°$ in the direction of the red arrow before regluing. It takes any K' crossing through D to a possibly different knot.

The automorphism h fixes λ_K and takes μ_K to $\mu_K + \lambda_K$. Furthermore, if $\mu_{K'}$ and $\lambda_{K'}$ are a meridian and longitude for K' (represented by knots on the boundary of a thin tubular neighborhood of K') then the images $h\mu_{K'}$ and $h\lambda_{K'}$ will represent curves $\mu_{h(K')}$ and $\lambda_{h(K')} + \mathrm{link}(K, K')\mu_{h(K')}$ in terms of a meridian and longitude $\mu_{h(K')}$, $\lambda_{h(K')}$ for the knot $h(K')$. To see this, consider the knot K' as it crashes through the meridianal disk D of the solid torus $S^3 - N(K)$ one strand at a time and think about the effect of h on a neighborhood of this strand; see Figure 15.

Example 2.7. The (2-component) Whitehead link admits an isotopy interchanging the two components, each of which are individually unknots. Doing 1 surgery on the first

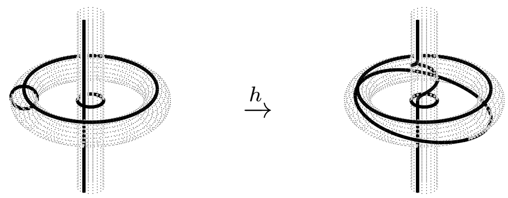

FIGURE 15. The automorphism h carries μ_K to $\mu_K + \lambda_K$ and fixes λ_K. It carries $\mu_{K'}$ to $\mu_{h(K')}$ and $\lambda_{K'}$ to $\lambda_{h(K')} + \text{link}(K,K')\mu_{h(K')}$ (in this figure $\text{link}(K,K') = 1$).

component changes the second component into a right-handed trefoil, while doing -1 surgery on the second component changes the first component into a figure 8 knot. Thus the manifold obtained from S^3 by -1 surgery on the right-handed trefoil is homeomorphic to the manifold obtained from S^3 by 1 surgery on the figure 8 knot. See Figure 16.

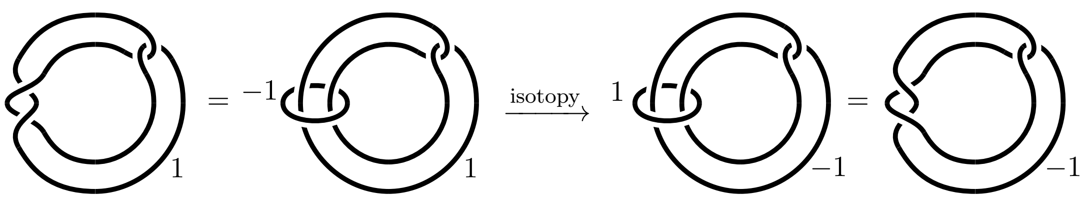

FIGURE 16. -1 surgery on the right-handed trefoil is the same as 1 surgery on the figure 8 knot.

Example 2.8 (Slam dunk move). Suppose L is a link consisting of two components $L = K_1 \cup K_2$ where K_2 links K_1 as a tiny meridian. Suppose we do integral surgery n on the knot K_1; i.e. drill out a neighborhood $N(K_1)$ and glue in a solid torus H whose meridian is the $(n,1)$ curve on $\partial N(K_1)$. We may push the knot K_2 into H and observe that it intersects the meridian disk transversely exactly once; in particular, K_2 is isotopic to the core of H. Thus performing p/q surgery on K_2 just drills out H and glues it back in by a twist, changing the surgery coefficient on K_1 to $n - q/p$.

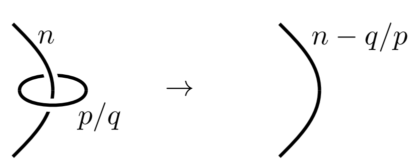

FIGURE 17. The slam dunk move.

Example 2.9 (Dehn twist on Heegaard surface). Let M be a closed oriented 3-manifold with a Heegaard splitting $M = H_1 \cup H_2$, and let S be the splitting surface. We may obtain a new 3-manifold by cutting along S and regluing by a mapping class in $\text{Mod}(S)$.

One particularly simple mapping class is a *Dehn twist*, defined as follows. If $\gamma \subset S$ is any essential simple closed curve, and A is an annulus neighborhood parameterized as

$S^1 \times [0,1]$ with $S^1 \times 1/2 = \gamma$, we obtain a self-homeomorphism of S which is equal to the identity outside A, and on A it is given by the formula $(\theta, t) \to (\theta + 2\pi t, t)$; see Figure 18.

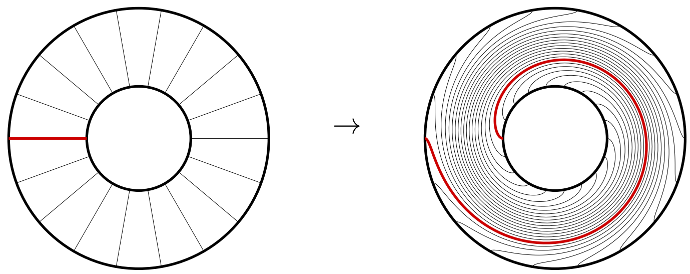

FIGURE 18. A (right-handed) Dehn twist.

The mapping class of this map is called the *right-handed Dehn twist in* γ and usually denoted τ_γ (the inverse is a *left-handed Dehn twist*). Let M' be obtained from M by cutting along S and regluing by τ_γ. I claim M and M' are related by Dehn surgery on the curve γ, thought of as a knot in M.

To see this, let's let N be the result of cutting M open along the annulus A; i.e. N is obtained by gluing H_1 to H_2 by identifying $S - A \subset \partial H_1$ with $S - A \subset \partial H_2$ by the identity. The boundary of N is a torus, made from two copies of A, one in each of ∂H_1 and ∂H_2, and we may choose a basis μ, λ on this torus in such a way that μ intersects each copy of A in the arc $0 \times [0,1]$, and $\lambda = S^1 \times 1/2$ in the top copy of A (say). To obtain the manifold M back from N we glue in the mapping cylinder of the identity map from A to itself, whereas to obtain M' we glue in the mapping cylinder of $\tau_\gamma | A$. These mapping cylinders are both solid tori; the meridian of the first solid torus is μ, and the meridian of the second solid torus is $\mu + \lambda$ in the given coordinates. Thus M' is obtained by 1 surgery on γ in M.

This procedure is powerful enough to produce all closed oriented 3-manifolds:

Proposition 2.10. *Every closed oriented 3-manifold may be obtained by Dehn surgery on some link in* S^3.

Proof. If M is a closed oriented 3-manifold, it admits a Heegaard splitting of some genus g. The 3-sphere also admits some Heegaard splitting of genus g that we denote $S^3 = H_1 \cup H_2$ with splitting surface S. It follows that there is some $\phi \in \text{Mod}(S)$ so that M is obtained from S^3 by cutting along S and regluing by ϕ.

We shall see in the sequel that the entire mapping class group $\text{Mod}(S)$ may be generated by Dehn twists. Thus we may obtain a factorization $\phi = \tau_1 \tau_2 \cdots \tau_n$ where each τ_i is a right or left handed Dehn twist in some essential simple curve γ_i in S. Take n parallel copies $S_1, \ldots, S_n$ of S contained in a collar neighborhood, and for each i let K_i be the copy of γ_i in S_i. The union of the K_i is a link L, and as in Example 2.9, the result of ± 1 surgery (in suitable coordinates) on the components of L takes S^3 to M. □

2.5. **Wirtinger presentation.** Let K be a knot in S^3, and let us consider a knot projection with n crossings. We may imagine the plane of the projection to be an equatorial S^2 in S^3, and think of K as being decomposed into segments, two for each crossing, that are properly embedded in the components of $S^3 - S^2$. These segments are 'bridges', one above and one below the equatorial sphere of the projection; see Figure 19.

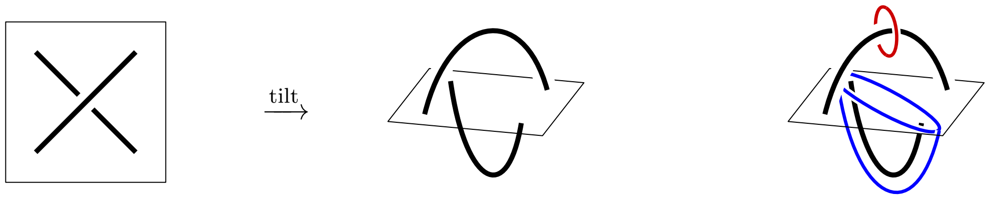

FIGURE 19. Crossings may be interpreted as 'bridges', one above and one below the equatorial sphere of the projection. A meridian loop (in red) going around a positive bridge is the core of a 1-handle of a handlebody H^+; a disk (in blue) going under a negative bridge is the core of a 2-handle of the compression body J.

If $N(K)$ is an open neighborhood of K, the compact manifold $S^3 - N(K)$ decomposes along the equatorial S^2 into two B^3s with n tubes drilled out. Evidently these tubes are unknotted in each B^3, so the two sides are *handlebodies* of genus n; we denote them H^+ and H^-. These two handlebodies are glued together along a common planar subsurface P of their boundary, namely $S^2 - (N(K) \cap S^2)$, a sphere with $2n$ open disks removed.

We may think of H^+ as being made from one 0-handle and n 1-handles. The core of each 1-handle is a loop that loops like a meridian around a positive bridge. Dually, we may think of H^- as being made from n 2-handles and one 3-handle. The 2-handles are disks that go underneath a negative bridge and isolate it from all the others; the result of attaching these disks to P creates a boundary 2-sphere, and this 2-sphere is capped off by the 3-handle.

There is another way to think of this picture: let's let J denote the union of H^- together with a collar neighborhood of ∂H^+. Thus we may think of J as being obtained from a collar of ∂H^+ by attaching n 2-handles and one 3-handle. In this presentation, we may *cancel* the 3-handle with any one of the 2-handles; the reason is that each 2-handle $D^2 \times I$ is attached to the boundary of the 3-handle along only one $D^2 \times$ point face, and we may simply 'push' this face across the 3-handle by an isotopy. Thus we may think of J after all as the union of a collar of ∂H^+ with *only* $(n-1)$ *2-handles*; in particular, J is a *compression body*.

From this picture one may read off a rather elegant presentation for $\pi_1(S^3 - N(K))$ called the *Wirtinger presentation*. Orient the knot K, and choose one generator x_i for each positive bridge which links it positively from above. Each negative bridge gives a relation between the three generators that meet at the crossing (see Figure 20). Any one of these relations follows from all the others and may be excluded (this is equivalent to the fact that any one of the n 2-handles may be canceled by the 3-handle); thus we obtain a presentation with n generators and $n - 1$ relations. In the abelianization, every generator maps to the generator 1 of $H_1(S^3 - N(K)) = \mathbb{Z}$.

Example 2.11. From the standard projection of the right-handed trefoil K (see Figure 21) we may read off a presentation

$$\pi_1(S^3 - N(K)) = \langle a, b, c \mid aca^{-1} = b, cbc^{-1} = a \rangle$$

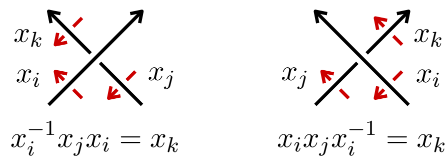

FIGURE 20. Each crossing gives a relation between three generators.

which may be rewritten symmetrically as $\langle a, b, c \mid ac = ba = cb\rangle$.

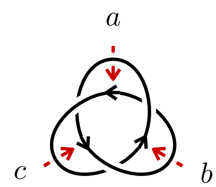

FIGURE 21. Wirtinger generators for $\pi_1(S^3 - \text{right-handed trefoil})$.

One advantage of the Wirtinger presentation is that it is easy to read off representatives for the meridian μ and longitude λ; we explain how to do this in the case of the trefoil. Any of the Wirtinger generators may be chosen to be the meridian, since they are all conjugate; let's choose a basepoint somewhere around the 'uppermost' point on the projection so that $\mu = a$. For a general knot, we may choose any generator as the meridian. To find the longitude, first trace around the outline of the knot, picking up one generator or its inverse at each undercrossing; for example, if we start at our basepoint and proceed positively around K we obtain $\lambda' = cab$. For a general knot we obtain a word of length equal to the number of crossings. This is one choice of longitude, but not the canonical choice which is homologically trivial in $S^3 - K$; it may be corrected by multiplying by a suitable choice of the meridian, hence $\lambda = caba^{-3}$. As a sanity check we verify that μ and λ commute:

$$\lambda\mu = caba^{-2} = ca^2ca^{-3} = cacba^{-3} = cbaba^{-3} = acaba^{-3} = \mu\lambda$$

Performing 1 surgery on K glues in a solid torus. At the level of π_1 this kills the loop on the boundary torus represented by $\lambda\mu = caba^{-2}$. Thus a presentation for the fundamental group of this 3-manifold is

$$\pi_1(M) = \langle a, b, c \mid ac = ba = cb, caba^{-2} = 1\rangle$$

Eliminating the generator b by $b = aca^{-1}$ gives

$$\pi_1(M) = \langle a, c \mid caca^{-1}c^{-1}a^{-1}, ca^2ca^{-3}\rangle$$

Adding back a new generator $b := a^{-1}c^{-1}$ and eliminating c gives

$$\pi_1(M) = \langle a, b \mid b^{-2}a^{-1}ba^{-1}, b^{-1}ab^{-1}a^{-4}\rangle$$

which is equivalent to the presentation for the fundamental group of the Poincaré Homology Sphere given in Example 1.18.

