Option Pricing and Portfolio Optimization Solutions

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Chapter 2: The Continuous-Time Market Model

Exercise 1. Prove: If Y is a modification of the stochastic process X and if X and Y both have continuous paths then they are indistinguishable.

Proof. Fix a continuous stochastic process X and suppose that Y is a continuous modification of X. By definition, we have that $\mathbb{P}\{X_t \neq Y_t\} = 0$ for each $t \in [0, \infty)$. Define $A := \{\omega \in \Omega \quad | \quad \exists t \in [0, \infty) : X_t(\omega) \neq Y_t(\Omega)\}$ and observe that, by the continuity of both X and Y, for each $\omega \in A$, there exists a collection of open interval $\{I_\omega\}$ such that $\omega \in \{X_t \neq Y_t\}$ for all $t \in I_\omega$. Define $\mathcal{I} := \Big\{\bigcup\{I_\omega\} : \omega \in A\Big\}$. By this standard properties of $\mathbb{R}$, the set that this collection covers has a countable subcovering $\{I_{\omega_n}\}_{n\in\mathbb{N}}$. By the same reasoning, for each I_{ω_n} there exists a countable collection $\{t_{n_k}\}_{k\in\mathbb{N}} \subset [0, \infty)$ such that, for any $I_\omega \in \mathcal{I}$ with $I_\omega \cap I_{\omega_n} \neq \varnothing$, there exists k with $t_{n_k} \in I_\omega \cap I_{\omega_n}$. I claim that $A \subset \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}}\{X_{t_{n_k}} \neq Y_{t_{n_k}}\}$. Indeed, for any $\omega \in A$, there exists $n \in \mathbb{N}$ such that $I_\omega \cap I_{\omega_n} \neq \varnothing$, and therefore some $k \in \mathbb{N}$ such that $t_{n_k} \in I_\omega \cap I_{\omega_n}$, so that $\omega \in \{X_{t_{n_k}} \neq Y_{t_{n_k}}\}$. It follows that $\mathbb{P}A \le \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}}\{X_{t_{n_k}} \neq Y_{t_{n_k}}\}\Big) = 0$, proving that X and Y are indistinguishable. □

Exercise 2. Let τ be a stopping time and $\{(X_t, \mathcal{F}_t)\}_{t\ge 0}$ a right-continuous (sub-)martingale. Show that under these assumptions the stopped process $\{(X_{t\wedge\tau}, \mathcal{F}_t)\}_{t\ge0}$ is again a (sub-)martingale.

Proof. Fix a stopping time τ. Observe that it suffices to prove the statement for locally right constant (sub-)martingales. Indeed, using the fact that any right-continuous (sub-)martingale can be approximated from below by locally right constant (sub-)martingales, the dominated convergence theorem will imply that the statement holds also for right-continuous (sub-)martingales. To this end, fix a locally right constant martingale $X_t = \sum_{n=0}^\infty \varphi_n 1_{[t_n, t_{n+1})}$, with $t_0 = 0$. Note that

$$X_{t\wedge\tau} = \varphi_{m+1} 1_{\{\tau \ge m\}} + \sum_{n=0}^{m-1} \varphi_n 1_{\{\tau\in[t_n, t_{n+1})\}}, \quad m := \max\{k : t_k < t\},$$

which is $\mathcal{F}_{t_m}$-measurable, and therefore $\mathcal{F}_t$ measurable. Moreover, $\mathbb{E}[|X_{t\wedge\tau}|] \le \sum_{n=0}^{m+1} \mathbb{E}[|\varphi_n|] < \infty$. Finally, observe that, by induction and the fact that each φ_n and $1_{\{\tau\in[t_n, t_{n+1})\}}$ are $\mathcal{F}_{t_{n+1}}$ measurable, to prove that $\mathbb{E}[X_{t\wedge\tau} \mid \mathcal{F}_s] = X_{s\wedge\tau}$ for any $0 \le s < t$, it suffices to prove that $\mathbb{E}[X_{t_{n+1}\wedge\tau} \mid \mathcal{F}_{t_n}] = X_{t_n\wedge\tau}$ for each n. We have that

$$\begin{aligned}
\mathbb{E}[X_{t_{n+1}\wedge\tau} \mid \mathcal{F}_{t_n}] &= \mathbb{E}\left[\varphi_{n+1} 1_{\{\tau\ge t_n\}} + \sum_{k=0}^{n-1} \varphi_k 1_{\{\tau\in[t_k, t_{k+1})\}} \mid \mathcal{F}_{t_n}\right] \\
&= 1_{\{\tau\ge t_n\}} \mathbb{E}[\varphi_{n+1} \mid \mathcal{F}_{t_n}] + \sum_{k=0}^{n-1} \varphi_k 1_{\{\tau\in[t_k, t_{k+1})\}} \\
&= 1_{\{\tau\ge t_n\}} \varphi_n + \sum_{k=0}^{n-1} \varphi_k 1_{\{\tau\in[t_k, t_{k+1})\}} \\
&= \varphi_n 1_{\{\tau\ge t_{n-1}\}} + \sum_{k=0}^{n-1} \varphi_k 1_{\{\tau\in[t_k, t_{k+1})\}} \\
&= X_{t_n\wedge\tau}.
\end{aligned}$$

□

Exercise 3. Let the process $\{P(t)\}_{t\geq 0}$ be defined by

$$P(t) = p \cdot e^{(b - \frac{1}{2}\sigma^2)t + \sigma W(t)},$$

where $W(t)$ is a one-dimensional Brownian motion, p, b, $\sigma \in \mathbb{R}$ are real constants with $\sigma \neq 0$.

Show:

(a) $Var(P(t)) = p^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right)$.

(b) $P(t) \xrightarrow{t\to\infty} \begin{cases} \infty \text{ a.s. } \mathbb{P} & \text{if } b > \frac{1}{2}\sigma^2 \\ 0 \text{ a.s. } \mathbb{P} & \text{if } b < \frac{1}{2}\sigma^2 \end{cases}$.

(c) Compare the result of (b) with the limiting behavior of $E(P(t))$, $Var(P(t))$ for $t \to \infty$.

Proof. (a) Noting that $\sigma W(t) \sim \mathcal{N}(0, \sigma^2 t)$, it follows that

$$\begin{aligned} \mathbb{E}(P(t)^2) &= p^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \cdot e^{(2b-\sigma^2)t + 2\sigma x} \cdot e^{-\frac{x^2}{2t}}\, dx \\ &= p^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \cdot e^{(2b+\sigma^2)t} \cdot e^{-\frac{(x-2\sigma t)^2}{2t}}\, dx \\ &= p^2 \cdot e^{2bt + \sigma^2 t}. \end{aligned}$$

From Lemma 2.25, we have that $\mathbb{E}(P(t)) = p \cdot e^{bt}$, and so

$$Var(P(t)) = \mathbb{E}(P(t)^2) - \mathbb{E}(P(t))^2 = p^2 \cdot e^{2bt}(e^{\sigma^2 t} - 1).$$

(b) We have that $\frac{\ln P(t)}{t} = \frac{\ln p}{t} + b - \frac{1}{2}\sigma^2 + \sigma \frac{W(t)}{t}$. Now by the law of large numbers for Brownian motion, we have that $\frac{W(t)}{t} \xrightarrow{t\to\infty} 0$ a.s. $\mathbb{P}$. Hence, $\frac{\ln P(t)}{t} \xrightarrow{t\to\infty} b - \frac{1}{2}\sigma^2$ a.s. $\mathbb{P}$. If $b > \frac{1}{2}\sigma^2$, it follows that $\ln P(t) \xrightarrow{t\to\infty} \infty$ a.s. $\mathbb{P}$, so that $P(t) \xrightarrow{t\to\infty} \infty$ a.s. $\mathbb{P}$. Otherwise if $b < \frac{1}{2}\sigma^2$, then $\ln P(t) \xrightarrow{t\to\infty} -\infty$ a.s. $\mathbb{P}$, so that $P(t) \xrightarrow{t\to\infty} 0$ a.s. $\mathbb{P}$.

(c) Observe that $\mathbb{E}(P(t)) = pe^{bt} \xrightarrow{t\to\infty} \begin{cases} \infty & \text{if } b > 0 \\ p & \text{if } b = 0 \\ 0 & \text{if } b < 0 \end{cases}$, and $Var(P(t)) = p^2 e^{2bt}\left(e^{\sigma^2 t} - 1\right) \xrightarrow{t\to\infty} \begin{cases} \infty & \text{if } b > -\frac{1}{2}\sigma^2 \\ p^2 & \text{if } b = -\frac{1}{2}\sigma^2 \\ 0 & \text{if } b < -\frac{1}{2}\sigma^2 \end{cases}$. □

Exercise 4. Let $\{(X(t), \mathcal{F}_t)\}_{t\geq 0}$ be a stochastic process with a filtration $\{\mathcal{F}_t\}_t$ satisfying the usual conditions. Show that for all $n \in \mathbb{N}$ the random variable $\tau(\omega) := \inf\{t \geq 0 : X(t, \omega) \geq n\}$ is a stopping time.

Proof. I am fairly certain we need the additional assumption that X is at least left or right path continuous a.s. $\mathbb{P}$. Since $\{\mathcal{F}_t\}_t$ is a complete filtration, we may assume w.l.o.g. that X is simply left or right path continuous. For the case where X is right path continuous, note that $\omega \in \{\tau \leq t\}$ if and only if $\omega \in X_{t'}^{-1}([n, \infty))$ for some $t' \in [0, t]$. By the right continuity of $t \mapsto X_t$, it follows that $\{\tau \leq t\} = X_t^{-1}([n,\infty)) \cup \bigcup_{q\in\mathbb{Q}\cap[0,t]} X_q^{-1}([n,\infty))$, and since $X_q^{-1}([n,\infty)) \in \mathcal{F}_t$ for all $q \in [0, t]$, it follows that $\{\tau \leq t\} \in \mathcal{F}_t$.