In fact, the manifold obtained from S^3 by 1 surgery on a right-handed trefoil *is* the Poincaré Homology Sphere! This observation is due to Max Dehn, and is a rather challenging exercise in cut-and-paste topology. One way to see it is to first drill out an unknotting arc from $S^3 - N(K)$ to create a genus 2 handlebody, as in Example 1.19. Then 1 surgery on the trefoil is the result of attaching another genus 2 handlebody, whose attaching circles are the boundary of the cocore of the unknotting arc we just drilled out, and the loop $\lambda\mu$ on $\partial S^3 - N(K)$. Finally, one may move these two curves around on the boundary of the handlebody by an isotopy to put them in the form of Figure 5.

2.6. **Branched covers.** Let L be a link in S^3. As in § 2.5, any projection of L with n crossings gives rise to a splitting of S^3 as a union of 3-balls $S^3 = B^+ \cup B^-$ along an equatorial S^2 for which each intersection $L^\pm := L \cap B^\pm$ is a union of n unknotted arcs joining up the $2n$ points of $L \cap S^2$.

Let $M \to S^3$ be a branched cover of degree d, branched over L. Topologically such a cover is obtained by taking an ordinary d-fold cover of $S^3 - L$ and then gluing back covers of components of L to close up the holes. Each pair $(B^\pm, L^\pm)$ is topologically a disk with n points times I, so the preimages $H^\pm$ of $B^\pm$ are handlebodies.

Example 2.12. In the case of a knot K, there is a unique cyclic branched cover of each degree d, given by the regular cyclic cover of $S^3 - K$ whose fundamental group is the kernel of the composition $\pi_1(S^3 - K) \to H_1(S^3 - K) = \mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$. For this regular cover, the handlebodies $H^\pm$ have genus $(d-1)n - d + 1$.

In fact, Alexander [1] showed that *every* closed oriented 3-manifold arises as a branched cover of some link in S^3:

Proposition 2.13. *Every closed oriented 3-manifold M may be obtained as a branched cover of S^3 over some link.*

Proof. The first step is to exhibit a map $M \to S^3$ branched over a graph in a particularly simple way, and then to modify the map locally so that the branch locus is a link. Let τ be a triangulation of M, and let τ' denote the barycentric subdivision of τ. Every vertex of τ' is the barycenter of an i-simplex of τ, and in this way we obtain a labeling of the vertices of τ' by numbers $0, 1, 2, 3$ in such a way that every simplex of τ' has all four labels on its vertices.

Build a triangulation of S^3 by taking a single tetrahedron and doubling it in its boundary. Label the four vertices of this triangulation by $0, 1, 2, 3$. Then there is an obvious map f from M to S^3 taking each simplex of τ' homeomorphically to one of the two simplices in this triangulation of S^3 in the unique manner which is orientation-preserving on each simplex. Evidently $f : M \to S^3$ is branched over the 1-skeleton of the triangulation of S^3, which is topologically the complete graph on 4 vertices.

The *branch locus* of f is the subset of M where f is not a local covering. The next step is to perturb f so that the branch locus is a link in M. Let Δ denote the 1-skeleton of the triangulation of S^3 so that $f^{-1}(\Delta)$ is the 1-skeleton of τ'. If e is an edge of $f^{-1}(\Delta)$ mapping to an edge $f(e)$ of Δ, the *local degree* at e is the degree with which a meridian circle for e maps to a meridian circle for $f(e)$. This is evidently equal to $n/2$ where n simplices of τ' meet along the edge e (note this is always even, since simplices of τ' that share a face are oppositely oriented by their vertex labels). We may erase edges of $f^{-1}(\Delta)$ with local degree 1 (if any) to get a subgraph Γ which is exactly the branch locus of f.

Now, let's parameterize neighborhoods of e and $f(e)$ as oriented cylinders $f : e \times \mathbb{D} \to f(e) \times \mathbb{D}$ where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ so that in these coordinates the map f is just

$(p, z) \to (p, z^{d(e)})$ where $d(e)$ is the local degree of e. We may perturb this map to $(p, z) \to (f(p), z^{d(e)} + \epsilon)$. This splits apart the branch locus near e into $d - 1$ parallel edges, each running between the vertices of e. After performing this modification near each edge of Γ, we get a new branched map $f' : M \to S^3$ whose branch locus Γ' is a graph, every edge of which maps with local degree 2. Note that the image $f(\Gamma')$ will not typically be equal to Δ.

If v is a vertex of Γ' mapping to $f'(v)$ then an open neighborhood of v is a ball mapping to a ball neighborhood of $f'(v)$. The restriction of f' to the boundary of this ball is a branched cover $g : S^2 \to S^2$ with simple branch points, and the map on balls is the cone on g. If we identify the target with $\mathbb{CP}^1$ and pull back the Riemann surface structure under g, then we may think of g as a rational map of some degree $d(v)$ — the local degree of v — with $2d(v) - 2$ simple critical points. The space of rational maps of degree $d(v)$ with simple critical points is Zariski open and therefore connected in the (connected) variety of all degree $d(v)$ rational maps. Let $\beta : \mathbb{D} \to \mathbb{D}$ be any Blaschke product — i.e. a holomorphic degree $d(v)$ map from the unit disk to itself — with simple critical points. Then β extends to $\beta : \mathbb{CP}^1 \to \mathbb{CP}^1$ by (Schwartz) reflection in the unit circle. Now think of $\mathbb{CP}^1$ as the boundary of B^3, and foliate B^3 by arcs joining points in $\mathbb{D}$ to their (Schwartz) reflections. The map β extends yet again to B^3, respecting this product structure, to a degree d map $\beta : B^3 \to B^3$ branched over $d(v) - 1$ unknotted arcs that join the simple critical points of β to their reflections. Modify the map f' in a neighborhood of v and replace it by a suitable conjugate of the map β, and do this for all vertices of Γ'. The result is a new branched map $f'' : M \to S^3$ whose branch locus is a link. If we perturb f'' to put it in general position, we may arrange that the image of this link is a link in S^3 over which M is branched. $\square$

2.7. **Seifert surfaces.** If K is an oriented knot in S^3, linking number with K defines a homomorphism from $\pi_1(S^3 - K)$ to $\mathbb{Z}$, or equivalently an element of $H^1(S^3 - K; \mathbb{Z})$. For any reasonable topological space X (for instance, a CW complex), $H^1(X; \mathbb{Z})$ is naturally in bijection with free homotopy classes of maps from X to S^1; thus, associated to an orientation on any knot K we obtain a *canonical* homotopy class of map from $S^3 - K$ to S^1.

Let's let $N(K)$ be an open tubular neighborhood of K, and parameterize the boundary torus as $S^1 \times S^1$ where each $S^1 \times$ point is an oriented meridian, and each point $\times S^1$ is an oriented longitude. Then we may choose a smooth representative map $f : S^3 - N(K) \to S^1$ whose restriction to $\partial(S^3 - N(K))$ is projection to the first factor $S^1 \times S^1 \to S^1$. If S is the preimage of a regular value of f, then S will be a proper, two-sided oriented surface with ∂S equal to a longitude for K. Such an S is called a *Seifert surface*. Seifert surfaces are not unique (even up to isotopy). We usually insist that S is *connected*; it turns out this is always possible (we shall see why shortly), and is equivalent to the fact that linking number represents a *primitive* class in $H^1(S^3 - K; \mathbb{Z})$. Thus a Seifert surface is a compact oriented surface of some genus $g \geq 0$ with one boundary component. The least genus over all Seifert surfaces is called the *genus* of K and is denoted $g(K)$. It is not easy to compute.

Example 2.14 (Genus 0). If $g(K) = 0$ then K is an unknot. For, a genus 0 Seifert surface is a disk, and we may shrink K across this disk to a round neighborhood of a point.

Example 2.15 (Connect sum). Genus is additive under connect sums; that is, $g(K\#K') = g(K)+g(K')$. Seifert surfaces for K and K' may be glued together along a pair of boundary arcs to produce a Seifert surface for $K\#K'$; this shows $g(K\#K') \le g(K)+g(K')$. To prove the other inequality one must take a Seifert surface S for $K\#K'$ and a decomposing sphere S^2 in general position with respect to S and intersecting $K\#K'$ in two points, and isotop S^2 to inductively eliminate innermost circles of intersections of $S^2 \cap S$ until this intersection is a single arc decomposing S into a pair of Seifert surfaces for K and K'. This gives another proof of Proposition 2.3.

Seifert gave an algorithm to produce a (connected!) Seifert surface directly from an oriented knot projection. The algorithm begins with a knot projection and has three steps:

(1) resolve the crossings compatibly with the orientation to produce a disjoint union of embedded loops;
(2) span the loops with embedded disks oriented compatibly with their boundary circles, innermost disks 'in front of' outermost ones (with respect to the projection); and
(3) connect up these embedded disks with twisted strips at each of the original crossings to produce S.

The meaning of this algorithm is best explained by an example. Figure 22 shows the (oriented) knot 9_{33}. The crossings are resolved consistently with the orientation to produce four oriented circles: three positively oriented, one negatively oriented. These are spanned by disks (the co-orientation of the green disks points out of the page and the co-orientation of the yellow disk points into the page), and then the disks are connected up with twisted strips to produce an oriented genus 3 Seifert surface for the knot, which turns out to be the minimum possible.

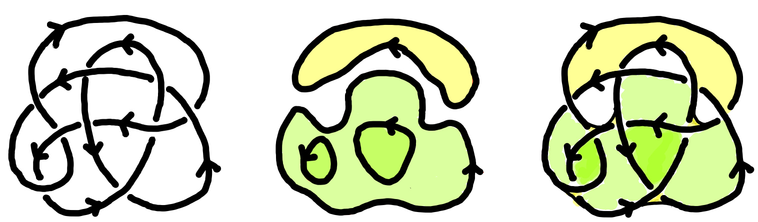

FIGURE 22. Seifert's algorithm produces a genus 3 Seifert surface for this projection of the knot 9_{33}.

A Seifert surface S defines a relative homology class $[S] \in H_2(S^3 - N(K), \partial(S^3 - N(K)))$ dual to the class in $H^1(S^3 - N(K))$ defined by linking number.

2.8. **Knot and link invariants.** Any two projections of a knot (resp. link) are related by a finite sequence of Reidemeister moves. Any function of a projection that does not change under one of these moves therefore defines a *knot (resp. link) invariant*.

Example 2.16 (3-colorability). A knot or link projection is *3-colorable* if we can color each segment of the projection with one of three fixed colors (let's call them red, green and blue) in such a way that at least two colors are used, and at each crossing the incident segments are either all the same color, or all different colors. This property is preserved under the Reidemeister moves; see Figure 23.

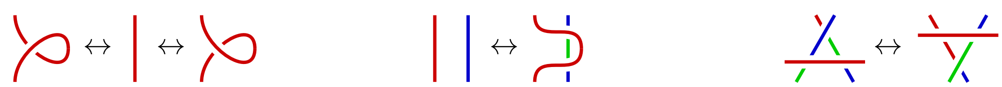

FIGURE 23. The Reidemeister moves preserve the property of 3-colorability.

The trefoil is 3-colorable; the unknot and the figure 8 knot are not.

Example 2.17 (Alexander–Conway polynomial). The *Alexander–Conway polynomial* $\nabla(L)$ is an invariant of oriented links taking values in the ring $\mathbb{Z}[z]$. This invariant is defined by setting $\nabla(O) = 1$ where O is any unknot, and if three oriented links L_+, L_-, L_0 have diagrams which are the same away from some little region where they differ as indicated in Figure 24, their polynomials are related by $\nabla(L_+) - \nabla(L_-) = z\nabla(L_0)$. This is called (by Conway) a *skein relation*.

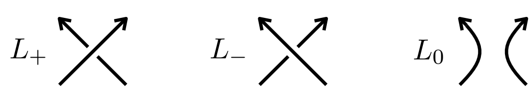

FIGURE 24. Defining skein relation of the Alexander–Conway polynomial.

Setting aside for the moment the question of whether any such ∇ with this property exists, we see from Reidemeister 1 applied to an unknot that ∇ vanishes on unlinks of two (and recursively, any $n > 1$) components. Define a complexity on links to be the ordered pair (a, b) where a is the minimal number of crossings in any diagram, and b is the minimal number of crossings that need to be changed (in some specific projection with a crossings) to change the diagram into a projection of an unlink. Evidently this complexity is well-ordered, and for any nontrivial oriented link L we can always arrange that L is one of $L_\pm$ in such a way that the complexity of the other element of $L_\pm$ and of L_0 are strictly smaller. Thus by induction, ∇ is uniquely determined.

To see that it is well-defined at all is harder; in fact, it is not even clear that it is well-defined on a knot or link *diagram*! One way to do it is to show that $\nabla(L)(t-t^{-1}) = \Delta_L(t^2)$ where Δ_L is the *Alexander polynomial* which may be defined in several ways, some of which are manifestly topologically invariant; the interested reader may look ahead to Chapter 6. Closer to the theme of this subsection, Hoste [3] gives an algorithm to evaluate ∇ uniquely on an oriented link diagram by choosing some specific sequence of simplifying crossings as above, and then shows that this (now well-defined) invariant of a diagram is invariant under Reidemeister moves. In fact, one obtains in this way a homogeneous polynomial invariant in $\mathbb{Z}[x, y, z]$ that specializes to ∇ when $-x = y = 1$. This multi-variable polynomial was discovered independently and almost simultaneously by several different groups, and goes under the name of the HOMFLYPT polynomial, made up from the initials of the various discoverers.

3. Seifert Fibered Spaces

3.1. **Circle bundles.**

Definition 3.1 (Euler number)**.** Let E be a smooth orientable circle bundle over a closed oriented surface S. If we scale a Riemannian metric on E so that each circle has length 2π we may realize E as the boundary of a smooth unit D^2 bundle W over S. Note that W is a smooth compact orientable 4-manifold with boundary E.

Since D^2 is contractible, we may find a smooth section $\sigma : S \to W$. The (oriented) self-intersection number of this section is an integer, called the *Euler number* of the bundle.