For the case where X is left continuous, observe that $\omega \in \{\tau \leq t\}$ if and only if $\omega \in X_{t'}^{-1}([n,\infty))$ for some $t' \in [0,t]$, or for each $t' > t$, there exists some $t'' \in (t, t')$ such that $\omega \in X_{t''}^{-1}([n,\infty))$. Take a sequence $\{t_k\}_k$ such that $t_k \xrightarrow{k\to\infty} t$ and $t_k > t$ for all k. By left continuity, we see that $\{\tau \leq t\} = \bigcup_{q\in\mathbb{Q}\cap[0,t]} X_q^{-1}([n,\infty)) \cup \bigcap_{k\geq 1}\bigcup_{m\geq k} X_{t_m}^{-1}([n,\infty))$. Observe that for any $\varepsilon > 0$, there exists some k such that $t < t_m < t + \varepsilon$ for all $m \geq k$, so that $\bigcup_{m\geq k} X_{t_m}^{-1}([n,\infty)) \in \mathcal{F}_{t+\varepsilon}$. It follows that $\bigcap_{k\geq 1}\bigcup_{m\geq k} X_{t_m}^{-1}([n,\infty)) \in \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, proving that $\{\tau \leq t\} \in \mathcal{F}_t$. Thus, if X is either left or right path continuous a.s. $\mathbb{P}$, then τ is a stopping time. □

Exercise 5. Let $\{(X(t), \mathcal{F}_t)\}_{t\geq 0}$ be a one-dimensional Itô process. Prove that its representation

$$X(t) = X(0) + \int_0^t K(s)\, ds + \int_0^t H(s)\, dW(s)$$

is uniquely determined. More precisely, if

$$X(t) = Y(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s)$$

is another representation, then we have

- $X(0) = Y(0)$ a.s. $\mathbb{P}$
- $K(s)$ and $\mu(s)$ as well as $H(s)$ and $\sigma(s)$ are equivalent with respect to $\lambda \otimes \mathbb{P}$.

Proof.

Lemma 1. Suppose that $\{(M(t), \mathcal{F}_t)\}_{t\in[0,T]}$ is a continuous martingale of the form

$$M(t) = \int_0^t v(s)\,ds \quad \text{with} \quad \int_0^T |v(s)|\,ds \le C < \infty.$$

Then $M(t) = 0$ for all $t \in [0,T]$ a.s. $\mathbb{P}$.

Suppose first that $v \in L^2[0,T]$. Then for any $t \in [0,T]$ and partition π of $[0,t]$, by repeated applications of Jensen's inquality, we have that

$$\begin{aligned}
\sum_\pi (M(t_{i+1}) - M(t_i))^2 &= \sum_\pi \left(\int_{t_i}^{t_{i+1}} v(s)\,ds \right)^2 \\
&\le \sum_\pi (t_{i+1} - t_i) \int_{t_i}^{t_{i+1}} v(s)^2\,ds \\
&\le \|\pi\| \sum_\pi \int_{t_i}^{t_{i+1}} v(s)^2\,ds \\
&= \|\pi\| \int_0^t v(s)^2\,ds \xrightarrow{\|\pi\|\to 0} 0.
\end{aligned}$$

Thus, when $v \in L^2[0,T]$, M has zero quadratic variation. In the case where $v \notin L^2[0,T]$, define $v_n = v \cdot 1_{\{|v|\le n\}}$. Then each $v_n \in L^2[0,T]$ and so $M_n(t) = \int_0^t v_n(s)\,ds$ has zero quadratic variation. Applying the dominated convergence theorem to $|v_n| \le |v|$, we see that

$$\sup_{t\in[0,T]} |M_n(t) - M(t)| \le \int_0^T |v_n(s) - v(s)|\,ds \xrightarrow{n\to\infty} 0.$$

Thus, $M_n \to M$ uniformly and it follows that M must also have zero quadratic variation. In particular, for any $t \in [0,T]$ and for any partition π of $[0,t]$, we have that

$$\mathbb{E}\left[M(t)^2\right] = \mathbb{E}\left[\sum_\pi (M(t_{t_{i+1}}) - M(t_i))^2\right] \xrightarrow{\|\pi\|\to 0} 0,$$

and it follows that $M(t) = 0$ a.s. $\mathbb{P}$ for all $t \in [0,T]$.

Lemma 2. Let $\{(M(t), \mathcal{F}_t)\}_{t\in[0,T]}$ be as above, but with the weakened condition:

$$\int_0^T |v(s)|\,ds < \infty \quad \text{a.s. } \mathbb{P}.$$

Then $M(t) = 0$ for all $t \in [0,T]$ a.s. $\mathbb{P}$.

For each $n \in \mathbb{N}$, define $\tau_n := \inf\{t \in [0,T] : \int_0^t |v(s)|\,ds \ge n\}$. By Exercise 4, each τ_n is a stopping time. Observe that the stopped martingale $M_{t\wedge\tau_n} = \int_0^{t\wedge\tau_n} v(s)\,ds$ has the property that $\int_0^{T\wedge\tau_n} |v(s)|\,ds \le C < \infty$, and we can apply the same reasoning as in Lemma 1 to conclude that $M(t) = 0$ for all $t \in [0, T\wedge\tau_n]$ a.s. $\mathbb{P}$. Given that $\int_0^T |v(s)|\,ds < \infty$ a.s. $\mathbb{P}$, it follows that a.s. $\mathbb{P}$ there exists some $N(\omega)$ such that $\tau_N = T$, and Lemma 2 follows after some obvious $\mathbb{P}$-null set arguments.

Finally, suppose that for some one-dimensional Itô process $\{(X(t), \mathcal{F}_t)\}_{t\ge 0}$, we have two representations:

- $X(t) = X(0) + \int_0^t K(s)\,ds + \int_0^t H(s)\,dW(s)$
- $X(t) = Y(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s)$

Then $X(0) = Y(0) + \sigma(0)W_0 = Y(0)$ a.s. $\mathbb{P}$. Now define the continuous martingale $M(t) := \int_0^t H(s) - \sigma(s)\, dW(s)$. Observe that $M(t) = \int_0^t \mu(s) - K(s)\, ds$ a.s. $\mathbb{P}$. Since $\int_0^T |\mu(s) - K(s)|\, ds < \infty$ for all $T > 0$, we can apply Lemma 2 to conclude that $M(t) = 0$ for all $t \in [0, \infty)$ a.s. $\mathbb{P}$. It follows that H and σ, as well as K and μ are equivalent with respect to $\lambda \otimes \mathbb{P}$. □

Exercise 6. Show that the processes M_t and H_t occurring in the proof of Itô's formula satisfy

$$\mathbb{E}\left(\sum_{k=1}^m \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2\, ds\right)\right)^2 = \mathbb{E}\left(\sum_{k=1}^m \left((M_{t_k} - M_{t_{k-1}})^2 - \int_{t_{k-1}}^{t_k} H_s^2\, ds\right)^2\right)$$

Proof. Define $X_i := (M_{t_i} - M_{t_{i-1}})^2 - \int_{t_{i-1}}^{t_i} H_s^2\, ds$. By the Itô isometry,

$$\mathbb{E}X_i = \mathbb{E}\left(\int_{t_{i-1}}^{t_i} H_s\, dW(s)\right)^2 - \mathbb{E}\left(\int_{t_{i-1}}^{t_i} H_s^2\, ds\right) = 0.$$

Thus, it suffices to prove that $Cov(X_i, X_j) = 0$ for $i \neq j$, for then $\mathbb{E}\left(\sum_{k=1}^m X_k\right)^2 = \mathbb{E}\left(\sum_{k=1}^m X_k^2\right)$. Fix $i < j$ and observe that

$$\begin{aligned}
\mathbb{E}(X_i X_j) &= \mathbb{E}(\mathbb{E}(X_i X_j \mid \mathcal{F}_{t_i})) \\
&= \mathbb{E}(X_i \mathbb{E}(X_j \mid \mathcal{F}_{t_i})) \\
&= \mathbb{E}(X_i \cdot 0) \\
&= 0,
\end{aligned}$$

where the third equality follows from another application of Itô's isometry. □

Exercise 7. Let $\{(X(t), \mathcal{F}_t)\}_{t \geq 0}$ be an Itô process. Let τ be a stopping time. Prove that for suitable f we have:

$$\int_0^s f(X(t \wedge \tau))\, dX(t \wedge \tau) = \int_0^{s \wedge \tau} f(X(t))\, dX(t).$$

Proof. Since X is an Itô process, there exist progressively measurable processes K and H with $\int_0^t |K(s)|\, ds < \infty$ and $\int_0^t H^2(s)\, ds < \infty$ a.s. $\mathbb{P}$ for all $t \geq 0$, such that $X(t) = X(0) + \int_0^t K(s)\, ds + \int_0^t H(s)\, dW(s)$. Thus,

$$\begin{aligned}
X(t \wedge \tau) &= X(0) + \int_0^{t \wedge \tau} K(s) ds + \int_0^{t \wedge \tau} H(s)\, dW(s) \\
&= X(0) + \int_0^t K(s) 1_{[0,\tau]}\, ds + \int_0^t H(s) 1_{[0,\tau]}\, dW(s),
\end{aligned}$$

and so $X(t \wedge \tau)$ is an Itô process. It follows that for suitable f we have

$$\begin{aligned}
\int_0^s f(X(t \wedge \tau))\, dX(t \wedge \tau) &= \int_0^s f(X(t \wedge \tau)) K(t) 1_{[0,\tau]}\, dt + \int_0^s f(X(t \wedge \tau)) H(t) 1_{[0,\tau]}\, dW(t) \\
&= \int_0^{s \wedge \tau} f(X(t)) K(t)\, dt + \int_0^{s \wedge \tau} f(X(t)) H(t)\, dW(t) \\
&= \int_0^{s \wedge \tau} f(X(t))\, dX(t).
\end{aligned}$$

□

Exercise 8. Prove the product rule, Corollary 2.53.

Proof. Fix one-dimensional Itô processes X_t and Y_t with $X_t = X_0 + \int_0^t K_s\, ds + \int_0^t H_s\, dW_s$, and $Y_t = Y_0 + \int_0^t \mu_s\, ds + \int_0^t \sigma_s\, dW_s$. Define the two-dimensional Itô process $Z_t = (X_t, Y_t)$ and let $f(t, x, y) = xy \in C^{1,2}([0, \infty) \times \mathbb{R}^2)$. Then by the multi-dimensional Itô formula,

$$\begin{aligned}
X_t \cdot Y_t &= X_0 \cdot Y_0 + \int_0^t Y_s\, dX_s + \int_0^t X_s\, dY_s + \int_0^t d\langle X, Y \rangle_s \\
&= X_0 \cdot Y_0 + \int_0^t Y_s K_s + X_s \mu_s + H_s \sigma_s\, ds + \int_0^t Y_s H_s + X_s \sigma_s\, dW_s.
\end{aligned}$$

□

Exercise 9. Let $\{(W(t),\mathcal{F}_t)\}_{t\in[0,T]}$ be a one-dimensional Brownian motion. Show that the following processes are martingales with respect to $\{\mathcal{F}_t\}_t$:

(a) $X(t)=\exp\left(\frac{t}{2}\right)\cdot\cos(W(t))$;

(b) $X(t)=\exp\left(\frac{t}{2}\right)\cdot\sin(W(t))$;

(c) $X(t)=(W(t)+t)\cdot\exp\left(-W(t)-\frac{t}{2}\right)$.

Proof. I believe Observe that $W(t)$ is an Itô process, with $W(t)=W(0)+\int_0^t 0\,ds+\int_0^t 1\,dW_s$. Define $f(t,x)=\exp\left(\frac{t}{2}\right)\cdot\cos(x)$, $g(t,x)=\exp\left(\frac{t}{2}\right)\cdot\sin(x)$, and $h(t,x)=(x+t)\cdot\exp\left(-x-\frac{t}{2}\right)$. Because $f,g,h\in C^{1,2}([0,\infty)\times\mathbb{R})$, we can apply the multi-dimensional Itô formula to obtain that

(a)

$$\begin{aligned}\exp\left(\frac{t}{2}\right)\cdot\cos(W(t))&=f(t,W(t))\\&=1+\frac{1}{2}\int_0^t\exp\left(\frac{s}{2}\right)\cdot\cos(W(s))\,ds-\int_0^t\exp\left(\frac{s}{2}\right)\cdot\sin(W(s))\,dW_s-\frac{1}{2}\int_0^t\exp\left(\frac{s}{2}\right)\cdot\cos(W(s))\,ds\\&=1-\int_0^t\exp\left(\frac{s}{2}\right)\cdot\sin(W(s))\,dW_s;\end{aligned}$$

(b)

$$\begin{aligned}\exp\left(\frac{t}{2}\right)\cdot\sin(W(t))&=f(t,W(t))\\&=\frac{1}{2}\int_0^t\exp\left(\frac{s}{2}\right)\cdot\sin(W(s))\,ds+\int_0^t\exp\left(\frac{s}{2}\right)\cdot\cos(W(s))\,dW_s-\frac{1}{2}\int_0^t\exp\left(\frac{s}{2}\right)\cdot\sin(W(s))\,ds\\&=\int_0^t\exp\left(\frac{s}{2}\right)\cdot\cos(W(s))\,dW_s;\end{aligned}$$

(c)

$$\begin{aligned}(W(t)+t)\cdot\exp\left(-W(t)-\frac{t}{2}\right)&=\int_0^t\exp\left(-W(s)-\frac{s}{2}\right)-\frac{1}{2}(W(s)+s)\exp\left(-W(s)-\frac{s}{2}\right)ds\\&\quad+\int_0^t\exp\left(-W(s)-\frac{s}{2}\right)-(W(s)+s)\exp\left(-W(s)-\frac{s}{2}\right)dW_s\\&\quad+\frac{1}{2}\int_0^t-2\exp\left(-W(s)-\frac{s}{2}\right)+(W(s)+s)\exp\left(-W(s)-\frac{s}{2}\right)ds\\&=\int_0^t(1-W(s)-s)\exp\left(-W(s)-\frac{s}{2}\right)dW_s.\end{aligned}$$

Since $\exp\left(\frac{t}{2}\right)\cdot\sin(W(t))$, $\exp\left(\frac{t}{2}\right)\cdot\cos(W(t))$, $(1-W(s)-s)\exp\left(-W(s)-\frac{s}{2}\right)\in L^2[0,T]_{\{\mathcal{F}_t\}_t}$, and because the Itô integral maps $L^2[0,T]$ into the space of continuous $\{\mathcal{F}_t\}_t$ martingales with expectation equal to 0, it follows that each of the given processes are martingales with respect to $\{\mathcal{F}_t\}_t$. □

Exercise 10. Define

$$H(t):=\exp\left(-\int_0^t r(s)+\frac{1}{2}\|\theta(s)\|^2\,ds-\int_0^t\theta(s)'\,dW(s)\right),\quad\theta(t):=\sigma^{-1}(t)(b(t)-r(t)\underline{1}).$$

(a) Show that $1/H(t)$ is the wealth process corresponding to the pair

$$(\pi(t),c(t))=(\sigma^{-1}(t)'\sigma^{-1}(t)(b(t)-r(t)\underline{1}),0)$$

and an initial wealth of $x=1/H(0)=1$.

Proof. We need to verify that $1/H(t)$ solves the wealth equation for the given self-financing pair (π, c). Define the Itô process $Y_t := -\int_0^t r(s) + \frac{1}{2}\|\theta(s)\|^2 \, ds - \int_0^t \theta(s)' \, dW(s)$ and apply the Itô formula to $f(x) = e^{-x}$ to get that

$$\begin{aligned}
d(1/H(t)) &= -f(Y_t)\, dY_t + \frac{1}{2} f(Y_t)\, d\langle Y \rangle_t \\
&= \frac{1}{H(t)} \left((r(t) + \frac{1}{2}\|\theta(t)\|^2)\, dt + (\theta(t)')\, dW(t) \right) + \frac{1}{H(t)} \left(\frac{1}{2}\|\theta(t)\|^2 \, dt \right) \\
&= \left(r(t) \frac{1}{H(t)} \right) dt + \frac{1}{H(t)} (b(t) - r(t)\underline{1})' \sigma^{-1}(t)' \sigma^{-1}(t) \Big((b(t) - r(t)\underline{1})\, dt + \sigma(t)\, dW(t) \Big) \\
&= \left(r(t) \frac{1}{H(t)} - c(t) \right) dt + \frac{1}{H(t)} \pi(t)' \Big((b(t) - r(t)\underline{1})\, dt + \sigma(t)\, dW(t) \Big).
\end{aligned}$$

Thus, $1/H(t)$ is the unique solution for the wealth equation corresponding to the self-financing pair (π, c) and initial wealth $x = 1$. □

(b) Let $(\pi, c) \in \mathcal{A}(1)$ with $c \equiv 0$ and

$$\mathbb{E}\left(\int_0^T \pi(t)' \sigma(t)\, dW(t) \right) = 0, \qquad \int_0^T \|\pi(t)^2\| \, dt < \infty.$$

Show that if for the wealth process $X(t)$ corresponding to $(\pi, 0)$ the expected value $\mathbb{E}(\ln(X(T))$ exists then we have

$$\mathbb{E}(\ln(X(T)) \le \mathbb{E}\left(\ln\left(\frac{1}{H(T)} \right) \right).$$

Proof. Since $X(t)$ is the wealth process corresponding to $(\pi, 0)$, $X(t)$ must satisfy the wealth equation:

$$dX(t) = (r(t)X(t) - c(t))\, dt + X(t)\pi(t)'((b(t) - r(t)\underline{1})\, dt + \sigma(t)\, dW(t)).$$

By the Itô formula,

$$\begin{aligned}
d(\ln(X(t))) &= \frac{1}{X(t)}\, dX(t) - \frac{1}{2X(t)^2}\, d\langle X \rangle_t \\
&= \left(r(t) + \pi(t)'(b(t) - r(t)\underline{1}) - \frac{1}{2}\pi(t)'\sigma(t)\sigma(t)'\pi(t) \right) dt + \pi(t)'\sigma(t)\, dW(t),
\end{aligned}$$

and it follows that

$$\begin{aligned}
\mathbb{E}(\ln(X(T)) &= \mathbb{E}\left(\ln(X(0)) + \int_0^T r(t) + \pi(t)'(b(t) - r(t)\underline{1}) - \frac{1}{2}\|\sigma(t)'\pi(t)\|^2 \, dt + \int_0^T \pi(t)'\sigma(t)\, dW(t) \right) \\
&= \mathbb{E}\left(\int_0^T r(t) + \pi(t)'(b(t) - r(t)\underline{1}) - \frac{1}{2}\|\sigma(t)'\pi(t)\|^2 \, dt \right).
\end{aligned}$$

The same line of reasoning shows that

$$\begin{aligned}
\mathbb{E}\left(\ln\left(\frac{1}{H(T)} \right) \right) &= \mathbb{E}\left(\ln\left(\frac{1}{H(0)} \right) + \int_0^T r(t) + \|\sigma^{-1}(t)(b(t) - r(t)\underline{1})\|^2 \right. \\
&\qquad \left. - \frac{1}{2}\|\sigma^{-1}(t)(b(t) - r(t)\underline{1})\|^2 \, dt + \int_0^T (b(t) - r(t)\underline{1})'\sigma^{-1}(t)'\, dW(t) \right) \\
&= \mathbb{E}\left(\int_0^T r(t) + \frac{1}{2}\|\sigma^{-1}(t)(b(t) - r(t)\underline{1})\|^2 \, dt \right).
\end{aligned}$$

Hence, the problem is reduced to proving the inequality

$$\mathbb{E} \int_0^T \|\sigma^{-1}(t)(b(t) - r(t)\underline{1})\|^2 - 2\pi(t)'(b(t) - r(t)\underline{1}) + \|\pi(t)'\sigma(t)\|^2 \, dt \ge 0.$$

Writing $\theta(t) = \sigma^{-1}(t)(b(t) - r(t)\underline{1})$, we have

$$\begin{aligned}\mathbb{E}\int_0^T \|\sigma^{-1}(t)(b(t) - r(t)\underline{1})\|^2 - 2\pi(t)'(b(t) - r(t)\underline{1}) + \|\pi(t)'\sigma(t)\|^2\,dt \\ = \mathbb{E}\int_0^T \|\theta(t)\|^2 - 2\pi(t)'\sigma(t)\theta(t) + \|\pi(t)'\sigma(t)\|^2\,dt \\ = \mathbb{E}\int_0^T \|\theta(t) - \pi(t)'\sigma(t)\|^2\,dt \geq 0.\end{aligned}$$

□

Exercise 11. Let $B \geq -K$ be an $\mathcal{F}_T$-measurable random variable with $K > 0$ and $T > 0$ fixed. Show that under suitable assumptions there exist an initial wealth of $x \geq -K$ and a trading strategy φ such that the corresponding wealth process $X(t)$ satisfies

$$X(t) \geq -K \text{ for all } t \in [0,T],$$
$$X(T) = B \text{ a.s. } \mathbb{P}.$$

Proof. Define $y := \mathbb{E}\Big(H(T)(B + K)\Big)$ and assume that $y < \infty$. Then by Theorem 2.63 (2) there exists a portfolio process $\pi(t)$, $t \in [0,T]$, with $(\pi, 0) \in \mathcal{A}(y)$ and the corresponding wealth process $Y(t)$ satisfies $Y(T) = B + K$ a.s. $\mathbb{P}$. Now define $X(t) := Y(t) - K$ and note that $X(t)$ also satisfies the same wealth equation that $Y(t)$ satisfies and so by the Variation of Constants Theorem, $X(t)$ is the unique wealth process corresponding to the self-financing pair $(\pi, 0)$ with initial wealth $X(0) = Y(0) - K = y - K \geq -K$. Moreover, we have that $X(T) = Y(T) - K = B$ a.s. $\mathbb{P}$. Thus, the trading strategy φ given by $\varphi_i(t) := \frac{\pi_i(t)X(t)}{P_i(t)}$ suffices. □

Exercise 12. By suitable localization deduce Corollary 2.70 from the martingale representation theorem.