The Euler number is a complete invariant of orientable circle bundles over closed surfaces:

Proposition 3.2 (Euler number classifies)**.** *Let $E \to S$ be an oriented circle bundle over a compact oriented connected surface S. If S has boundary, E is a product $S \times S^1$. Otherwise, E is classified up to isomorphism (as an oriented circle bundle) by the Euler number, which may be any integer $\mathbb{Z}$.*

Proof. For an oriented circle bundle, a trivialization is the same as a section. Any oriented circle bundle over a graph has a section, so it is trivial. If S has boundary, it deformation retracts to an embedded subgraph and therefore $E = S \times S^1$ in this case.

Otherwise we may find a cell decomposition of S with one face, and trivialize E over a neighborhood of the 1-skeleton. In other words, we may find an embedded disk $D \subset S$ and a section $\sigma : S - D \to E|(S - D)$. Now, $E|D$ is just a solid torus, which admits its own trivialization as a circle bundle $\phi : D \times S^1 \to E|D$. If we choose points $p \in S^1$ and $q \in \partial D$ the curve $\mu' := \phi(\partial D \times p)$ is a meridian on this solid torus, and the fiber $\lambda := \phi(q \times S^1)$ is a longitude. In these coordinates the section $\sigma(\partial D)$ is a curve μ that may be written (up to isotopy) as $\mu' + e\lambda$ for some integer e.

We claim e is the Euler number; in particular, it is independent of the choice of trivialization over $S - D$. To see this, think of $E = \partial W$ for a disk bundle W as above. The map σ is a section over $S - D$. The trivialization ϕ extends to a trivialization $\phi : D \times D^2 \to W|D$. In ϕ coordinates, σ is the graph of $z \to z^e$ over the unit circle S^1 in $\mathbb{C}$. This extends as a section of W over D to $z \to z^e$ if $e \geq 0$ or as $z \to \bar{z}^{-e}$ if $e < 0$. Thus the intersection number of $\sigma(S)$ with the zero section in W (which is the self-intersection number of $\sigma(S)$) is e, so that e is the Euler number.

We may reverse the construction above to exhibit E as the result of Dehn surgery on a product bundle as follows. Start with $S \times S^1$ and let K be any oriented fiber. A neighborhood of K is a solid torus foliated by circles, and on the boundary we may choose a meridian μ and a longitude λ which is also a fiber. If we perform $-1/e$ surgery in these coordinates, the meridian μ' of the new solid torus satisfies $\mu' = \mu - e\lambda$ so that $\mu = \mu' + e\lambda$ as above. In particular, E is determined by e up to isomorphism, and any $e \in \mathbb{Z}$ may arise. □

Changing the orientation of the fiber or the base changes the Euler number to its negative. By van Kampen's theorem, we may obtain a presentation for $\pi_1(E)$. Pick a point $p \in S$ and let $N(p)$ be an open disk neighborhood of p. If S has genus g, the fundamental group $\pi_1(S - N(p))$ is free on $2g$ generators $a_1, \cdots, a_g, b_1, \cdots, b_g$. One may choose these generators so that the signed intersection of a_i with b_i is 1 for all i, and then the product of commutators $\gamma := \prod_i [a_i, b_i]$ represents the *negatively oriented* boundary of $S - N(p)$ which is the oriented image of both curves μ and μ' that wind *positively* around the solid torus $E|N(p)$. See Figure 25.

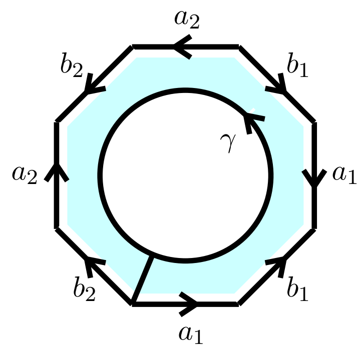

FIGURE 25. If a_i, b_i are a standard surface basis for a genus g surface with one boundary component, the product $\prod [a_i, b_i]$ respresents the *negatively oriented* boundary γ.

Since $E|(S-N(p))$ is a product, its fundamental group is just the product of $\pi_1(S-N(p))$ with a $\mathbb{Z}$ factor generated by a central element z representing the oriented fiber. Gluing back in a solid torus adds a relation $\gamma = z^e$. Thus a presentation is given by

$$\pi_1(E) = \langle a_1, \cdots, a_g, b_1, \cdots, b_g, z \mid [a_i, z] = [b_i, z] = 1, \prod_i [a_i, b_i] = z^e \rangle$$

In particular, $H_1(E) = \mathbb{Z}^{2g} \oplus \mathbb{Z}/e\mathbb{Z}$ which gives another way to see that e is well-defined independent of choices, and gives an easy method to compute it, at least up to sign.

Let's consider some examples.

Example 3.3 (Hopf fibration). The 3-sphere S^3 may be thought of as the unit sphere in $\mathbb{C}^2$ with its standard Hermitian metric. There is a free action of S^1 on S^3 given by multiplication of coordinates:

$$e^{i\theta} \cdot (w, z) = (e^{i\theta} w, e^{i\theta} z)$$

and the orbits are precisely the intersection of S^3 with the complex lines in $\mathbb{C}^2$. It follows that the quotient space is the Riemann sphere $\mathbb{CP}^1$, which is diffeomorphic to S^2 with its standard orientation. This exhibits S^3 as the total space of a circle bundle over S^2 called the *Hopf fibration*.

S^3 is the boundary of the unit B^4 in $\mathbb{C}^2$. Complex lines in $\mathbb{C}^2$ meet B^4 in disks, and these disks all intersect only at the origin. It follows that we may obtain W as in Definition 3.1 by blowing up B^4 at the origin. Thus W is $\overline{\mathbb{CP}}^2 -$ point, and a section of $W^4 \to S^2$ takes S^2 (with the standard orientation) to the exceptional divisor in $\overline{\mathbb{CP}}^2$. In particular, the Euler number of the Hopf fibration is -1.

Example 3.4 (SO(3)). The group SO(3) acts transitively on the unit sphere S^2 in $\mathbb{R}^3$ and point stabilizers are the conjugates of $\mathrm{SO}(2) = S^1$. Thus SO(3) is the total space of an S^1 bundle over S^2.

To compute the Euler number we find a section away from a point. Let p^+ be the north pole and p^- the south pole. For $0 \leq t \leq \pi$ let α_t be the rotation of the yz plane and β_t the rotation of the xy plane through positive angle t. Let (θ, t) for $\theta \in S^1$ and $t \in [0, \pi]$ be polar coordinates on S^2 for which $(\theta, 0)$ is the north pole and $(\theta, 1)$ is the south pole. The map $(\theta, t) \to \beta_\theta \alpha_t \beta_{-\theta}$ is a section from $S^2 - p^+$ to SO(3). As $t \to \pi$ the rotations $\beta_\theta \alpha_t \beta_{-\theta}$ limit to $\iota_{yz} \beta_{-2\theta}$ where ι_{yz} is the involution $(x, y, z) \to (x, -y, -z)$ taking p^+ to p^-. Thus as we wind around a tiny *positively oriented* circle around p^- the angle θ winds *negatively* once around the circle and the section winds *positively* twice around a fiber. Hence the Euler number is 2.

Another way to see this is to note that SO(3) acts freely and transitively on the unit tangent bundle UTS^2, and after choosing a base vector, the orbit map is an identification $\mathrm{SO}(3) = UTS^2$. The unit tangent bundle bounds the unit disk bundle. A section of this bundle is just a vector field where the vectors have length ≤ 1. The intersection of a generic section with the zero section is simply the number of zeros of a generic vector field, counted with sign, which by the Poincaré-Hopf index formula is equal to the Euler characteristic, which is 2.

As a 3-manifold SO(3) is homeomorphic to $\mathbb{RP}^3$. Here's one way to see this. Let $I \subset \mathrm{SO}(3)$ be the set of involutions. A nontrivial rotation γ of $\mathbb{R}^3$ has a unique axis. This axis intersects S^2 at two points, and if γ is not an involution, there is exactly one of these points p at which γ acts by (oriented) rotation through angle $\alpha < \pi$. The map that takes γ to $\alpha p \in \mathbb{R}^3$ extends continuously to the identity and maps $\mathrm{SO}(3) - I$ homeomorphically to the open unit ball in $\mathbb{R}^3$ of radius π. Each involution fixes two antipodal points on S^2, and acts as multiplication by -1 in the tangent planes at these two points. Thus I is an $\mathbb{RP}^2$, compactifying the ball $\mathrm{SO}(3) - I$ to $\mathbb{RP}^3$.

Example 3.5 (Unit quaternions). We may identify $\mathbb{C}^2$ with Hamilton's quaternions by $(z, w) \leftrightarrow z + wj$. The group of unit quaternions may be thought of as the unit S^3, and acts on itself by left multiplication. The subgroup S^1 of unit complex numbers sits inside S^3 as a subgroup and acts freely on the left, and we may think of the Hopf fibration $S^3 \to \mathbb{CP}^1$ as the map to the space of right cosets of S^1 in S^3.

On the other hand, there is the adjoint action of S^3 on the space of purely imaginary quaternions: $q \cdot q' = qq'\bar{q}$. The center acts trivially, so this action factors through $S^3 \to \mathrm{SO}(3) = S^3/\pm 1$. The induced action of SO(3) is the standard action considered in Example 3.4. We may identify the north pole with the quaternion i, and the stabilizer of i is again S^1. But now this exhibits S^2 as the space of *left* cosets of S^1 in S^3. The Euler number of *this* circle bundle is 1, since each fiber of $\mathrm{SO}(3) \to S^2$ is double-covered by a fiber of $S^3 \to S^2$.

Example 3.6 (unit tangent bundles). Let S be any surface. For any Riemannian surface S the unit tangent bundle UTS is a circle bundle over S. This bundle is oriented if S is, and if S is closed and oriented its Euler number is $\chi(S)$. In particular, if S has genus g, the Euler number is $2 - 2g$.

If S is a torus, it is parallelizable and UTS is just the 3-torus T^3. If S is a hyperbolic surface we may realize S as the quotient of the hyperbolic plane $\mathbb{H}^2$ by the deck group, a discrete subgroup Γ of the isometries of $\mathbb{H}^2$ isomorphic to $\pi_1(S)$. The group of orientation-preserving isometries of $\mathbb{H}^2$ is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$; it acts transitively with point stabilizers isomorphic to S^1 acting on $\mathbb{H}^2$ by rotations. Thus we may identify $\mathbb{H}^2$ with the space of right cosets of some S^1 subgroup of $\mathrm{PSL}(2,\mathbb{R})$, and S with the double coset space of S^1 and the subgroup Γ, so that the fibration $UTS \to S$ may be written in the form

$$\Gamma\backslash\mathrm{PSL}(2,\mathbb{R}) \to \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})/S^1$$

Example 3.7 (Heisenberg group). Let H be the group of 3×3 real matrices of the form:

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ for } x,y,z \in \mathbb{R} \right\}$$

As a topological space, H is evidently diffeomorphic to $\mathbb{R}^3$. There is a discrete subgroup $H_{\mathbb{Z}}$ consisting of matrices for which $x,y,z\in\mathbb{Z}$. There is a homomorphism $H \to \mathbb{R}^2$ taking a matrix to the pair (x,y), and this homomorphism restricts on $H_{\mathbb{Z}}$ to a map to $\mathbb{Z}^2$. Thus there is a natural projection to a torus $H_{\mathbb{Z}}\backslash H \to \mathbb{R}^2/\mathbb{Z}^2$. The fiber is a circle, and $E := H_{\mathbb{Z}}\backslash H$ is the total space of an orientable circle bundle over the torus. Note that H is the universal cover of E, and $H_{\mathbb{Z}} = \pi_1(E)$.

The Euler number e of this bundle is 1. One way to see this is to consider the fundamental group. Under the surjection $H_{\mathbb{Z}} \to \mathbb{Z}^2$ the standard generators lift to elements a,b, and the kernel is central and generated by z, represented by a fiber. As matrices we have

$$a := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In particular we may compute $[a,b] = z^e = z$ so that $e = 1$.

Example 3.8. The manifold $S^2 \times S^1$ is obtained by 0 surgery on an unknot in S^3. Thus the unique oriented S^1 bundle over S^2 with Euler number e is given by surgery on a Hopf link in S^3, one component with coefficient 0 and one with coefficient $-1/e$. By the slam dunk move (Example 2.8) this is equivalent to e surgery on an unknot; i.e. it is the Lens space $L(e,1)$. For example, taking $e=2$ gives $\mathbb{RP}^3 = L(2,1)$.

Example 3.9 (Nonorientable bundles). The total space of a circle bundle E might be oriented even if it is not oriented as a circle bundle. Let $E \to S$ be a circle bundle over a nonorientable surface S. Then E is orientable if and only if for each one-sided embedded loop γ in S the restriction $E|\gamma$ is a Klein bottle fibering over a circle.

A not necessarily orientable circle bundle E over S still bounds a disk bundle W over S, and the total space of W is orientable if E is. There is still a section $\sigma : S \to W$ but if S is not orientable, the self-intersection number is only defined mod 2.

We may still find a disk $D \subset S$ and a section σ of $E|S - D$ so that $\sigma(\partial D)$ represents $\mu' + e\lambda$ in some trivialization of $E|D$, but now a different choice of section will change e by an arbitrary multiple of 2.

Arguing as in Proposition 3.2 one may show that nonorientable circle bundles over nonorientable surfaces with orientable total space are classified by the topology of the base surface, and the Euler number $e \in \mathbb{Z}/2\mathbb{Z}$.

3.2. **Exceptional fibers.** The total space of a circle bundle (orientable or not) admits a foliation by the circle fibers. However, not every foliation of a 3-manifold by circles is a circle bundle, even locally.