Proof. I believe Corollary 2.70 needs the further assumption that there exists a localization $\{\tau_n\}_n$ for the local Brownian martingale $\{(M_t, \mathcal{F}_t)\}_{t\in[0,T]}$ such that $M_{t\wedge\tau_n}$ is square integrable for each n. Thus, fix some local Brownian martingale $\{(M_t, \mathcal{F}_t)\}_{t\in[0,T]}$ with localization $\{\tau_n\}_n$ such that $\mathbb{E}M^2_{t\wedge\tau_n} < \infty$ for all $t \in [0,T]$ and $n \in \mathbb{N}$. Then by the martingale representation theorem, for each n there exists some progressively measurable $\mathbb{R}^m$-valued process $\psi^{(n)}(t)$, $t \in [0,T]$, with

$$\mathbb{E}\left(\int_0^T \|\psi^{(n)}(t)\|^2\,dt\right) < \infty, \quad M_{t\wedge\tau_n} = M_0 + \int_0^{t\wedge\tau_n} \psi^{(n)}(s)'\,dW(s) \text{ a.s. } \mathbb{P}.$$

Define the progressively measurable $\mathbb{R}^m$-valued process ψ by $\psi(s,\omega) := \psi^{(n)}(s,\omega)$ for $\omega \in \mathcal{F}_t$ with $s \in [0, \tau_n(\omega)]$. Note that for all $0 \leq s \leq \tau_{n-1}(\omega)$,

$$\int_0^s \psi^{(n-1)}(t)'\,dW(t)(\omega) = M_{s\wedge\tau_{n-1}}(\omega) - M_0(\omega) = M_{s\wedge\tau_n}(\omega) - M_0(\omega) = \int_0^s \psi^{(n)}(t)'\,dW(t)(\omega) \quad \text{a.s. } \mathbb{P},$$

and it follows that $\psi^{(n)}(s,\omega) = \psi^{(n-1)}(s,\omega)$ a.s. $\mathbb{P}$. Thus, ψ is well-defined up to some null set, and we can arbitrarily set $\psi(s,\omega) = 0$ for all $s \in [0,T]$ and all ω in this null set. Now for any $s \in [0,T]$, since $\tau_n \xrightarrow{n\to\infty} \infty$ a.s. $\mathbb{P}$, we see that for every $t \in [0,T]$ and for a.e. $\omega \in \mathcal{F}_t$, there exists some n such that $s \leq \tau_n(\omega)$, and so

$$M_s(\omega) = M_{s\wedge\tau_n}(\omega) = M_0(\omega) + \int_0^{s\wedge\tau_n} \psi^{(n)}(h)\,dW(h)(\omega) = M_0(\omega) + \int_0^s \psi(h)\,dW(h)(\omega) \quad \text{a.s. } \mathbb{P}.$$

Finally, by the definition of ψ, we see that for all $t \in [0,T]$ and $\omega \in \mathcal{F}_t$ either there exists some n such that $\int_0^T \|\psi(s)\|^2\,ds(\omega) = \int_0^T \|\psi^{(n)}(s)\|\,ds(\omega) < \infty$, or $\psi(s,\omega) = 0$ for all $s \in [0,T]$ and so $\int_0^T \|\psi(s)\|^2\,ds(\omega) = 0$. □

Chapter 3: Option Pricing

Exercise 1. Under the assumptions of the Black-Scholes model determine the fair prices of the following options given by their payoff diagrams.

(a) Butterfly spread with mean basis price $2K$

Solution. The payoff diagram for the butterfly spread can be replicated by buying two calls on the security, one with strike price K and another with strike price $3K$, and selling two calls with strike price $2K$. Thus, all together we have the time T payoff $B = (P_1(T) - K)^+ - 2(P_1(T) - 2K)^+ + (P_1(T) - 3K)^+$. Applying Corollary 3.15 followed by the Black-Scholes formula, the fair price process $\hat{X}(t)$ for the contingent claim B is therefore given by

$$\begin{aligned}
\hat{X}(t) &= \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)\,ds\right)\cdot B \quad\middle|\quad \mathcal{F}_t\right)\\
&= \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)\,ds\right)\cdot (P_1(T)-K)^+ \quad\middle|\quad \mathcal{F}_t\right) - 2\,\mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)\,ds\right)\cdot (P_1(T)-2K)^+ \quad\middle|\quad \mathcal{F}_t\right)\\
&\quad + \mathbb{E}_Q\left(\exp\left(-\int_t^T r(s)\,ds\right)\cdot (P_1(T)-3K)^+ \quad\middle|\quad \mathcal{F}_t\right)\\
&= P_1(t)\Big(\Phi(d_{1,K}(t)) - 2\Phi(d_{1,2K}(t)) + \Phi(d_{1,3K}(t))\Big) - e^{-r(T-t)}\Big(K\Phi(d_{2,K}(t)) - 4K\Phi(d_{2,2K}(t)) + 3K\Phi(d_{2,3K}(t))\Big).
\end{aligned}$$

(b) Straddle with basis price K

Solution. The payoff diagram for the straddle can be replicated by buying a put and a call, both with strike price K. Thus, the time T payoff is given by $B = (P_1(T) - K)^+ + (K - P_1(T))^+$. Again applying Corollary 3.15 and the Black-Scholes formula, we get that the price process $\hat{X}(t)$ for the contingent claim B is given by

$$\begin{aligned}
\hat{X}(t) &= X_C(t) + X_P(t)\\
&= P_1(t)\Big(\Phi(d_1(t)) - \Phi(-d_1(t))\Big) - K\cdot e^{-r(T-t)}\Big(\Phi(d_2(t)) - \Phi(-d_2(t))\Big)\\
&= P_1(t)\mathrm{sgn}(d_1(t))(2\Phi(|d_1(t)|) - 1) - Ke^{-r(T-t)}\mathrm{sgn}(d_2(t))(2\Phi(|d_2(t)|) - 1).
\end{aligned}$$

(c) Strangle with basis prices $K_1 < K_2$

Solution. The payoff diagram for the strangle can be replicated by buying a put with strike price K_1 and a call with strike price K_2. Thus, we have that the price process $\hat{X}(t)$ is given by

$$\begin{aligned}
\hat{X}(t) &= X_{C,K_1}(t) + X_{P,K_2}(t)\\
&= P_1(t)\Big(\Phi(d_{1,K_1}(t)) - \Phi(-d_{1,K_2}(t))\Big) - K_1\cdot e^{-r(T-t)}\Phi(d_{2,K_1}(t)) + K_2\cdot e^{-r(T-t)}\Phi(-d_{2,K_2}(t)).
\end{aligned}$$

(d) Bull spread with basis prices $K_1 < K_2$

Solution. The payoff diagram is replicated by buying a call with strike price K_1 and selling a call with strike price K_2, resulting in the price process

$$\begin{aligned}
\hat{X}(t) &= X_{C,K_1}(t) - X_{C,K_2}(t)\\
&= P_1(t)\Big(\Phi(d_{1,K_1}(t)) - \Phi(d_{1,K_2}(t))\Big) - K_1\cdot e^{-r(T-t)}\Phi(d_{2,K_1}(t)) + K_2\cdot e^{-r(T-t)}\Phi(d_{2,K_2}(t)).
\end{aligned}$$

Exercise 2. Show that in the Black-Scholes setting the price $X_C(t)$ of a European call satisfies:

(a) $X_C(t)$ decreases in t

Proof. Writing the one-dimensional Black-Scholes call price process given time t and security price p as

$$f(t,p) = p\cdot\Phi(d_1(t)) - K\cdot e^{-r(T-t)}\Phi(d_2(t)),$$

the task is to prove that $f_t < 0$. We may assume that $r \geq 0$. Using the identities: $d_2(t) = d_1(t) - \sigma\sqrt{T-t}$ and $P_1(t)\varphi(d_1(t)) = Ke^{-r(T-t)}\varphi(d_2(t))$, where φ is defined to be the density function of the standard normal distribution, we have that

$$\begin{aligned}
f_t(t,p) &= p\varphi(d_1(t))\frac{\partial d_1}{\partial t}(t) - rKe^{-r(T-t)}\Phi(d_2(t)) - Ke^{-r(T-t)}\varphi(d_2(t))\frac{\partial d_2}{\partial t}(t) \\
&= p\varphi(d_1(t))\frac{\partial d_1}{\partial t}(t) - rKe^{-r(T-t)}\Phi(d_2(t)) - p\varphi(d_1(t))\left(\frac{\partial d_1}{\partial t}(t) + \frac{\sigma}{2\sqrt{T-t}}\right) \\
&= -p\varphi(d_1(t))\frac{\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi(d_2(t)) \\
&< 0.
\end{aligned}$$