Example 3.10 (Exceptional fiber). For coprime integers p, q with $0 < q < p$ let $f : \mathbb{D} \to \mathbb{D}$ be the map $z \to e^{2\pi i \cdot q/p} z$. This has order p, and acts freely on $\mathbb{D} - 0$. The mapping torus $M_f := \mathbb{D} \times I/(z, 1) \sim (f(z), 0)$ is a solid torus, fibered by circles γ_Q which are the images of $Q \times I$ in M_f, where $Q \subset \mathbb{D}$ is an orbit of the group $G := \mathbb{Z}/p\mathbb{Z}$ generated by f. Note that we may also think of an orbit Q as a point of the quotient space $\mathbb{D}/G$, which is topologically a disk.

We may change coordinates on M_f and think of it as a product $\mathbb{D} \times S^1$ and now the circles γ_Q are just the curves in $\mathbb{D} \times S^1$ of the form $(e^{2\pi i t q/p} z, e^{2\pi i t})$ for $t \in [0, p]$. In these coordinates, a meridian μ' for M_f is just the circle $S^1 \times 0$. Let $T := \partial M_f$ denote the boundary torus. Let λ be one of the γ_Q on T, and let μ be a curve on T made from a pair of segments, one in μ' and one in λ, the segment in μ' winding $1/p$ of the way around and the segment in λ winding $-q/p$ of the way around. Then μ and λ may be perturbed to intersect transversely once positively, so that they form a basis for $H_1(T)$. In these coordinates, $\mu' = p\mu + q\lambda$; see Figure 26.

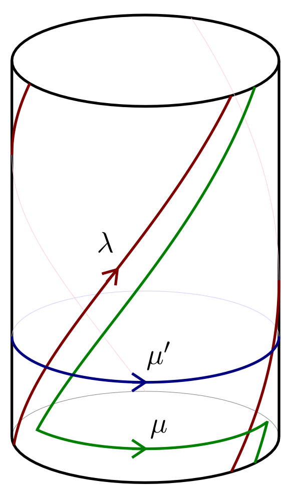

FIGURE 26. A basis μ, λ on the boundary of an exceptional fiber in green and red respectively. A meridian μ' represents $p\mu + q\lambda$; in this figure $p = 3$ and $q = 1$.

Note that a different trivialization of T with λ as circle fiber would pick a new $\hat{\mu} := \mu + n\lambda$. In these new coordinates we would have $\mu' = p\mu + q\lambda = p\hat{\mu} + (np + q)\lambda$. The core circle of the foliation of M_f in Example 3.10 is called an *exceptional fiber* of type p, q.

Definition 3.11 (Seifert fibered space)**.** A *Seifert fibered space* is a compact oriented 3-manifold M with a foliation $\mathcal{F}$ by circles (called a *Seifert fibration* of M) so that every leaf γ either has a neighborhood foliated as a product, or has a neighborhood foliated like M_f in Example 3.10 with γ as the exceptional fiber.

If M is Seifert fibered, the boundary of M consists of tori with a product foliation by circles. Note that although we require M to be oriented, the foliation by circles doesn't need to be. If it is, we say that the *Seifert fibration is oriented*. This is equivalent to the existence of an action of the group S^1 on M which is locally free, in the sense that point stabilizers are discrete in S^1.

Example 3.12 (Brieskorn spheres). For integers $p, q, r \geq 2$ let V be the complex algebraic surface in $\mathbb{C}^3$ defined by the equation

$$z_1^p + z_2^q + z_3^r = 0$$

This is singular at the origin, but its intersection with the unit sphere $S^5 \subset \mathbb{C}^3$ is a smooth 3-manifold $M(p,q,r) = V \cap S^5$. There is an action of the circle on $M(p,q,r)$ given by

$$e^{i\theta} \cdot (z_1, z_2, z_3) = (e^{qri\theta} z_1, e^{pri\theta} z_2, e^{pqi\theta} z_3)$$

This action is locally free, and factors through an action of a quotient circle which is free away from the three orbits where one of the $z_j = 0$. These are exceptional fibers whose type is a rather complicated function of p, q, r in general.

For any $(z_1, z_2, z_3) \in M(p,q,r)$ there is a unique real $t > 0$ for which $(t^{qr} z_1, t^{pr} z_2) \in S^3 \subset \mathbb{C}^2$. The map

$$(z_1, z_2, z_3) \to (t^{qr} z_1, t^{pr} z_2)$$

takes $M(p,q,r)$ surjectively to S^3, and is an r-fold cyclic branched cover, branched over the subset of S^3 where $z_1^p + z_2^q = 0$. This is a link, lying on the torus $|z_1|^p = |z_2|^q$, called the (p,q) *torus link*, and has $n = \gcd(p,q)$ components. The $(2,3)$ torus link is a (right-handed) trefoil.

We can extend the definition of Euler number to Seifert fibered spaces, although in general it will takes values in $\mathbb{Q}$ (at least when the Seifert fibration is oriented). Suppose M has an oriented Seifert fibration with exceptional fibers $\delta_1, \cdots, \delta_n$ of type p_i, q_i. Let N_i be a fibered solid torus neighborhood of δ_i and let $N := \cup N_i$. The complement $M - N$ is an ordinary oriented circle bundle over a compact oriented surface S' of some genus g with n boundary components, and we may choose a trivialization $M - N = S' \times S^1$. This trivialization picks out a choice of meridian μ_i in the boundary of each N_i, that intersects each oriented circle fiber on ∂N_i positively in one point. Choose one such circle fiber λ_i. Finally, let μ_i' be the meridian of N_i. Calculating as above, we see that $\mu_i' = p_i\mu_i + (n_i p_i + q_i)\lambda_i$ for some integer n_i. We define the Euler number e by the formula $-e = \sum_i (n_i + q_i/p_i)$.

Proposition 3.13 (e is well-defined)**.** *The Euler number e of an oriented Seifert fibration is well-defined.*

Proof. With notation as above, $\pi_1(S')$ is a free group of rank $2g+n-1$ with a presentation of the form

$$\pi_1(S') = \langle a_1, b_1, \cdots, a_g, b_g, c_1, \ldots, c_n \mid \prod [a_i, b_i] = \prod c_j \rangle$$

where the a_i, b_i are a standard basis for π_1 of a closed surface of genus g, and the c_j are conjugacy classes of the *negatively* oriented boundary loops, which are the projections of the μ_j.

Two different trivializations of a circle bundle over S' differ over each loop in S' by some multiple of the fiber; in other words, the difference between two trivializations is measured by a *homomorphism* ϕ from $\pi_1(S')$ to $\mathbb{Z}$. In particular, all trivializations agree on any commutator, hence on the loop $\prod[a_i, b_i]$. Thus, although we might change the trivialization over each c_i individually, changing n_i to n_i' where $n_i' - n_i = \phi(c_i)$, the sum $\sum n_i$ is independent of all choices, and e is well-defined. □

In case M is an ordinary oriented circle bundle we may pick a single (ordinary) fiber as δ_1 and (by abuse of notation) define $p_1, q_1 = 1, 0$. In this case we get $\mu' = \mu - e\lambda$ which agrees with our definition $\mu = \mu' + e\lambda$ in the proof of Proposition 3.2.

If S is a closed surface of genus g, we may obtain M by Dehn surgery on $S \times S^1$ as follows. Take n fibers $K_i := x_i \times S^1$ and on the boundary of a neighborhood of each of them let μ_i be the meridian, and choose a fiber as the longitude λ_i. In these coordinates, do p_i/q_i surgery on each K_i. Finally, pick one last fiber $x \times S^1$ and (if necessary) do $-1/m$ surgery where $e = m - \sum q_i/p_i$. This construction shows that every Euler number may arise, subject only to the constraint $-e = \sum q_i/p_i \bmod \mathbb{Z}$.

We may read off a presentation of $\pi_1(M)$ from this picture. Recall that $\pi_1(S')$ is free on $2g + n - 1$ generators, and $\pi_1(S' \times S^1)$ is isomorphic to $F_{2g+n-1} \times \mathbb{Z}$. Gluing back in the neighborhoods of the exceptional fibers adds n relations. Thus we have a presentation for $\pi_1(M)$ of the form

$$\pi_1(M) = \langle a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_n, z \mid \prod[a_i, b_i] = \prod c_j, [a_i, z] = [b_i, z] = [c_j, z] = 1, c_j^{p_j} = z^{-n_j p_j - q_j} \rangle$$

where $-e = \sum(n_j + q_j/p_j)$ as before. It is a common convention to replace the generators c_j with new generators $\hat{c}_j := c_j z^{-n_j}$ for which the presentation becomes

$$\pi_1(M) = \langle a_1, b_1, \cdots, a_g, b_g, \hat{c}_1, \cdots, \hat{c}_n, z \mid \prod[a_i, b_i](\prod \hat{c}_j)^{-1} = z^m, [a_i, z] = [b_i, z] = [\hat{c}_j, z] = 1, \hat{c}_j^{p_j} = z^{-q_j} \rangle$$

where $m = -\sum n_i$ as above.

3.3. **Orbifolds.** The fundamental group $\pi_1(M)$ of an oriented Seifert fibration has a central subgroup Z, generated by the class z of a generic (i.e. non-exceptional) circle fiber. This central subgroup is usually isomorphic to $\mathbb{Z}$, but there are some exceptional cases where it is finite cyclic. The quotient has presentation

$$\pi_1(M)/Z := \langle a_1, b_1, \cdots, a_g, b_g, c_1, \ldots, c_n \mid c_j^{p_j} = 1, \prod[a_i, b_i] = \prod c_j \rangle$$

This is very nearly, but not quite a surface group. What is it?

Example 3.14 (Triangle group). Let's look at a very simple case, that $g = 0$ and $n = 3$. In other words, we are considering a group with a presentation of the form

$$G := \langle c_1, c_2 \mid c_1^p = c_2^q = (c_1 c_2)^r = 1 \rangle$$

for positive integers p, q, r. Let's suppose Δ is a triangle with angles $\pi/p, \pi/q, \pi/r$. The triangle Δ may be realized as a geodesic triangle in a Riemannian surface X where

$$X^2 = \begin{cases} \text{round } S^2 \text{ if } 1/p + 1/q + 1/r > 1 \\ \mathbb{E}^2 \text{ if } 1/p + 1/q + 1/r = 1 \\ \mathbb{H}^2 \text{ if } 1/p + 1/q + 1/r < 1 \end{cases}$$

Take the edge with angles $1/p$ and $1/q$ and double Δ across it, to produce a convex quadrilateral Q in X, with two opposite vertices at which there are angles of $2\pi/p$ and $2\pi/q$ respectively. Let C_1 and C_2 be rotations, centered at these vertices, through angles $2\pi/p$ and $2\pi/q$ respectively; thus C_1 and C_2 have orders p and q. Then $C_1 C_2$ is rotation through an angle of $-2\pi/r$, and therefore has order r. In particular, we have obtained a (faithful!) representation of G into the group of (orientation-preserving) isometries of X. See Figure 27 for the case $p = 3$, $q = 5$, $r = 2$.

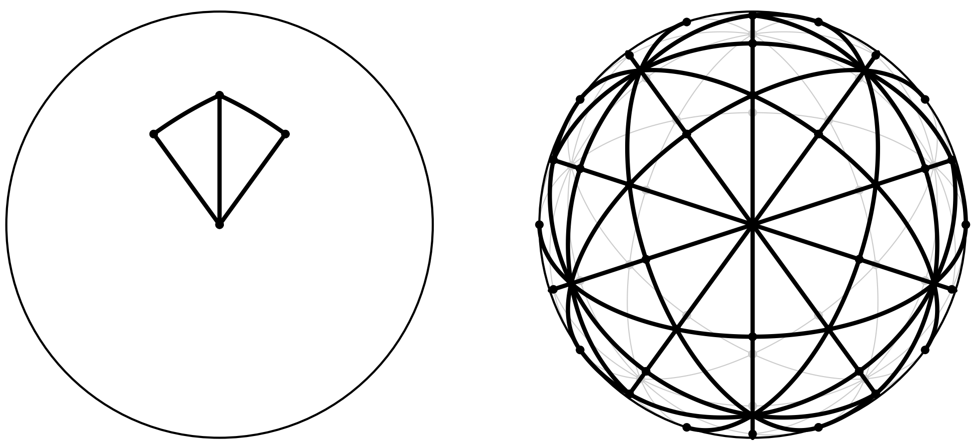

FIGURE 27. The quadrilateral Q made by doubling Δ, and its images under the group G.

The quotient X/G is topologically a 2-sphere, but metrically it is locally modeled on X except at three points where it looks like the quotient of X by a finite cyclic group of rotations.

The space X/G in Example 3.14 is an *orbifold* — a topological space O locally modeled on the quotient of a manifold by a finite group of diffeomorphisms. The three exceptional points are *orbifold points*, which look locally like the quotient of X by rotations of order p, q and r respectively.

A finite group of diffeomorphisms of $\mathbb{R}^2$ is either conjugate to $\mathbb{Z}/n\mathbb{Z}$ acting by rotations, or a dihedral group D_n acting by rotations and reflections. For an orientable surface orbifold, only rotations are possible, and the quotient of $\mathbb{R}^2$ by $\mathbb{Z}/n\mathbb{Z}$ is homeomorphic to $\mathbb{R}^2$. Thus an orientable surface orbifold O is determined by its underlying topological space (which is an orientable surface) together with a discrete set of *orbifold points* labeled by integers $p_i > 1$ where O is locally modelled on $\mathbb{R}^2$ quotiented by a rotation group $\mathbb{Z}/p_i\mathbb{Z}$.

Example 3.15 (Orbit space of a Seifert fibration). If M admits an oriented Seifert fibration, we may consider the quotient space of M whose fibers are the (Seifert) fibers of the fibration. This quotient space O admits the natural structure of an orbifold: away from the exceptional fibers M is a product $S' \times S^1$ for some surface S' with boundary, and near an exceptional fiber of type p, q the space of circles looks like a disk with one orbifold point with label p (compare with Example 3.10).

One may define covering maps between orbifolds:

Definition 3.16 (Covering space)**.** A map between orbifolds $\phi : \tilde{O} \to O$ is a covering space if every point in O is contained in a neighborhood locally modelled on U/Γ for some open $U \subset \mathbb{R}^n$ and finite group of diffeomorphisms Γ, so that the preimages under ϕ are a disjoint union of open subsets of the form U/Γ' for some subgroup Γ' of Γ (that might vary from subset to subset).

In 2 dimensions this just means that $\tilde{O} \to O$ is a branched cover of the underlying topological surfaces and if $\tilde{x} \in \tilde{O}$ maps to $x \in O$ with local degree d, then if x has label $n \ge 1$ (where $n = 1$ corresponds to an ordinary point) the label at $\tilde{x}$ is n/d. In particular, the branch locus is a subset of the set of orbifold points. Covers are regular or irregular as for ordinary covering spaces, and a map $S \to O$ where S is a surface is a regular cover if and only if it is the quotient by a discrete properly discontinuous (but not necessarily free!) group action.

Example 3.17 (Orbifold Euler characteristic). If O is a surface orbifold with orbifold points x_i with labels p_i, we may define the *orbifold Euler characteristic* $\chi_o(O)$ to be equal to $\chi_o(O) = \chi(S) - \sum(p_i - 1)/p_i$. Heuristically, we may think of a point with label p_i as only $1/p_i$ of a point. With this definition, orbifold Euler characteristic is multiplicative under orbifold covers of finite degree, where the degree of an orbifold covering is its ordinary degree as a branched cover between the underlying topological surfaces.

The Galois correspondence lets us define the *orbifold fundamental group* $\pi_1^o(O)$ of an orbifold O in terms of its system of (orbifold) coverings. If O is an orbifold with orbifold points $x_1, \cdots, x_m$ with labels $p_1, \cdots, p_n$ then an orbifold cover of O is the same thing as an ordinary cover $\tilde{S}'$ of the surface $S' := O - \cup x_i$ for which the fundamental group $\pi_1(\tilde{S}')$, thought of as a subgroup of $\pi_1(S')$, contains every conjugate of $\gamma_i^{p_i}$ where $\gamma_i \subset S'$ is a small loop winding once around x_i. If S' has genus g, it follows that $\pi_1^o(O)$ has a presentation of the form

$$\pi_1^o(O) = \langle a_1, b_1, \cdots, a_g, b_g, c_1, \ldots, c_n \mid c_j^{p_j} = 1, \prod [a_i, b_i] = \prod c_j \rangle$$

This answers the question we asked at the start of § 3.3: if M is an oriented Seifert fibration whose quotient space is an orbifold O, then the quotient of $\pi_1(M)$ by the central subgroup Z generated by the class z of the fiber is the orbifold fundamental group $\pi_1^o(O)$. In other words, we have a central extension $Z \to \pi_1(M) \to \pi_1^o(O)$.

Example 3.18 (Good and bad orbifolds). An orbifold O is *good* if it is finitely covered by a manifold. Otherwise it is *bad*. An orbifold O with underlying space S^2 and one orbifold point with label n (a *teardrop*) is bad, since its orbifold fundamental group is trivial. Likewise an orbifold O with underlying space S^2 and two orbifold points with labels n, m

and $n \neq m$ (a *spindle*) is bad, since its orbifold fundamental group is $\mathbb{Z}/\gcd(n,m)\mathbb{Z}$ so no orbifold cover can completely unwrap either of the orbifold points (if $n = m$ the orbifold is good, since it is n-fold cyclically covered by S^2).

Another way to see that these orbifolds are bad is to use Euler characteristic. If O is a teardrop then $\chi_o(O) = 1 + 1/n$ and if O is a spindle then $\chi_o(O) = 1/n + 1/m$. In neither case is there a positive integer d so that $d\chi_o(O)$ is $\chi(S)$ for a closed oriented surface S, so no degree d cover $S \to O$ can exist.

Proposition 3.19. *Every oriented surface orbifold is good, except for teardrops and spindles.*

Proof. To see this first observe that if O is good, so is a connect sum $O\#T$ (where the connect sum is performed at an ordinary point in O) since we may obtain a degree d orbifold cover of $O\#T$ by taking a degree d orbifold cover of O, and connect summing it with d tori. If O is an orbifold whose underlying surface S has positive genus, then by taking an (ordinary) cover of S we may ensure that O has as many orbifold points as we like. Thus we are reduced to showing that O is good, where O is an orbifold with underlying space S^2 and $n \geq 3$ orbifold points with labels $p_1, \cdots, p_n$. Just as in Example 3.14 we may find a convex geodesic n-gon in a space X with angles π/p_i at the vertices where X is round S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$ according to the sign of $\sum 1/p_i + 2 - n$. The group generated by rotations through angles $2\pi/p_i$ at the vertices of this n-gon is a discrete group Γ of isometries of X with quotient $X/\Gamma = O$. Let Γ' be a torsion-free normal subgroup of Γ of finite index (that such a subgroup exists follows from a theorem of Selberg). Then X/Γ' is a surface that covers O with degree equal to the index of Γ' in Γ. ☐

Example 3.20 (UTO as a Seifert fibered space). If O is a surface orbifold, the unit tangent bundle of O naturally has the local structure of the unit tangent bundle of an open subset of $\mathbb{R}^2$ modulo a finite group of rotations. This is a Seifert fibered space, whose exceptional fibers correspond precisely to the orbifold points of O! The operation that takes an orbifold O to its unit tangent bundle UTO has a (one-sided) inverse, that takes a Seifert fibered space to its space of circles (i.e. Example 3.15).

We claim if x is an orbifold point of O with label p, the circle UT_xO is an exceptional fiber in UTO of type $p,(p-1)$. To see this, let's model O locally on a quotient of a disk D by the action of $\mathbb{Z}/p\mathbb{Z}$ generated by a rotation α through positive angle $2\pi/p$.

The unit tangent bundle UTD is a solid torus $D^2 \times S^1$ fibered as a product by unit circles. Let $D' := D/\langle\alpha\rangle$ denote the quotient of D in the orbifold. The unit tangent bundle $UTD' = UTD/\langle\alpha\rangle$ is also a solid torus, and the circle fibration on UTD descends to a foliation on the quotient with an exceptional fiber of type $p,(p-1)$; see Figure 28.

Getting the signs right is a headache: thinking of UTD' as a quotient gives it coordinates $D^2 \times I/(z,0) \sim (e^{2\pi i/p}z, 1)$ which as a mapping torus is $D^2 \times I/(z,1) \sim (e^{2\pi i\cdot(p-1)/p}z, 0)$.

The Euler number e of UTO may now be calculated as above. If S' is the underlying topological surface of O then we may put all the orbifold points in a disk D, and trivialize $UT|S' - D$ in such a way that the trivialization winds $\chi(S')$ times around the fibers relative to the 'obvious' trivialization of $UT|\partial D$. Thus if the orbifold points have orders p_i,

$$e = \chi(S') - \sum_i (p_i - 1)/p_i = \chi_o(O)$$

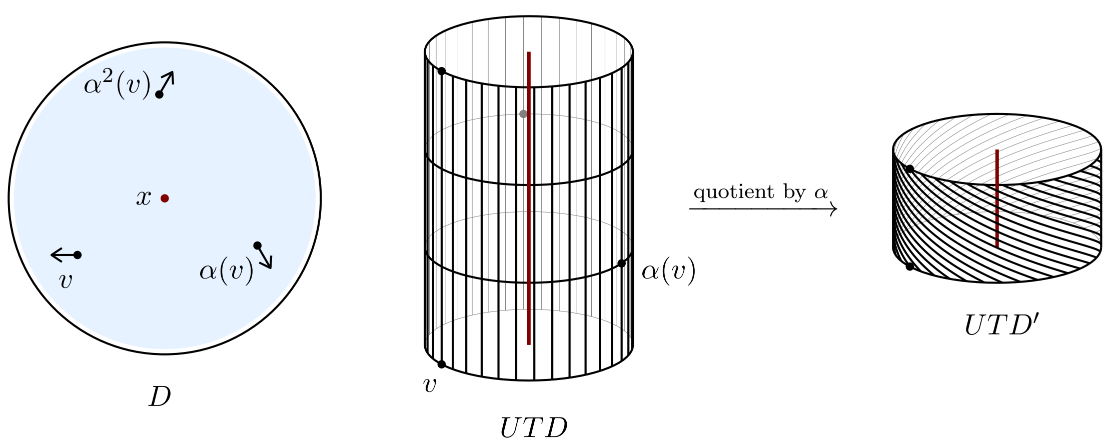

FIGURE 28. The quotient of UTD by a positive rotation through angle $2\pi/p$ is a neighborhood of an exceptional fiber of type $p,(p-1)$ in UTD'.

This recovers and generalizes the statement that the Euler number of UTS is $\chi(S)$ for an ordinary oriented surface S.

As a concrete example, if O is the spherical triangle orbifold with three orbifold points of orders $2, 3, 5$ then $\chi_o(O) = 1/30$. The orbifold O is covered by S^2 with order 60, so UTO has a degree 60 cover which is $UTS^2 = \mathbb{RP}^3$; in particular, the order of $\pi_1(UTO)$ is 120. What is it?

As above, we have a presentation

$$\pi_1(UTO) = \langle c_1, c_2, c_3, z \mid c_1^{-2} = c_2^3 = c_3^5 = z, c_1c_2c_3 = 1\rangle$$

The generators z and c_1 may be eliminated and we may relabel the generators by $a^{-1} = c_2$, $b = c_3$ simplifying the presentation to

$$\langle a, b \mid (a^{-1}b)^2 = b^3 = a^{-5}\rangle$$

which recovers the presentation for the fundamental group of the Poincaré Homology Sphere from Example 1.18. In fact, UTO *is* the Poincaré Homology Sphere; this is a rather challenging exercise in the manipulation of Dehn surgery diagrams.

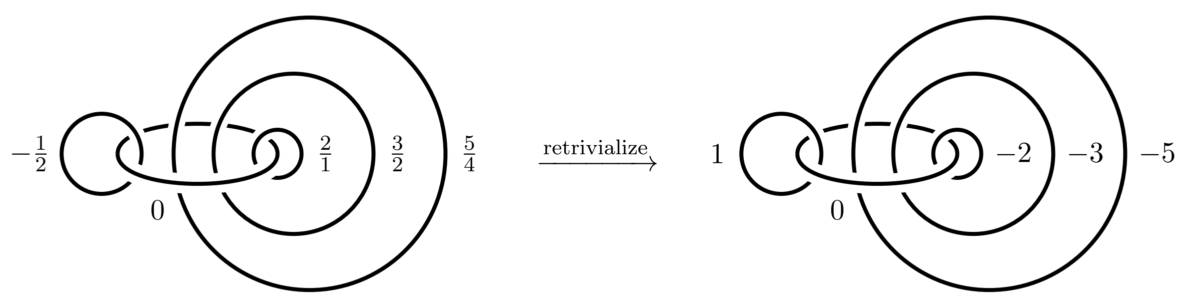

FIGURE 29. Change the trivialization to change the surgery coefficients.

We start with $S^2 \times S^1$ presented as 0 surgery on an unknot K, and then surger four fibers (unlinked unknots linking K once). The coefficients of the three exceptional fibers are 2/1, 3/2 and 5/4, and the other fiber has coefficient $-1/2$ where 2 comes from the Euler characteristic of S^2. By changing the trivialization of the fibering away from the exceptional fibers, we may replace $2/1, 3/2, 5/4$ with $-2, -3, -5$ and replace $-1/2$ with 1. See Figure 29.

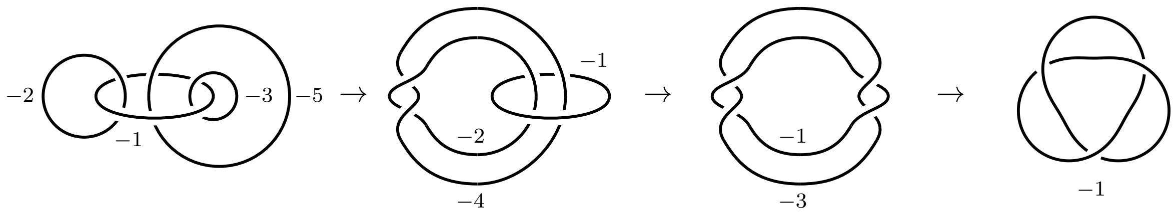

FIGURE 30. Untwist -1 circles repeatedly.

Slam-dunk this 1 circle into the knot K, changing its coefficient to -1. Then repeatedly untwist the spanning disks of unknots with coefficient -1 as in Example 2.6 until we are left with -1 surgery on the left-handed trefoil, which is indeed the Poincaré Homology Sphere (with the opposite orientation to Example 2.11). See Figure 30.

Euler number behaves well under covering maps. Suppose M admits an oriented Seifert fibration, and we have a degree d covering map $\pi : N \to M$. We may pull back the Seifert fibers of M to give N the structure of an oriented Seifert fibration. If O_N and O_M are the quotient orbifolds of the fibration, there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\pi} & M \\ {\scriptstyle q_N}\downarrow & & {\scriptstyle q_M}\downarrow \\ O_N & \xrightarrow{\pi^o} & O_M \end{array}$$

where π^o is an orbifold covering map of degree d_o. A generic fiber of M is covered by a generic fiber of N, and since the set of generic fibers is open and connected, the covering degree is some fixed number d_f independent of the choice.

With this notation, $d = d_o d_f$ and we have equalities

$$\chi_o(O_N) = d_o \chi_o(O_M) \text{ and } e(N) = e(M) \cdot d_o/d_f$$

which follow more or less immediately from the definitions.

Example 3.21 (Cyclic quotient). Let N be a solid torus neighborhood of an exceptional fiber of type (p, q). There is a locally free S^1 action on N that rotates the circles at constant speed, and the subgroup $\mathbb{Z}/p\mathbb{Z}$ stabilizes the exceptional fiber. If n is coprime to p, the group $\mathbb{Z}/n\mathbb{Z}$ acts freely, and the quotient is a solid torus neighborhood of an exceptional fiber of type (p, nq).