□

(b) $X_C(t)$ increases in r

Proof. Observe that

$$\begin{aligned}
\frac{\partial X_C(t)}{\partial r} &= p\varphi(d_1(t))\frac{\partial d_1}{\partial r}(t) + (T-t)Ke^{-r(T-t)}\Phi(d_2(t)) - Ke^{-r(T-t)}\varphi(d_2(t))\frac{\partial d_2}{\partial r}(t) \\
&= p\varphi(d_1(t))\frac{\partial d_1}{\partial r}(t) + (T-t)Ke^{-r(T-t)}\Phi(d_2(t)) - p\varphi(d_1(t))\frac{\partial d_1}{\partial r}(t) \\
&= (T-t)Ke^{-r(T-t)}\Phi(d_2(t)) \\
&\geq 0.
\end{aligned}$$

□

(c) $X_C(t)$ increases in $P_1(t)$

Proof. Observe that

$$\begin{aligned}
\frac{\partial X_C(t)}{\partial p} &= \Phi(d_1(t)) + p\varphi(d_1(t))\frac{\partial d_1}{\partial p}(t) - Ke^{-r(T-t)}\varphi(d_2(t))\frac{\partial d_2}{\partial p}(t) \\
&= \Phi(d_1(t)) + p\varphi(d_1(t))\frac{\partial d_1}{\partial p}(t) - p\varphi(d_1(t))\frac{\partial d_1}{\partial p}(t) \\
&= \Phi(d_1(t)) \\
&> 0.
\end{aligned}$$

□

(d) $X_C(t)$ increases in σ for $\sigma > 0$

Proof. Observe that

$$\begin{aligned}
\frac{\partial X_C(t)}{\partial \sigma} &= p\varphi(d_1(t))\frac{\partial d_1}{\partial \sigma}(t) - Ke^{-r(T-t)}\varphi(d_2(t))\frac{\partial d_2}{\partial \sigma}(t) \\
&= p\varphi(d_1(t))\frac{\partial d_1}{\partial \sigma}(t) - p\varphi(d_1(t))\left(\frac{\partial d_1}{\partial \sigma}(t) - \sqrt{T-t}\right) \\
&= p\varphi(d_1(t))\sqrt{T-t} \\
&\geq 0.
\end{aligned}$$

□

Exercise 3. Compute the price of a European call with the help of the equivalent martingale measure in a market model with $d = 2$, $\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22}\end{pmatrix}$, where the call is a call on the first stock, i.e. the final payment B is given by $B = (P_1(T) - K)^+$

Solution. Observe that in $(\Omega, \mathcal{F}_T, Q)$ we have

$$dP_1(t) = P_1(t) \cdot \left(r(t)\, dt + \sigma_{11}\, dW_1^Q(t) + \sigma_{12}\, dW_2^Q(t) \right).$$

Thus, by the Variation of Constants Theorem, $P_1(t) = P_1(0) \cdot \exp\left(\int_0^t r(s)\, ds - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)t + \sigma_{11} W_1^Q(t) + \sigma_{12} W_2^Q(t) \right)$. By Corollary 3.15, the fair price of the contigent claim B is given by

$$\begin{aligned}
\hat{p} &= \mathbb{E}_Q \left(\exp\left(-\int_0^T r(s)\, ds \right) (P_1(T) - K)^+ \right) \\
&= \mathbb{E}_Q \left(\exp\left(-\int_0^T r(s)\, ds \right) \left(P_1(0) \cdot \exp\left(\int_0^T r(s)\, ds - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T + \sigma_{11} W_1^Q(T) + \sigma_{12} W_2^Q(T) \right) - K \right)^+ \right) \\
&= \mathbb{E}_Q \left(\frac{(P_1(0) p_T e^{-\frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T + \sigma_{11} W_1^Q(T) + \sigma_{12} W_2^Q(T)} - K)^+}{p_T} \right),
\end{aligned}$$

where we define $p_T := \exp\left(\int_0^T r(s)\, ds \right)$. Define $Z := \sigma_{11} W_1^Q(T) + \sigma_{12} W_2^Q(T)$ and observe that since $W_1^Q(T)$ and $W_2^Q(T)$ are normal i.i.d. with respect to Q, $Z \sim \mathcal{N}_Q(0, (\sigma_{11}^2 + \sigma_{12}^2)T)$. Moreover,

$$P_1(0) p_T e^{-\frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T + Z} - K > 0$$

if and only if

$$Z > \ln\left(\frac{K}{P_1(0) p_T} \right) + \frac{1}{2}\left(\sigma_{11}^2 + \sigma_{12}^2 \right) T =: \hat{K}.$$

Thus, we have that

$$\begin{aligned}
\hat{p} &= \mathbb{E}_Q \left(\frac{(P_1(0) p_T e^{-\frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2) + Z} - K)^+}{p_T} \right) \\
&= \int_{\hat{K}}^\infty \frac{1}{\sqrt{2\pi(\sigma_{11}^2 + \sigma_{12}^2)T}} P_1(0) e^{-\frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T + z - \frac{z^2}{2(\sigma_{11}^2 + \sigma_{12}^2)T}}\, dz - \frac{1}{p_T} K \left(\Phi\left(\frac{-\hat{K}}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} \right) \right) \\
&= P_1(0) \int_{\hat{K}}^\infty \frac{1}{\sqrt{2\pi(\sigma_{11}^2 + \sigma_{12}^2)T}} e^{-\frac{(z - (\sigma_{11}^2 + \sigma_{12}^2)T)^2}{2(\sigma_{11}^2 + \sigma_{12}^2)T}}\, dz - \frac{1}{p_T} K \left(\Phi\left(\frac{-\hat{K}}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} \right) \right) \\
&= P_1(0) \Phi\left(\frac{-\hat{K}}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} + \sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T} \right) - \frac{1}{p_T} K \Phi\left(\frac{\ln\left(\frac{P_1(0) p_T}{K} \right) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} \right) \\
&= P_1(0) \Phi\left(\frac{\ln\left(\frac{P_1(0) p_T}{K} \right) + \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} \right) - \frac{1}{p_T} K \Phi\left(\frac{\ln\left(\frac{P_1(0) p_T}{K} \right) - \frac{1}{2}(\sigma_{11}^2 + \sigma_{12}^2)T}{\sqrt{(\sigma_{11}^2 + \sigma_{12}^2)T}} \right).
\end{aligned}$$

Exercise 4. Let

$$\varphi(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t} \right).$$

(a) Show that $\varphi(t, x)$ is a solution of the partial differential equation

$$u_t = \frac{1}{2} u_{xx}.$$

Proof. Observe that

$$\begin{aligned}
\frac{1}{2} \varphi_{xx} &= -\frac{\partial}{\partial x} \left(\frac{x}{2\sqrt{2\pi}} t^{-3/2} \exp\left(-\frac{x^2}{2t} \right) \right) \\
&= \left(\frac{x^2}{\sqrt{2\pi}} t^{-5/2} - \frac{1}{2\sqrt{2\pi}} t^{-3/2} \right) \exp\left(-\frac{x^2}{2t} \right) \\
&= \varphi_t.
\end{aligned}$$

□

(b) Show that the problem

$$u_t(t,x) = u_{xx}(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$
$$u(0,x) = g(x), \quad x \in \mathbb{R},$$

with a bounded function g is solved by

$$u(t,x) = \mathbb{E}\left(g\left(\sqrt{2t} \cdot Y + x\right)\right)$$

for some random variable $Y \sim \mathcal{N}(0,1)$.

Proof. Clearly $u(0,x) = \mathbb{E}[g(x)] = g(x)$. Observe that $Z := \sqrt{2t} \cdot Y + x \sim \mathcal{N}(x, 2t)$ and so, since g is bounded, we can apply Dominated Convergence twice to get that

$$\begin{aligned} u_t(t,x) &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} g(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(z-x)^2}{4t}} \, dz \\ &= 2 \int_{-\infty}^{\infty} g(z) \varphi_t(2t, z-x) \, dz \\ &= 2 \int_{-\infty}^{\infty} g(z) \left(\frac{1}{2} \varphi_{xx}(2t, z-x)\right) dz \\ &= \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} g(z) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(z-x)^2}{4t}} \, dz \\ &= u_{xx}(t,x). \end{aligned}$$

□

Exercise 5. Prove Proposition 3.28, part (2): The price $P_A(t, P_1(t))$ of an American put with strike $K \geq 0$ satisfies

$$(K - P_1(t))^+ \leq P_A(t, P_1(t)) \leq K.$$

Proof. In the case $P_A(t, P_1(t)) > K$, the strategy "sell the option at time t and immediately invest the proceeds at the riskless rate r" is an arbitrage opportunity: If the buyer of the option exercises the option at some point $s \in [t, T]$, the time T value of the strategy is $P_A(t, P_1(t))e^{r(s-t)} + (P_A(t, P_1(t)) + P_1(t) - K)e^{r(T-s)} > 0$, and if the buyer of the option never exercises the option, the time T value of the strategy is $P_A(t, P_1(t))e^{r(T-t)} > 0$.

In the case $(K - P_1(t))^+ > P_A(t, P_1(t))$, the strategy "buy the option and immediately exercise it" yields a riskless time t gain $K - P_1(t) - P_A(t, P_1(t)) > 0$ and incurs no further costs, which is impossible in an arbitrage free market. The desired inequality follows by the principle of no-arbitrage. □

Exercise 6. Prove Proposition 3.29, part (2): For the price $P_E(t, P_1(t))$ of a European put with strike price $K \geq 0$ and exercise date T, we have

$$\left(e^{-r(T-t)}K - P_1(t)\right)^+ \leq P_E(t, P_1(t)) \leq K,$$

if there will be no dividend payments on the stock in $[0, T]$.

Proof. Observe that $P_E(t, P_1(t)) \leq P_A(t, P_1(t)) \leq K$, proving the right hand inequality. Now suppose that

$$\left(e^{-r(T-t)}K - P_1(t)\right)^+ > P_E(t, P_1(t)).$$

I claim that the follows strategy constitutes an arbitrage strategy: "Take a loan of value $e^{-r(T-t)}K$ at the riskless rate r, buy the put for $P_E(t, P_1(t))$ and one unit of stock for $P_1(t)$, and invest the positive rest $e^{-r(T-t)}K - P_1(t) - P_E(t, P_1(t))$ at the riskless rate r". The riskless investment leads to a capital of $K - e^{r(T-t)}(P_1(t) + P_E(t, P_1(t)))$ at $t = T$.

If $P_1(T) < K$, the option buyer exercises the put, selling their one unit of stock for the strike price K and uses this money to close out their loan, realizing a gain of $K - e^{r(T-t)}(P_1(t) + P_E(t, P_1(t))) > 0$.