If M has an oriented Seifert fibration, these locally free S^1 actions may be glued together via a partition of unity to obtain a global locally free circle action on M. Suppose the exceptional fibers are of type (p_i, q_i) and n is coprime to every p_i. Then the group $\mathbb{Z}/n\mathbb{Z}$ acts freely on M with Seifert fibered quotient M'. Evidently $O_{M'} = O_M$ and $e(M') = n \cdot e(M)$.

As an example, let's suppose M is the manifold UTO from Example 3.20 where O is the spherical triangle orbifold with three orbifold points of orders $2, 3, 5$. This has $e = \chi_o(O) = 1/30$. Suppose M' is another Seifert fibered space with orbifold O. The exceptional fibers must be of type $(2, 1)$, $(3, a)$ and $(5, b)$ for some a coprime to 3 and b coprime to 5, so that $e(M')$ is of the form $n - 1/2 - a/3 - b/5 = m/30$ for some arbitrary integer m coprime to 30. Thus M' is isomorphic as a Seifert fibered space to the quotient of M by the free action of $\mathbb{Z}/|m|\mathbb{Z}$ on circle fibers, possibly up to change of orientation.

Example 3.22 (Circle bundle cover). If M has an oriented Seifert fibration with quotient orbifold O, and if O is good, then O has a finite orbifold cover (and therefore a regular cover) which is a surface S, coming from a surjective homomorphism $\pi_1^o(O) \to \Delta$ where Δ is a finite group. Composing with $\pi_1(M) \to \pi_1^o(O)$ gives a surjective homomorphism from $\pi_1(M)$ to Δ, and therefore a finite cover $\tilde{M}$ of M. Evidently the quotient orbifold of $\tilde{M}$ is S; in particular, $\tilde{M}$ is an oriented circle bundle over S.

The long exact sequence in homotopy groups ends in

$$0 \to \pi_2(M) \to \pi_2(S) \to \mathbb{Z} \to \pi_1(M) \to \pi_1(S) \to 0$$

and the central subgroup Z generated by the fiber z is the quotient of $\mathbb{Z}$ by the image of $\pi_2(S)$. Thus we are in exactly one of the following situations:

(1) S is not a sphere, and the class of the fiber generates a $\mathbb{Z}$ central subgroup of $\pi_1(M)$; or
(2) S is a sphere; in this case the image of the generator of $\pi_2(S)$ is the Euler number e (up to sign) and there are two subcases:
   (a) $\pi_2(M) = \mathbb{Z}$ mapping isomorphically to $\pi_2(S)$; this implies the Euler number is 0 so that $M = S^2 \times S^1$ fibered as a product; or
   (b) $\pi_2(M) = 0$ and the Euler number is nonzero; in this case M is a Lens space $L(e, 1)$ as in Example 3.8.

If z generates a $\mathbb{Z}$ central subgroup in a cover, then it generates one in $\pi_1(M)$. Thus if O is a good orbifold, the center of $\pi_1(M)$ contains a $\mathbb{Z}$ unless the quotient orbifold O is covered by S^2 and the Euler number is nonzero, in which case M is finitely covered by S^3. If O is a bad orbifold, it has no honest surface cover, and M has no finite cover for which the Seifert fibration lifts to an honest circle bundle. Note in this case that the Euler number is necessarily non-integral (and therefore in particular nonzero).

Example 3.23 (Seifert fibered spaces with bad orbifolds). Suppose M has an oriented Seifert fibration with quotient orbifold O. Either O is a teardrop with label p or a spindle with labels p_1, p_2. Thus M has one exceptional fiber of type p, q in the first case, and two exceptional fibers of type p_1, q_1 and p_2, q_2 in the second case. Thus M has a Dehn surgery description as in Figure 31.

In either case we can apply the slam-dunk move twice. This reduces the first diagram to surgery on an unknot with coefficient $m-q/p$, and the second to surgery on a Hopf link with coefficients $m - q_1/p_1$ and p_2/q_2. In the first case we have a Lens space $L(mp - q, p)$, and in the second case we have a Lens space $L(a, b)$ where a and b are somewhat complicated functions of the p_i, q_i and m.

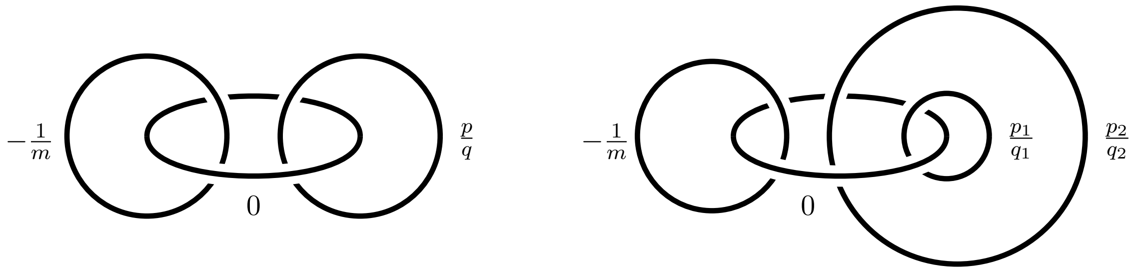

FIGURE 31. Surgery diagrams for oriented Seifert fibrations with bad quotient orbifold. These are all Lens spaces.

3.4. **The geometry of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.** The Lie group $\mathrm{SL}(2,\mathbb{R})$ is 3 dimensional and orientable, and homeomorphic to an open solid torus. To see this, consider the standard linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$; the space of oriented lines in $\mathbb{R}^2$ through the origin is a circle, and the stabilizer of an oriented line (for instance the x-axis) is the subgroup of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right)$ where $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ which is evidently diffeomorphic to $\mathbb{R}^2$.

The subgroup $K := \mathrm{SO}(2,\mathbb{R})$ is a maximal compact subgroup, abstractly isomorphic to a circle. Right multiplication of K on $\mathrm{SL}(2,\mathbb{R})$ is free, and its orbits foliate the open solid torus as a product. This action commutes with the left action; thus for any discrete cocompact subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ the quotient space $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$ is a Seifert fibered 3-manifold.

The left multiplication of $\mathrm{SL}(2,\mathbb{R})$ on itself is free and transitive, so we may choose any Euclidean inner product on the tangent space at the identity and transport this around to obtain a (left-invariant) Riemannian metric. This left multiplication commutes with the right action of K, and since K is compact, we may average the Riemannian metric under the K action to obtain a Riemannian metric on $\mathrm{SL}(2,\mathbb{R})$ which is invariant under the full $\mathrm{SL}(2,\mathbb{R}) \times K$ action (note that this action is not faithful, since the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is central, and acts in the same way on the left and the right).

The universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is diffeomorphic to $\mathbb{R}^3$. As a Lie group it is a central extension of $\mathrm{SL}(2,\mathbb{R})$ by the group $\mathbb{Z}$. In this covering group the subgroup K is unwrapped to $\tilde{K}$, which is isomorphic to $\mathbb{R}$. We may pull back an invariant Riemannian metric on $\mathrm{SL}(2,\mathbb{R})$ to obtain a Riemannian metric on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \tilde{K}$.

Example 3.24 (Hyperbolic orbifold). As already remarked in Example 3.6, the group $\mathrm{PSL}(2,\mathbb{R})$ (i.e. the quotient of $\mathrm{SL}(2,\mathbb{R})$ by its (order 2) center) is isomorphic to the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2$. The point stabilizers in $\mathrm{PSL}(2,\mathbb{R})$ are double covered by the conjugates of the subgroup K. Thus we may identify $\mathbb{H}^2$ with the space of cosets $\mathrm{PSL}(2,\mathbb{R})/K$ and if Γ is a discrete group of isometries of $\mathbb{H}^2$, the quotient orbifold O satsifies $O = \Gamma\backslash\mathbb{H}^2 = \Gamma\backslash\mathrm{PSL}(2,\mathbb{R})/K$. In particular, we may identify UTO with $\Gamma\backslash\mathrm{PSL}(2,\mathbb{R})$. By varying O and taking covers or cyclic quotients as in Example 3.21 one may obtain many examples of Seifert fibered spaces isomorphic to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ modulo a discrete properly discontinuous group of isometries.

3.5. **Geometrization of Seifert Fibered Spaces.** We have encountered a bewildering assortment of Seifert fibered spaces throughout this section and it is natural to want to try to impose some sort of order. At least for oriented Seifert fibrations of closed manifolds M we have two invariants: the (orbifold) Euler characteristic of the quotient orbifold $\chi_o(O)$, and the Euler number $e(M)$. In all the cases we have encountered — and it turns out, in every case — the manifold M may be obtained as the quotient of a complete homogeneous simply-connected Riemannian manifold X by a discrete group Γ acting freely and isometrically on X (*homogeneous* means that the metric on X looks the same at every point; i.e. that the group of isometries of X acts transitively).

Which Riemannian manifold one takes for X depends on the two invariants. If one insists that the Riemannian metric should be chosen so that the isometry groups are as big as possible, there are six possibilities for any X (up to a suitable notion of isomorphism) that can cover Seifert fibered spaces:

$$S^2 \times \mathbb{R}, \mathbb{E}^3, \mathbb{H}^2 \times \mathbb{R}, S^3, \text{Nil}, \widetilde{\text{SL}}(2, \mathbb{R})$$

where Nil is the 3-dimensional nilpotent Lie group H (i.e. the Heisenberg group) we encountered in Example 3.7. The relationship to our two invariants is captured in the following table:

	$\chi_o(O) > 0$	$\chi_o(O) = 0$	$\chi_o(O) < 0$
$e(M) = 0$	$S^2 \times \mathbb{R}$	$\mathbb{E}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e(M) \neq 0$	S^3	Nil	$\widetilde{\text{SL}}(2, \mathbb{R})$

This fact goes by the name of the Geometrization Theorem for Seifert Fibered Spaces, and it extends to non-orientable Seifert fibrations and non-orientable manifolds (the condition $e(M) \neq 0$ holds for such a manifold if it holds for some orientable finite cover). It was formulated and proved by Thurston, and it is a special (though in many ways atypical) case of his Geometrization Conjecture, which we shall return to many times in subsequent chapters. With some effort, a complete proof of the Geometrization Theorem for Seifert Fibered Spaces could be deduced from our discussion up to this point; however this would involve a detailed analysis of many cases, and it does not seem worthwhile to include it here. The interested reader may find all the details, and much more beside, in Peter Scott's excellent article [6].

4. Surface bundles

4.1. **Mapping class groups.** Let S_g be a closed oriented surface of genus g. The *mapping class group* Mod_g is the group of isotopy classes of orientation-preserving homeomorphisms from S_g to itself (see Example 1.20). If S has boundary and/or marked points or punctures one may define a mapping class group for S to be the group of isotopy classes of orientation-preserving homeomorphisms fixing (or permuting) the marked points or punctures, and fixing the boundary pointwise.

Surfaces and their automorphisms are a central object in mathematics, and mapping class groups and their properties are a vast subject. We confine ourselves here to the merest outline, and point the reader to the textbook [2] for details.

Example 4.1 (Braid groups). Let D be a (closed) disk with n marked points in the interior. The *braid group* B_n is the group of isotopy classes of orientation-preserving homeomorphisms of D fixing the boundary pointwise and permuting the n marked points. This is isomorphic to the group of isotopy classes of *compactly supported* homeomorphisms of $\mathbb{C}$ permuting n marked points.

The space of degree n monic polynomials f in one complex variable is $\mathbb{C}^n$, where the coordinates are the coefficients of f. The *discriminant* of f, denoted $\Delta(f)$, is a polynomial in the coefficients of f that vanishes if and only if f has a multiple root. By abuse of notation, we let $\Delta \subset \mathbb{C}^n$ denote the set of polynomials for which $\Delta(f) = 0$; this is a (rather singular) algebraic hypersurface in $\mathbb{C}^n$. The complement $\mathbb{C}^n - \Delta$ is a $K(\pi, 1)$ (i.e. its universal cover is contractible) with $\pi_1 = B_n$.

Example 4.2. The mapping class group of a disk is trivial by the Alexander trick, and the mapping class group of an annulus is $\mathbb{Z}$, generated by a *Dehn twist* in the core circle (see Example 2.9). The mapping class group of a pair of pants (i.e. a sphere minus three open disks) is $\mathbb{Z}^3$, generated by Dehn twists in circles parallel to the boundary components.

Example 4.3 (Outer automorphism). A homeomorphism of S_g fixing a basepoint induces an automorphism of π_1; ignoring basepoints introduces the ambiguity of an inner automorphism. Thus there is a homomorphism from Mod_g to $\text{Out}(\pi_1(S_g))$, the group of outer automorphisms of π_1. It turns out that this homomorphism is injective, and the image has index 2 (every automorphism may be achieved if one allows orientation-reversing homeomorphisms). This image is usually denoted $\text{Out}^+(\pi_1(S_g))$. This is known as the Dehn–Nielsen Theorem.

Injectivity is equivalent to the fact that a homeomorphism homotopic to the identity is isotopic to the identity. Since a homeomorphism must take simple curves to simple curves this amounts to showing that homotopic simple curves on a surface are isotopic (which can be proved by lifting to an annular covering space and finding the isotopy there), moving the images of a sufficiently complicated system of simple curves back to themselves by an isotopy, and applying the Alexander trick to the complementary disk regions.

Since S_g for any $g \geq 1$ is a $K(\pi, 1)$, any outer automorphism of the fundamental group is induced by a homotopy equivalence. Thus, surjectivity is equivalent to the fact that a homotopy equivalence of S_g is homotopic to a homeomorphism. The key is again to show that the image of an essential simple curve is homotopic to a simple curve, so that these curves can be straightened one by one as above. This is equivalent to giving a purely algebraic criterion for a conjugacy class in $\pi_1(S_g)$ to be represented by a simple curve. There are a few ways to do this; one way is to show that a conjugacy class has a simple representative if and only if $\pi_1(S_g)$ splits as an HNN extension or amalgamated free product over the $\mathbb{Z}$ subgroup it generates.

Example 4.4 (One marked point). Suppose S is a closed surface and $p \in S$ is a marked point. Let's denote the mapping class group of S by $\text{Mod}(S)$ and the mapping class group of S fixing p by $\text{Mod}(S, p)$. There is a surjective map $\text{Mod}(S, p) \to \text{Mod}(S)$ obtained by forgetting the point p. Since a mapping class in $\text{Mod}(S, p)$ fixes p it induces a well-defined *automorphism* of $\pi_1(S)$. In particular, $\text{Mod}(S, p)$ is isomorphic to the subgroup $\text{Aut}^+(\pi_1(S))$ of index 2 mapping to $\text{Out}^+(\pi_1(S))$ and the kernel is the group of inner automorphisms; equivalently, the quotient of $\pi_1(S)$ by its center.

This short exact sequence relating $\text{Mod}(S, p)$ to $\text{Mod}(S)$ is called the *Birman exact sequence*.

Example 4.5. The mapping class group of a torus T is $\mathrm{SL}(2,\mathbb{Z})$, generated by Dehn twists in a meridian and longitude. Since $\pi_1(T)$ is abelian, the mapping class group of a torus with one marked point is also $\mathrm{SL}(2,\mathbb{Z})$.

Theorems about mapping class groups may be proved inductively by cutting a surface S into simpler surfaces along essential simple (i.e. embedded) closed curves or arcs. The most important theorem of this kind is due to Max Dehn:

Theorem 4.6 (Dehn twists generate)**.** *Every mapping class group is generated by Dehn twists along essential simple closed curves.*

This theorem is proved inductively. Suppose γ is an essential simple closed curve, and ϕ is a mapping class. One first proves that there are a sequence of (left or right handed) Dehn twists $\tau_1,\cdots,\tau_n$ so that $\tau_1\cdots\tau_n\phi(\gamma)$ is isotopic to γ. Then one may cut open the surface along γ, restrict the mapping class $\tau_1\cdots\tau_n\phi$ to the cut open surface, and induct. At the end one is left with a collection of elementary cases (annuli or pairs of pants) where the theorem can be proved by hand.

Let us see that such a sequence of Dehn twists exists in the easier case that γ is non-separating. Any two non-separating curves γ, γ' on a surface may be interpolated by a sequence of non-separating curves $\gamma=\gamma_0,\gamma_1,\cdots,\gamma_n=\gamma'$ where each successive pair γ_i,γ_{i+1} intersect transversely in a single point. A neighborhood of $\gamma_i\cup\gamma_{i+1}$ is homeomorphic to a punctured torus, and we have already seen that the mapping class group of a punctured torus is $\mathrm{SL}(2,\mathbb{Z})$, generated by Dehn twists. So we may find a product of Dehn twists supported in this punctured torus that takes γ_i to γ_{i+1} as claimed.

4.2. **Surface bundles.** A *surface bundle over a circle* (or just *surface bundle* for short) is a 3-manifold obtained as the mapping torus of a surface homeomorphism. In other words, M is the mapping torus $M_f := S\times I/(z,1)\sim(f(z),0)$ associated to some homeomorphism $f:S\to S$ of a surface to itself. M is oriented if and only if S is oriented and f is orientation-preserving.

The homeomorphism type of M depends only on the mapping class of f by the isotopy extension theorem. The same 3-manifold may sometimes be realized as a surface bundle in infinitely many (not-isotopic) ways.

Example 4.7 (Closed nonsingular 1-form). Let M be closed and suppose there is a closed 1-form α on M which is nonsingular; i.e. the restriction of α to the tangent space at every point of M is nonzero. Since α is closed, it defines a de Rham cohomology class $[\alpha]\in H^1(M;\mathbb{R})$ and a homomorphism from $H_1(M;\mathbb{Z})$ to $\mathbb{R}$ obtained by integrating α over representative loops. The image of $H_1(M;\mathbb{Z})$ in $\mathbb{R}$ is the group of *periods* of α.

By adding a small multiple of another closed 1-form to α, we may perturb it to a closed nonsingular 1-form with rational periods, and we may then scale it so that the group of periods is exactly $\mathbb{Z}$. Fix a basepoint $p\in M$ and for every other $q\in M$ define $\pi(q):=\int_{\gamma_q}\alpha$ where γ_q is any oriented path from p to q. Different paths will give rise to different values, but the difference between any two values will be a period. Thus π is well-defined as a map from M to $\mathbb{R}/\mathbb{Z}=S^1$. Since α is nonsingular, this map is a submersion, and therefore M is fibered by the point preimages, exhibiting M as a surface bundle over a circle.

Example 4.8 (Torus bundles). A matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ acts linearly on $\mathbb{R}^2$ and on the quotient torus $T = \mathbb{R}^2/\mathbb{Z}^2$ and we may form the mapping torus M_A which is a torus bundle over the circle. It turns out in every case that M_A has a geometric structure — i.e. it may be obtained as the quotient of a complete homogeneous simply-connected Riemannian manifold X by a discrete group of isometries; compare the discussion in § 3.5. The fundamental group of M_A is a semidirect product $\mathbb{Z}^2 \to \pi_1(M_A) \to \mathbb{Z}$ where the conjugation action of the generator of $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by the matrix A. Since $\mathrm{tr}(A^2) = \mathrm{tr}(A)^2 - 2$, if $|\mathrm{tr}(A)| \ge 2$ then by replacing A with its square if necessary (which corresponds to taking a cyclic double cover of M_A) we may assume $\mathrm{tr}(A) \ge 2$.

A nontrivial matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ has eigenvalues λ, λ^{-1} which are either both real or both on the unit circle.

(1) If the trace of A is one of $-1, 0, 1$ then A has finite order (6, 4 or 6 respectively) and we may choose a Euclidean structure on T for which it acts by isometries. In this case the mapping torus M_A is finitely covered by T^3 and has a Euclidean metric.

(2) If the trace of A is 2 then A is conjugate to a matrix of the form $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Thus the foliation of T by meridians is preserved by A, and suspends to a foliation of M_A by oriented circles, exhibiting M_A as an oriented circle bundle over a torus with Euler number n. In particular, M_A has an $|n|$-fold cyclic cover which is homeomorphic (after possibly reversing orientations if $n < 0$) to the manifold $H_{\mathbb{Z}}\backslash H$ from Example 3.7, and M_A has a metric modeled on Nil.

(3) If the trace of A is > 2 then A has two real eigenvalues that we may order $0 < \lambda^{-1} < 1 < \lambda$ with associated eigenspaces V_s and V_u, which necessarily have irrational slope, or else they would contain primitive vectors in $\mathbb{Z}^2$ that were taken to non-primitive vectors by A or A^{-1}. The map A compresses the 'stable direction' V_s and expands the 'unstable direction' V_u; see Figure 32.

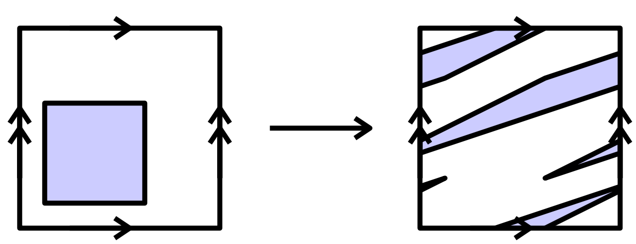

FIGURE 32. An Anosov map on a torus compresses the stable direction and expands the unstable direction.

Parallel translates of these eigenspaces give a pair of linear foliations $\tilde{\mathcal{F}}_s$ and $\tilde{\mathcal{F}}_u$ of $\mathbb{R}^2$ that are preserved by A; it shrinks leaves of $\tilde{\mathcal{F}}_s$ by λ^{-1} and stretches leaves of $\tilde{\mathcal{F}}_u$ by λ. These foliations descend to linear foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ on T of irrational slope. One says that an automorphism A of T is *Anosov* if it preserves such a pair of foliations and shrinks/stretches the leaves by complementary factors.

Now suppose we choose a linear isomorphism $B : \mathbb{R}^2 \to \mathbb{R}^2$ taking V_u, V_s to the x and y coordinate axes respectively. Then BAB^{-1} is the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ which preserves the irrational lattice $\Lambda := B\mathbb{Z}^2$.

Define Sol to be the 3-dimensional solvable Lie group obtained as an exact sequence $\mathbb{R}^2 \to \text{Sol} \to \mathbb{R}$ where the generator t of $\mathbb{R}$ acts by conjugation on $\mathbb{R}^2$ by $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. Any Euclidean metric on the tangent space at the identity defines a complete left-invariant Riemannian metric on Sol. There is a discrete faithful representation from $\pi_1(M_A)$ to Sol which takes $\pi_1(T)$ to $\Lambda \subset \mathbb{R}^2$ and takes the generator of the monodromy to $\log \lambda \in \mathbb{R}$. Thus M_A has a metric modeled on Sol.

Example 4.9 (Finite order elements). Let $\phi \in \text{Mod}(S)$ have finite order. This means there is a power n so that ϕ^n is isotopic to the identity. It turns out that ϕ has a representative homeomorphism f for which f^n is *equal* to the identity. This is a special case of the Nielsen realization theorem (which applies to all finite subgroups of $\text{Mod}(S)$, not just the cyclic ones).

Example 4.10 (Reducible mapping classes). A mapping class $\phi \in \text{Mod}(S)$ is said to be *reducible* if there is a finite family of disjoint essential simple closed curves $\Gamma \subset S$ which are permuted up to isotopy by ϕ. The suspension of Γ in M_ϕ is a finite collection of two-sided embedded tori T_i and one-sided Klein bottles K_j. These lift to infinite cylinders in the cyclic cover $S \times \mathbb{R}$ of M_ϕ, and thence to planes in the universal cover. In particular, the inclusions $T_i \to M_\phi$ and $K_j \to M_\phi$ are injective on π_1. An embedded π_1-injective surface is called *essential*; and a 3-manifold that contains an essential torus or Klein bottle is said to be *toroidal*.

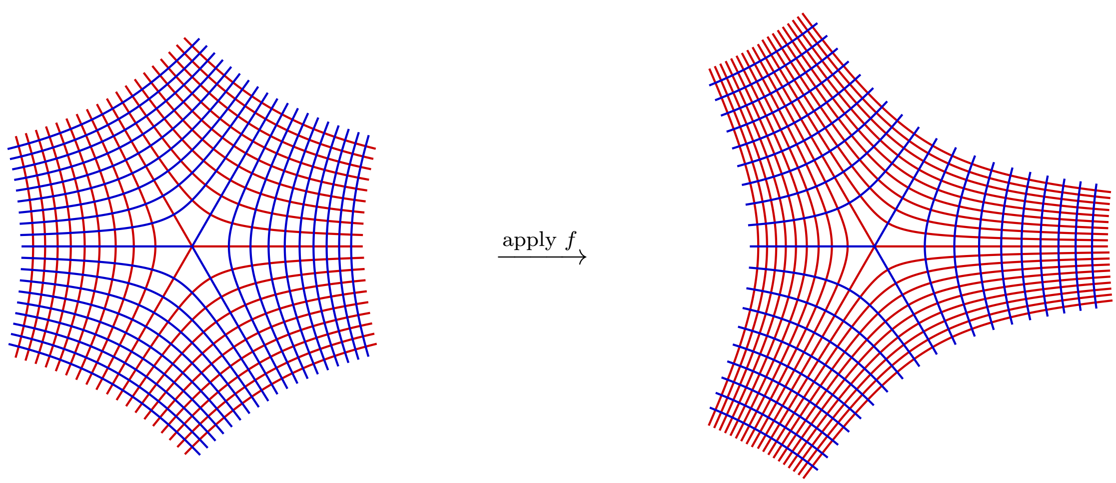

FIGURE 33. A pair of transverse singular foliations (in red and blue respectively) near a 3 prong singularity. A pseudo-Anosov map f stretches the red lines by λ and shrinks the blue lines by λ^{-1}.

Example 4.11 (Pseudo-Anosov mapping classes). An automorphism f of a closed surface S is said to be *pseudo-Anosov* if it admits a pair of transverse singular (in a sense to be defined) foliations $\mathcal{F}_s$ and $\mathcal{F}_u$ which are each taken themselves by f, and which admit local coordinates modeled on foliations of $\mathbb{R}^2$ by vertical and horizontal lines respectively in which f stretches the horizontal lines (i.e. the leaves of $\mathcal{F}_u$) by λ and shrinks the vertical lines (i.e. the leaves of $\mathcal{F}_s$) by λ^{-1} for some real $\lambda^{-1} < 1 < \lambda$.

At the (finitely many) singularities the local picture is obtained from that above near the origin $(0,0) \in \mathbb{R}^2$ by first quotienting by $(x,y) \to (-x,-y)$ and then taking an n-fold cover (for some $n > 2$) branched over the image of $(0,0)$. Figure 33 gives a picture of the pair of singular foliations near a 3-prong singularity (i.e. the case $n = 3$).

One example of a pseudo-Anosov automorphism may be obtained by taking an *Anosov* automorphism A of a torus T, letting P be a finite orbit of A, and letting $S \to T$ be a finite cover of T branched over P in such a way that A lifts to an automorphism $f : S \to S$. The singular f-invariant foliations on S are the preimages of the (nonsingular) A-invariant foliations on T. The singularities in this case are all even-pronged.

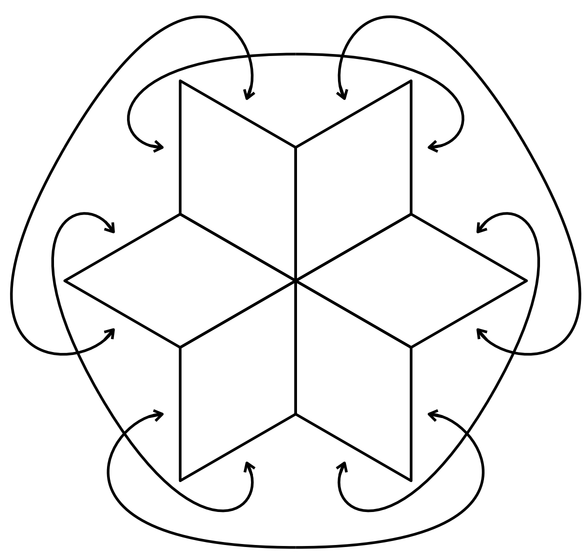

FIGURE 34. A genus 2 surface made from six Euclidean squares.