If instead $P_1(T) \geq K$, the option buyer sells their one unit of stock and closes out their loan, realizing a gain of

$$(P_1(T) - K) + (K - e^{r(T-t)}(P_1(t) + P_E(t, P_1(t)))) > K - e^{r(T-t)}(P_1(t) + P_E(t, P_1(t))) > 0.$$

Since both cases result in strictly positive gains without any initial capital, the no-arbitrage principle implies the desired inequality. □

Exercise 7. Prove Proposition 3.44: All martingale measures Q for $P_0(t), \dots, P_d(t)$ which are equivalent to P can be obtained by a Girsanov transformation with an m-dimensional progressively measurable stochastic process $\{(\theta(t), \mathcal{F}_t)\}_{t\in[0,T]}$ where for all $t \in [0,T]$ we have

$$\int_0^t \theta_i^2(s)\, ds < \infty \text{ a.s. } \mathbb{P}, \text{ for } i = 1, \dots, m$$

and where $Z(t,\theta)$, defined as in Excursion 5, p. 93, is martingale with respect to P. In particular, Q is given as

$$Q(A) := Q_T(A) := \mathbb{E}(1_A \cdot Z(T,\theta)) \text{ for all } A \in \mathcal{F}_T.$$

Proof. Fix a martingale measure space $(\Omega, \mathcal{F}_T, Q)$ for $P_0(t), \dots, P_d(t)$ such that Q is equivalent to P. Observe that since Q and P are equivalent on $\mathcal{F}_T$, they must also be equivalent measures on $\mathcal{F}_t$ for all $t \in [0,T]$. For each $t \in [0,T]$, define D_t to be the Radon-Nikodym derivative $\frac{dQ|_{\mathcal{F}_t}}{dP|_{\mathcal{F}_t}}$. For any $t \in [0,T]$ and $A \in \mathcal{F}_t$

$$\begin{aligned}
\mathbb{E}\Big(\mathbb{E}[D_T \cdot 1_A \mid \mathcal{F}_t]\Big) &= \int_A D_T\, dP \\
&= Q(A) \\
&= Q|_{\mathcal{F}_t}(A) \\
&= \int_A D_t\, dP|_{\mathcal{F}_t} \\
&= \int_A D_t\, dP \\
&= \mathbb{E}[D_t \cdot 1_A].
\end{aligned}$$

It follows that for all $t \in [0,T]$, $\mathbb{E}[D_T \mid \mathcal{F}_t] = D_t$, showing that $\{(D_t, \mathcal{F}_t)\}_{t\in[0,T]}$ satisfies the martingale property. Moreover, since $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are equivalent measures, it follows that $D_t > 0$ a.s. P and so $\mathbb{E}|D_t| = Q(\Omega) = 1 < \infty$ for all $t \in [0,T]$. Thus, $\{D_t\}_{t\in[0,T]}$ is a P-Brownian martingale and we can apply Corollary 2.70 to the Martingale Representation Theorem to get that there exists an m-dimensional progressively measurable process $\{(\Psi(t), \mathcal{F}_t)\}_{t\geq 0}$, $t \in [0,T]$, with

$$\int_0^T \|\Psi(t)\|^2\, dt < \infty$$

and

$$D_t = D_0 + \int_0^t \Psi(s)'\, dW(s) \quad \text{a.s. } P.$$

I claim that $D_0 = 1$. Observe that the statement $D_0 = 1$ is equivalent to the statement that $P|_{\mathcal{F}_0} = Q|_{\mathcal{F}_0}$, and so to prove the statement, it suffices to verify that $P(A) = Q(A)$ for all $A \in \mathcal{F}_0$. Since P and Q are equivalent measures, this statement holds for all P-null sets. Fix some $A \in \mathcal{F}_0$ such that $P(A) \neq 0$. Because $\mathcal{F}_0$ is defined to be the completion of $\sigma\{W(0)\}$, and $W(0)$ is constant a.s. P, it follows that for all $B \in \mathcal{F}_0$, $P(B) \in \{0,1\}$, and so $P(A) = 1$. Thus, $P(A^c) = 0 = Q(A^c)$, which implies that $Q(A) = 1 = P(A)$, and the claim follows. Hence, for all $t \in [0,T]$

$$D_t = 1 + \int_0^t \Psi(s)'\, dW(s) \quad \text{a.s. } P.$$

Since $Q|_{\mathcal{F}_t}$ and $P|_{\mathcal{F}_t}$ are equivalent measures, $D_t > 0$ a.s. P for all $t \in [0,T]$. Define $\theta(t) := -\frac{\Psi(t)}{D_t}$, so that $D_t = 1 - \int_0^t D_s \cdot \theta(s)'\, dW(s)$. Clearly $\{(\theta(t), \mathcal{F}_t)\}_{t\in[0,T]}$ is an m-dimensional progressively measurable stochastic process. Moreover, if we can show that $\int_0^t \|\theta(s)\|^2\, ds < \infty$ a.s. P for all $t \in [0,T]$, it will follow by the Variation of Constants Theorem that $D_t = \exp\Big(-\sum_{j=1}^m \int_0^t \theta_j(s)\, dW_j(s) - \int_0^t \|\theta(s)\|^2\, ds\Big) = Z(t,\theta)$, so that $Z(t,\theta)$ is a P-martingale and $Q(A) = \mathbb{E}[1_A \cdot D_T] = \mathbb{E}[1_A \cdot Z(T,\theta)]$ for all $A \in \mathcal{F}_T$.

Need to prove: $\int_0^t \|\theta(s)\|^2\, ds < \infty$ a.s. P for all $t \in [0,T]$. □

Exercise 8. Show: With the notations and assumptions of Section 3.6 we have the following equivalence for a trading strategy $\varphi(t)$:

$$\begin{aligned}
&\varphi(t) \text{ is self-financing} \iff \\
&\qquad \hat{X}(t) = \frac{x}{p_0} + \sum_{i=1}^d \int_0^t \varphi_i(s)\, d\hat{P}_i(s) \text{ a.s. P for all } t \in [0,T].
\end{aligned}$$

Proof. We have that

$$dP_i(t) = P_i(t)\left(b_i(t)\,dt + \sum_{j=1}^{m}\sigma_{ij}(t)\,dW_j(t)\right),$$

and, by the Itô formula

$$\frac{1}{P_0(t)} = \frac{1}{p_0} - \int_0^t \frac{1}{P_0(s)^2} r(s) ds.$$

Hence, by the product rule,

$$\begin{aligned} d\hat{P}_i &= dP_i(t)\frac{1}{P_0(t)} + P_i(t) d\left(\frac{1}{P_0(t)}\right) + \left\langle P_i\,, \frac{1}{P_0}\right\rangle_t dt \\ &= \hat{P}_i(t)\left(b_i(t)\,dt + \sum_{j=1}^{m}\sigma_{ij}(t)\,dW_j(t)\right) - \frac{\hat{P}_i(t)}{P_0(t)} r(t)\,dt. \end{aligned}$$

Suppose that $\varphi(t)$ is a self-financing trading strategy. Then by definition, the wealth process $X(t)$ corresponding to $\varphi(t)$ satisfies

$$X(t) = x + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\,dP_i(s) \text{ a.s. P for all } t \in [0,T].$$

Another application of the product rule gives that for all $t \in [0,T]$

$$\begin{aligned} \hat{X}(t) &= \frac{x}{p_0} + \int_0^t X(s)\,d\frac{1}{P_0(s)} + \int_0^t \frac{dX(s)}{P_0(s)} + \int_0^t \left\langle X\,, \frac{1}{P_0}\right\rangle_s ds \\ &= \frac{x}{p_0} - \sum_{i=1}^{d}\int_0^t \varphi_i(s)\frac{\hat{P}_i(s)}{P_0(s)} r(s)\,ds + \sum_{i=1}^{d}\left(\int_0^t \hat{P}_i(s)\varphi_i(s)b_i(s)\,ds + \sum_{j=1}^{m}\int_0^t \hat{P}_i(s)\varphi_i(s)\sigma_{ij}(s)\,dW_j(s)\right) \\ &= \frac{x}{p_0} + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\,d\hat{P}_i(s) \quad \text{a.s. P}. \end{aligned}$$

For the other direction, suppose that $\varphi(t)$ is a trading strategy such that

$$\hat{X}(t) = \frac{x}{p_0} + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\,d\hat{P}_i(s) \text{ a.s. P for all } t \in [0,T].$$

Then $X(t) = \hat{X}(t)P_0(t)$, and so by the product rule, we have that for all $t \in [0,T]$

$$\begin{aligned} X(t) &= \frac{x}{p_0}p_0 + \int_0^t P_0(s)\,d\hat{X}(s) + \int_0^t \hat{X}(s)\,dP_0(s) + \int_0^t \langle X\,, P_0\rangle_s\,ds \\ &= x + \sum_{i=1}^{d}\int_0^t P_0(s)\varphi_i(s)\,d\hat{P}_i(s) + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\hat{P}_i(s) r(s)\,ds \\ &= x + \sum_{i=1}^{d}\left(\int_0^t \varphi_i(s)(P_i(s)b_i(s) - \hat{P}_i(s)r(s))\,ds + \sum_{j=1}^{m}\int_0^t P_i(s)\varphi_i(s)\sigma_{ij}(s)\,dW_j(s)\right) + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\hat{P}_i(s)r(s)\,ds \\ &= x + \sum_{i=1}^{d}\int_0^t \varphi_i(s)\,dP_i(s) \quad \text{a.s. P}. \end{aligned}$$

It follows that $\varphi(t)$ is self-financing. □

Exercise 9. In the case of a two-dimensional Black-Scholes model compute the fair price of the contingent claim with the final payment

$$B = 1_{\{P_1(T) \geq P_2(T)\}}.$$

Solution. By Corollary 3.15, the price process $\hat{X}(t)$ of the contingent claim B satisfies

$$\begin{aligned} \hat{X}(t) &= \mathbb{E}_Q\left(e^{-r(T-t)} \cdot 1_{\{P_1(T) \geq P_2(T)\}} \mid \mathcal{F}_t\right) \\ &= e^{-r(T-t)} Q(P_1(T) \geq P_2(T) \mid P_1(t), P_2(t)). \end{aligned}$$

Observe that $P_1(T) \geq P_2(T)$ if and only if

$$P_1(t) e^{(T-t)\left(r - \frac{1}{2}\sum_{j=1}^2 \sigma_{1j}^2\right) + \sum_{j=1}^2 \sigma_{1j}(W_j^Q(T) - W_j^Q(t))} \geq P_2(t) e^{(T-t)\left(r - \frac{1}{2}\sum_{j=1}^2 \sigma_{2j}^2\right) + \sum_{j=1}^2 \sigma_{2j}(W_j^Q(T) - W_j^Q(t))},$$

if and only if

$$(\sigma_{11} - \sigma_{21})\left(W_1^Q(T) - W_1^Q(t)\right) + (\sigma_{12} - \sigma_{22})\left(W_2^Q(T) - W_2^Q(t)\right) \geq \ln\left(\frac{P_2(t)}{P_1(t)}\right) - \frac{1}{2}(T-t)\left(\sigma_{21}^2 + \sigma_{22}^2 - \sigma_{11}^2 - \sigma_{12}^2\right) =: \hat{K}.$$

Set $Z := (\sigma_{11} - \sigma_{21})\left(W_1^Q(T) - W_1^Q(t)\right) + (\sigma_{12} - \sigma_{22})\left(W_2^Q(T) - W_2^Q(t)\right)$. As $(\sigma_{11} - \sigma_{21})\left(W_1^Q(T) - W_1^Q(t)\right)$ and $(\sigma_{12} - \sigma_{22})\left(W_2^Q(T) - W_2^Q(t)\right)$ are independent normally distributed random variables with zero mean and variances $(\sigma_{11} - \sigma_{21})^2(T-t)$ and $(\sigma_{12} - \sigma_{22})^2(T-t)$, respectively, it follows that $Z \sim \mathcal{N}\Big(0, (T-t)\big((\sigma_{12} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{21})^2\big)\Big)$. Thus,

$$\begin{aligned}\hat{X}(t) &= e^{-r(T-t)} \int_{\hat{K}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)((\sigma_{12} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{21})^2)}} \exp\left(-\frac{x^2}{2(T-t)((\sigma_{12} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{21})^2)}\right) dx \\ &= e^{-r(T-t)} \Phi\left(\frac{\ln\left(\frac{P_1(t)}{P_2(t)}\right) + \frac{1}{2}(T-t)(\sigma_{21}^2 + \sigma_{22}^2 - \sigma_{11}^2 - \sigma_{12}^2)}{\sqrt{(T-t)((\sigma_{12} - \sigma_{22})^2 + (\sigma_{11} - \sigma_{21})^2)}}\right).\end{aligned}$$

Exercise 10 (Black-Scholes formula with dividend rates)**.** If a stock pays a dividend rate $\delta P_1(t)$ for some $\delta > 0$ per unit of time then its price in the Black-Scholes model is modelled as the solution of

$$\begin{aligned} dP_1(t) &= P_1(t)((b - \delta)\,dt + \sigma\,dW(t)), \\ P_1(t) &= p. \end{aligned}$$

Show that the price $C(t, P_1(t))$ of a European call on this stock with strike K is given by:

$$C(t, P_1(t)) = e^{-\delta(T-t)} P_1(t) \Phi(\delta_1(t)) - e^{-r(T-t)} K \Phi(\delta_2(t)),$$

with

$$\begin{aligned}\delta_1(t) &= \frac{\ln\left(\frac{P_1(t)}{K}\right) + (r - \delta + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ \delta_2(t) &= \delta_1(t) - \sigma\sqrt{T-t}.\end{aligned}$$

Proof. Note that

$$P_1(T) = P_1(t) \cdot \exp\left(\left(r - \delta - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W^Q(T) - W^Q(t))\right).$$

By Corollary 3.15 and the independence of $W^Q(T) - W^Q(t)$ from $\mathcal{F}_t$,

$$\begin{aligned} C(t, P_1(t)) &= \mathbb{E}_Q\left(e^{-r(T-t)}(P_1(T) - K)^+ \mid \mathcal{F}_t\right) \\ &= e^{-r(T-t)} \mathbb{E}_Q\left(\left(P_1(t) \exp\left((T-t)\left(r - \delta - \frac{1}{2}\sigma^2\right) + \sigma(W^Q(T) - W^Q(t))\right) - K\right)^+\right) \\ &= \int_{\hat{K}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \left(P_1(t) e^{(T-t)(-\delta - \frac{1}{2}\sigma^2) + \sigma x} - e^{-r(T-t)} K\right) e^{-\frac{x^2}{2(T-t)}}\,dx \\ &= e^{-\delta(T-t)} P_1(t) \int_{\hat{K}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} \exp\left(-\frac{(x - \sigma(T-t))^2}{2(T-t)}\right) dx - e^{-r(T-t)} K \Phi\left(\frac{\ln\left(\frac{P_1(t)}{K}\right) + \left(r - \delta - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &= e^{-\delta(T-t)} P_1(t) \Phi(d_1(t)) - e^{-r(T-t)} K \Phi(d_2(t)) - P_1(t)(e^{-\delta(T-t)} - 1), \end{aligned}$$

where $\hat{K} := \frac{\ln\left(\frac{K}{P_1(t)}\right) - (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma} \leq W^Q(T) - W^Q(t)$ if and only if $K \leq P_1(T)$. □

Exercise 11 (Garman-Kohlhagen model for currency options)**.** In the Garman-Kohlhagen model the exchange rate $S(t)$ between the domestic and a foreign currency (e.g. Euro/Dollar) in units of the domestic currency is given as the solution of

$$dS(t) = \mu\,dt + \sigma\,dW(t), \quad S(0) = s \text{ for } \mu, \sigma \in \mathbb{R}.$$

Let r_d denote the riskless domestic rate, r_f the foreign riskless rate. Show that under these assumptions the price of a call option with time to maturity $T-t$ and strike K on one unit of foreign currency is given by

$$C(t,S(t)) = \exp(-r_f(T-t))S(t)\Phi(\gamma_1(t)) - K\exp(-r_d(T-t))\Phi(\gamma_2(t))$$

with

$$\gamma_1(t) = \frac{\ln\left(\frac{S(t)}{K}\right) + (r_d - r_f + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$
$$\gamma_2(t) = \gamma_1(t) - \sigma\sqrt{T-t},$$

in the units of the domestic currency.

Proof. Observe that one unit of foreign currency appreciates at the riskless rate of r_f per unit of time in units of the foreign currency, which is worth $r_f S(t)$ per unit of time with respect to the domestic currency. It follows that the exchange rate $S(t)$ may be interpreted as a stock paying a dividend rate $r_f S(t)$ per unit of time with respect to a one-dimensional Black-Scholes model. The conclusion then follows directly by application of Exercise 10. □

Exercise 12. Compute the price of the "asset or nothing" option which is given by

$$B = P_1(T)\cdot 1_{\{P_1(T)\geq K\}}$$

in the one-dimensional Black-Scholes model.

Solution. By Corollary 3.15, the price process $\hat{X}(t)$ for the payout $B = P_1(T)\cdot 1_{\{P_1(T)\geq K\}}$ is given by

$$\begin{aligned}\hat{X}(t) &= e^{-r(T-t)}\mathbb{E}_Q(P_1(T)\cdot 1_{\{P_1(T)\geq K\}} \mid \mathcal{F}_t)\\ &= e^{-r(T-t)}\mathbb{E}_Q\left(P_1(t)e^{(T-t)(r-\frac{1}{2}\sigma^2)+\sigma(W^Q(T)-W^Q(t))}\cdot 1_{\{W^Q(T)-W^Q(t)\geq \hat{K}\}}\right)\\ &= P_1(t)\Phi\left(\frac{\ln\left(\frac{P_1(t)}{K}\right)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

Exercise 13. (a) In the one-dimensional Black-Scholes model compute both the gamma and the delta of a European call and a European put with maturity T and strike K.

Solution. I computed the delta for a European call in Exercise 2(c): $\Delta_{EC}(t) = \Phi(d_1(t))$. I will use the same identities as in Exercise 2, namely: $d_2(t) = d_1(t) - \sigma\sqrt{T-t}$ and $P_1(t)\varphi(d_1(t)) = Ke^{-r(T-t)}\varphi(D_2(t))$, where φ is defined to be the density function of the standard normal distribution. Computing the delta of a European put, we have that

$$\begin{aligned}\Delta_{EP}(t) &= Ke^{-r(T-t)}\varphi(-d_2(t))\frac{\partial(-d_2(t))}{\partial p} - \Phi(-d_1(t)) - P_1(t)\varphi(-d_1(t))\frac{\partial(-d_1(t))}{\partial p}\\ &= Ke^{-r(T-t)}\varphi(d_2(t))\frac{\partial(-d_1(t))}{\partial p} - \Phi(-d_1(t)) - P_1(t)\varphi(-d_1(t))\frac{\partial(d_1(t))}{\partial p}\\ &= -\Phi(-d_1(t)).\end{aligned}$$

Computing the gamma of a European call, we get

$$\begin{aligned}\Gamma_{EC}(t) &= \frac{\partial}{\partial p}\Phi(d_1(t))\\ &= \varphi(d_1(t))\frac{\partial d_1(t)}{\partial p}\\ &= \frac{\varphi(d_1(t))}{P_1(t)\sigma\sqrt{T-t}}.\end{aligned}$$

And finally computing the gamma of a European put, we get

$$\begin{aligned}\Gamma_{EP}(t) &= -\frac{\partial}{\partial p}\Phi(-d_1(t))\\ &= \frac{\varphi(-d_1(t))}{P_1(t)\sigma\sqrt{T-t}}.\end{aligned}$$

(b) Assume that an investor holds one European call with strike K_1 and maturity T_1. Further, he can trade in European puts with maturities T_2, T_3 and strikes of K_2, K_3. In the Black-Scholes model, determine the numbers $\varphi_1(t), \varphi_2(t)$ of the two different puts the investor has to hold such that the portfolio - consisting of the call and the put position - is both delta- and gamma-neutral at time t.