Another example is given in Figure 34 which shows an arrangement of six unit squares in Euclidean 3-space and a gluing of the twelve free edges by translations with quotient a genus 2 surface S. If we apply the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ to each square the result is a Euclidean parallelogram, and these six Euclidean parallelograms can be made to tile S (each parallelogram wraps over four squares). The map taking each square to its image pieces together on the edges, and defines a piecewise linear map $f : S \to S$. Each square admits a pair of foliations with slopes -1.618034 and 0.618034, and these foliations match together along the edges of the squares with 3-pronged singularities at the vertices. These foliations are preserved by f, one stretched with eigenvalue 2.618034 and one shrunk by a factor of 0.381966. See Figure 35.

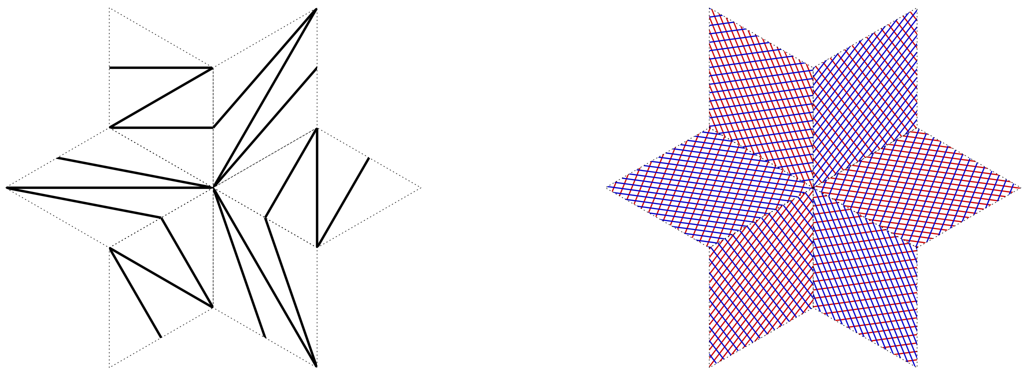

FIGURE 35. A matrix in $\mathrm{SL}(2,\mathbb{Z})$ takes each square to a parallelogram that wraps over four squares. This automorphism preserves a pair of singular foliations with slopes -1.618034 and 0.618034 on each square.

Although Example 4.11 may look very exotic, it turns out to be the *typical* example of a mapping class! Thurston [8] proved the following extraordinary theorem:

Theorem 4.12 (Thurston, classification of surface automorphisms)**.** *Let S be a closed, oriented surface of genus at least 2. Then every orientation-preserving self-homeomorphism of S is isotopic to a map f which is either finite order, reducible, or pseudo-Anosov.*

If $f : S \to S$ is reducible we may cut S along a family of invariant curves and obtain an induced map on the cut open surface. An analog of Theorem 4.12 applies to surfaces with boundary, and we may inductively cut up S into pieces so at the end the restriction of f (after isotopy) is either finite order, or pseudo-Anosov. We shall return to this theorem and its consequences in Chapter 2.

4.3. **Fibered knots and links.** A *fibered knot or link* is a knot or link L in S^3 whose complement is a surface bundle. One may also consider fibered knots or links in arbitrary 3-manifolds.

Example 4.13 (Links of plane singularities). Let z, w be coordinates on $\mathbb{C}^2$, and let $f(z,w)$ be a polynomial with no constant term. Let $V \subset \mathbb{C}^2$ be the variety where $f = 0$; thus $(0,0) \in V$. For small positive ϵ let S^3_ϵ denote the (real) sphere in $\mathbb{C}^2$ consisting of vectors of norm ϵ. Then for sufficiently small ϵ depending on f, the intersection $L := V \cap S^3_\epsilon$ is a link, and the map $f/|f| : S^3_\epsilon - L \to S^1$ is a fibration; in particular, L is a fibered link.

For example,

(1) if f is nonsingular at 0, then L is an unknot;
(2) if $f = zw$ then L is a Hopf link;

(3) if $f = z^3 + w^2$ then L is a right-handed trefoil and in general if $f = z^p + w^q$ then L is a (p,q) torus link (see Example 3.12).

Example 4.14 (Open book decomposition). Let S be a compact oriented surface with boundary, and let $f : S \to S$ be an orientation-preserving homeomorphism which is the identity on ∂S. The mapping torus M_f is a compact manifold whose boundary has the structure of a product $\partial M_f = \partial S \times S^1$. Form the quotient space N from M_f by collapsing every $p \times S^1$ in $\partial S \times S^1$ to a point. Topologically, N is obtained from M_f by doing Dehn filling on each boundary torus along the $p \times S^1$ curves; in particular, it is an oriented 3-manifold, and the image of ∂M_f becomes an oriented link $L \subset N$ whose complement is foliated by Seifert surfaces which are the images of the fibers of M_f. We call the result an *open book decomposition* of N with *binding* L, and the Seifert surface fibers are the *pages* of the book. Thus: an open book decomposition for a manifold N is an oriented fibered link L for which the fiber of $N - L$ may be taken to be a Seifert surface for L.

If $K \subset S^3$ is a fibered knot, for any fibration of $S^3 - K$ the oriented fibers S represent generators of the relative homology $H_2(S^3, K) = H^1(S^3 - K) = \mathbb{Z}$ and their boundary components are therefore longitudes for K; thus every fibered knot in S^3 is the binding of an open book decomposition.

Example 4.15 (Heegaard decompositions). Let $L \subset M$ be the binding of an open book decomposition with fibers homeomorphic to F. Let F^+ and F^- be two such fibers. The union $S := F^+ \cup F^-$ is a closed surface which is the Heegaard surface of a Heegaard splitting $M = H_1 \cup_S H_2$ where each of H_1 and H_2 is an interval of pages. If we take a union of annular neighborhoods A of L on S then each of H_1 and H_2 has a product structure $H_i := F \times I$ in which $A = \partial F \times I$.

Conversely, if $L \subset M$ is an oriented link which admits a Seifert surface F and N is the compact 3-manifold obtained by removing an open tubular neighborhood of F, then $\partial N = S$ is made from two copies $F^+ \cup F^-$ of F together with a union of annuli A whose cores are isotopic to L. If N is homeomorphic to a product $F \times I$ in which $A = \partial F \times I$, then M admits an open boook decomposition with binding L and pages isotopic to F.

Example 4.16 (Connect sum). Suppose L is the binding of an open book structure on S^3; i.e. L is a fibered link with Seifert surface fiber S. Suppose $\alpha \subset L$ is a closed arc and there is a little 3-ball B_L with $B_L \cap L = \alpha$. Then α is the binding of an open book structure on B_L whose pages are disks that bound arcs foliating the open annulus $\partial B_L - L$.

Now suppose L' is the binding of another open book structure on S^3, and choose an arc $\alpha' \subset L'$ and ball $B_{L'}$ as above. Then $S^3 - \text{int}(B_L)$ and $S^3 - \text{int}(B_{L'})$ are closed 3-balls with open book structures that match up on the boundaries, and can be glued up to form an open book structure on S^3 with $L \# L'$ as binding.

Example 4.17 (Murasugi sum). Let S_1 and S_2 be two compact, disjointly embedded oriented surfaces in S^3, and suppose we can isotop the two surfaces so that they can be made to intersect along a polygon P whose edges alternate between boundary edges of S_1 and S_2. We may glue them together along such a polygon to produce a new surface, which is called the *Murasugi sum* of S_1 and S_2, and denoted $S_1 \#_P S_2$.

Now suppose S_1 and S_2 are pages of open book decompositions of S^3. We claim the same is true for $S_1 \#_P S_2$. Split S^3 into two 3-balls $S^3 = B^+ \cup B^-$ where we can arrange for $S_1 \subset B^+$ and $S_2 \subset B^-$ and P is contained on the equatorial $S^2 = B^+ \cap B^-$.

The hypothesis that S_1 and S_2 are pages of open book decompositions means that the complements $M_1 := S^3 - N(S_1)$ and $M_2 := S^3 - N(S_2)$ are handlebodies with the structure of a product $S_1 \times I$ and $S_2 \times I$. The ball B^- is contained in M_1 and B^+ is contained in M_2; with respect to the product structure we can think of $M_1 - B^-$ as being obtained by taking a $P \times [1/2, 1]$ 'bite' out of the $S_1 \times I$ structure, and $B_2 - B^+$ obtained by taking a $P \times [0, 1/2]$ 'bite' out of $S_2 \times I$. After an isotopy we can think of P itself as $P \times [1/2]$ in either factor; gluing along this P produces

$$(S_1 \times I - P \times [1/2, 1]) \cup_{P \times 1/2} (S_2 \times I - P \times [0, 1/2]) = S_1 \#_P S_2 \times I$$

It follows that the complement $S^3 - N(S_1 \#_P S_2)$ is a handlebody with a product structure, so that $S_1 \#_P S_2$ is a page of an open book decomposition. See Figure 36 in the case that P is a square with two edges on each of ∂S_1 and ∂S_2.

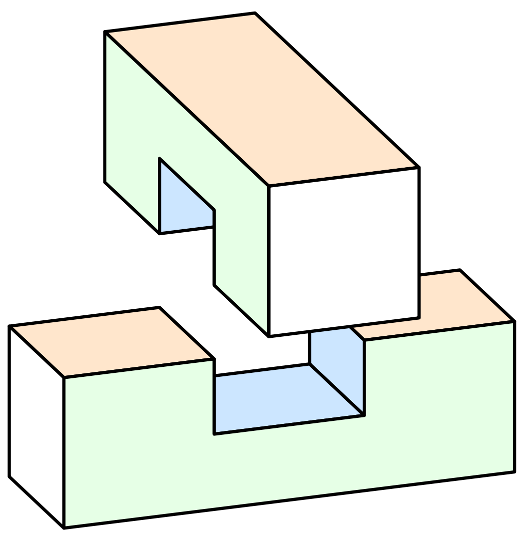

FIGURE 36. Murasugi sum of pages of open books on S^3 yields a page of a new open book. Surfaces are in orange, and annuli are in green. The blue faces are exposed by the 'bite' taken out of each product piece by removing B^- and B^+ respectively.

Many fibered knots and links may be obtained via Murasugi sum. Summing two Hopf bands of the same handedness produces a fiber for a right- or left-handed trefoil. Summing two Hopf bands of opposite handedness produces a fiber for the figure 8 knot.

Example 4.18 (Braids). A *braid representation* of an oriented link L in S^3 is a link projection in which every component winds monotonely and positively around some point in the projection plane. One may express this equivalently without reference to a projection: choose an unknot K in $S^3 - L$ and parameterize $S^3 - K$ as $D \times S^1$ in such a way that every component of L projects to the S^1 factor by a covering map (of positive degree); in this case we might say that L is in *braid position* with respect to the unknot K. A braid representation of L exhibits $S^3 - (L \cup K)$ as the mapping torus of some element of a braid group; in particular, $L \cup K$ is a fibered link.

Alexander showed that every link L in S^3 admits a braid representation. One way to see this is to choose an arbitrary projection which is in general position with respect to the foliation of the plane minus the origin by lines of constant argument. Each component of L decomposes into segments whose top and bottom points are tangencies with the foliation; we are concerned with the negatively oriented segments. Each negative segment may be broken up further into *long negative segments* that contain only overcrossings, and *short negative segments* that contain a single undercrossing. Take each long negative segment and drag it up and over the origin, replacing it by a long positive segment. Now each short negative segment has endpoints on positively oriented segments, and may be adjusted by an isotopy to be positively oriented, thus eliminating all tangencies. See Figure 37.

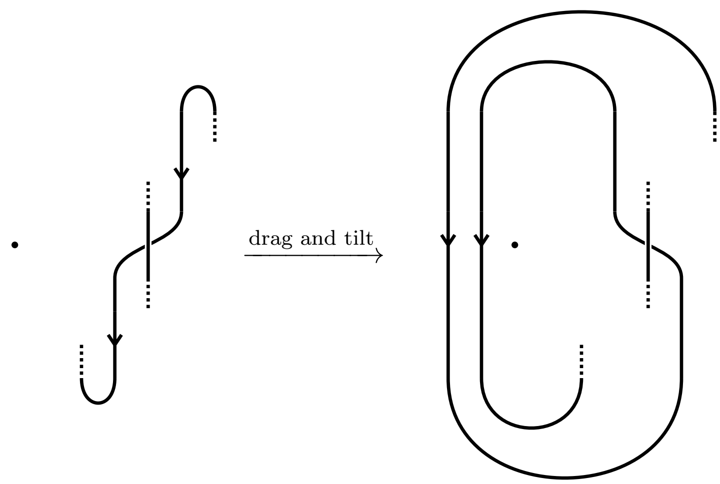

FIGURE 37. Drag long negative segments over the origin and tilt short negative segments to put a link in braid position.

Example 4.19. Every closed oriented 3-manifold M contains a fibered link. To see this first exhibit M as surgery on a link L in S^3, so that there is a link L' in M and an identification $M - L' = S^3 - L$. Now drill out an additional unknot K from S^3 so that L is in braid position with respect to K.

Example 4.20. Every closed oriented 3-manifold admits an open book decomposition. To see this first take a branched cover $M \to S^3$ branched over some link L. Put L in braid position with respect to an unknot K. Then S^3 has an open book decomposition with K as the binding, and with L transverse to the pages. Thus the preimages of the pages give an open book decomposition of M with binding the preimage of K.

5. Acknowledgments

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