Solution. Using part (a), the requirement that the portfolio is delta-neutral is equivalent to the relation that for all t,

$$\Phi(d_1(t,K_1,T_1)) + \frac{\partial\varphi_1(t)}{\partial p}X_{EP}(t,K_2,T_2) + \frac{\partial\varphi_2(t)}{\partial p}X_{EP}(t,K_3,T_3)$$
$$-\varphi_1(t)\Phi(-d_1(t,K_2,T_2)) - \varphi_2(t)\Phi(-d_1(t,K_3,T_3) = 0,$$

and the requirement that the portfolio is gamma-neutral is equivalent to the relation

$$\frac{\varphi(d_1(t,K_1,T_1))}{P_1(t)\sigma\sqrt{T_1-t}} + \varphi_1(t)\frac{\varphi(-d_1(t,K_2,T_2))}{P_1(t)\sigma\sqrt{T_2-t}} + \varphi_2(t)\frac{\varphi(-d_1(t,K_3,T_3))}{P_1(t)\sigma\sqrt{T_3-t}}$$
$$-\frac{\partial\varphi_1(t)}{\partial p}\Phi(-d_1(t,K_2,T_2)) - \frac{\partial\varphi_2(t)}{\partial p}\Phi(-d_1(t,K_3,T_3))$$
$$+\frac{\partial^2\varphi_1(t)}{\partial p^2}X_{EP}(t,K_2,T_2) + \frac{\partial^2\varphi_2(t)}{\partial p^2}X_{EP}(t,K_3,T_3) = 0.$$

The possible solutions $(\varphi_1(t), \varphi_2(t))$ are then determined by the general solution to the above system of second order linear differential equations.

Exercise 14. In a Black-Scholes market show that the absolute price change of a European call as a function of the price of the underlying stock is smaller than the absolute price change of the underlying itself.

Proof. Let $C(p)$ be the call price for a given price p, holding all else constant. By the mean value theorem, for any $p_1 < p_2$, there exists some $\tilde{p} \in (p_1, p_2)$ satisfying

$$\begin{aligned} C(p_2) - C(p_1) &= C_p(\tilde{p})(p_2 - p_1) \\ &= \Phi(d_1(t,\tilde{p}))(p_2 - p_1) \\ &< p_2 - p_1. \end{aligned}$$

Since $C_p > 0$, it follows that $|C(p_2) - C(p_1)| = C(p_2) - C(p_1) < |p_2 - p_1|$. □

Chapter 4: Pricing of Exotic Options and Numerical Algorithms

Exercise 1. Show that, with the notation of the proof of Proposition 4.1, we have

$$I_1 = P_1(t)\Phi^{(\rho_1)}(g_1(t), h_1(t))$$

$$I_2 = K_1 e^{-r(T_1-t)}\Phi^{(\rho_1)}(g_2(t), h_2(t)).$$

Proof. We have that

$$I_1 := P_1(t)\int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} e^{\sigma x - \frac{1}{2}\sigma^2(T-t)}\Phi(a)\,dx,$$

$$I_2 := \int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} e^{-r(T_1-t)} K_1\Phi(b)\,dx,$$

$$\tilde{w} := \frac{1}{\sigma}\cdot\left(\ln\left(\frac{p^*}{P_1(T)}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right),$$

$$a := \frac{\sigma x + \ln\left(\frac{P_1(t)}{K_1}\right) + (r+\frac{1}{2}\sigma^2)(T_1-T) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T_1-T}},$$

$$b := \frac{\sigma x + \ln\left(\frac{P_1(t)}{K_1}\right) + (r-\frac{1}{2}\sigma^2)(T_1-t)}{\sigma\sqrt{T_1-T}}.$$

Observe that

$$\begin{aligned} I_1 &= P_1(t)\int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-\sigma(T-t))^2}{2(T-t)}}\Phi(a)\,dx \\ &= P_1(t)\int_{\tilde{w}}^{\infty} \varphi_{\sigma(T-t),(T-t)}(x)\cdot\Phi\left(\frac{1}{\sqrt{T_1-T}}x + \beta\right)dx, \end{aligned}$$

where $\beta := \frac{\ln\left(\frac{P_1(t)}{K_1}\right)+(r+\frac{1}{2}\sigma^2)(T_1-T)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T_1-T}}$. Then by Lemma 4.2, $I_1 = P_1(t)\mathbb{P}(X \geq \tilde{w}, Z \leq \beta)$, where

$$(X,Z) \sim \mathcal{N}\left(\begin{pmatrix}\sigma(T-t)\\ -\frac{\sigma(T-t)}{\sqrt{T_1-T}}\end{pmatrix}, \begin{pmatrix}T-t & -\frac{T-t}{\sqrt{T_1-T}}\\ -\frac{T-t}{\sqrt{T_1-T}} & \frac{T_1-t}{T_1-T}\end{pmatrix}\right).$$

Let $Y_1 := -\frac{1}{\sqrt{T-t}}X + \sigma\sqrt{T-t}$ and $Y_2 := \frac{\sqrt{T_1-T}}{\sqrt{T_1-t}}Z + \frac{\sigma(T-t)}{\sqrt{T_1-t}}$, so that

$$(Y_1,Y_2) \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}\right).$$

Then

$$\begin{aligned} I_1 &= P_1(t)\mathbb{P}\left(Y_1 \leq -\frac{\tilde{w}}{\sqrt{T-t}} + \sigma\sqrt{T-t}, Y_2 \leq \frac{\sqrt{T_1-T}}{\sqrt{T_1-t}}\beta + \frac{\sigma(T-t)}{\sqrt{T_1-t}}\right) \\ &= P_1(t)\mathbb{P}(Y_1 \leq g_1(t), Y_2 \leq h_1(t)) \\ &= P_1(t)\Phi^{(\rho_1)}(g_1(t), h_1(t)). \end{aligned}$$

We also have that

$$\begin{aligned} I_2 &= K_1 e^{-r(T_1-t)}\int_{\tilde{w}}^{\infty}\varphi_{0,(T-t)}(x)\Phi\left(\frac{1}{\sqrt{T_1-T}}x + \beta + \sigma\sqrt{T_1-T}\right)dx \\ &= K_1 e^{-r(T_1-t)}\mathbb{P}(X \geq \tilde{w}, Z \leq \beta + \sigma\sqrt{T_1-T}), \end{aligned}$$

with

$$(X,Z) \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}T-t & -\frac{T-t}{\sqrt{T_1-T}}\\ -\frac{T-t}{\sqrt{T_1-T}} & \frac{T_1-t}{T_1-T}\end{pmatrix}\right).$$

Let $Y_3 := -\frac{1}{\sqrt{T-t}}X$ and $Y_4 := \sqrt{\frac{T_1-T}{T_1-t}}Z$, so that

$$(Y_3, Y_4) \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}\right).$$

Then

$$\begin{aligned} I_2 &= K_1 e^{-r(T_1-t)}\mathbb{P}\left(Y_3 \le -\frac{\tilde{w}}{\sqrt{T-t}}, Y_4 \le \sqrt{\frac{T_1-T}{T_1-t}}\beta + \frac{\sigma(T_1-T)}{\sqrt{T_1-t}}\right) \\ &= K_1 e^{-r(T_1-t)}\Phi^{(\rho_1)}(g_2(t), h_2(t)). \end{aligned}$$

□

Exercise 2. Prove Lemma 4.2: If X and Y are independent random variables with

$$X \sim \mathcal{N}(\mu, \sigma^2), \quad Y \sim \mathcal{N}(0,1),$$

then for $\tilde{x}, \alpha, \beta \in \mathbb{R}$, $\alpha > 0$, we have

$$\begin{aligned} \int_{\tilde{x}}^{\infty} \varphi_{\mu,\sigma^2}(x)\cdot\Phi(\alpha x+\beta)\,dx &= \mathbb{P}(X \ge \tilde{x}, Y \le \alpha X + \beta) \\ &= \mathbb{P}(X \ge \tilde{x}, Z \le \beta), \end{aligned}$$

where

$$(X, Z) \sim \mathcal{N}\left(\begin{pmatrix}\mu\\ -\alpha\mu\end{pmatrix}, \begin{pmatrix}\sigma^2 & -\alpha\sigma^2\\ -\alpha\sigma^2 & 1+\alpha^2\sigma^2\end{pmatrix}\right).$$

Here φ_{μ,σ^2} is rhe density function of the normal distribution with mean μ and variance σ^2.

Proof. Observe that

$$\begin{aligned} \int_{\tilde{x}}^{\infty} \varphi_{\mu,\sigma^2}(x)\cdot\Phi(\alpha x+\beta)\,dx &= \int_{-\infty}^{\infty} \varphi_{\mu,\sigma^2}(x)\mathbb{P}(x \ge \tilde{x}, Y \le \alpha x + \beta)\,dx \\ &= \mathbb{E}[\mathbb{P}(X \ge \tilde{x}, Y \le \alpha X + \beta)] \\ &= \mathbb{P}(X \ge \tilde{x}, Y \le \alpha X + \beta) \\ &= \mathbb{P}(X \ge \tilde{x}, Z \le \beta), \end{aligned}$$

where $Z := Y - \alpha X$. Observe that $\mathbb{E}Z = \mathbb{E}(Y-\alpha X) = -\alpha\mu$, and since X and Y are independent,

$$\mathrm{var}(Z) = \mathrm{var}(Y) + \alpha^2\mathrm{var}(X) = 1 + \alpha^2\sigma^2,$$

and

$$\mathrm{cov}(X,Z) = \mathbb{E}[X(Y-\alpha X)] + \alpha\mu^2 = \alpha(\mathbb{E}[X]^2 - \mathbb{E}[X^2]) = -\alpha\sigma^2.$$

Thus,

$$(X, Z) \sim \mathcal{N}\left(\begin{pmatrix}\mu\\ -\alpha\mu\end{pmatrix}, \begin{pmatrix}\sigma^2 & -\alpha\sigma^2\\ -\alpha\sigma^2 & 1+\alpha^2\sigma^2\end{pmatrix}\right),$$

as required. □

Exercise 3. Compute explicitly the price of the chooser option with maturity T and final payment

$$B_{Ch} = \max\left(X^{\mathrm{Call}}_{T_1,K_1}(P_1(T),T), X^{\mathrm{Put}}_{T_2,K_2}(P_1(T),T)\right).$$

Solution. Let $p^* \ge 0$ be the unique price such that $X^{\mathrm{Call}}_{T_1,K_1}(T,p^*) = X^{\mathrm{Put}}_{T_2,K_2}(T,p^*)$. Observe that, since $X^{\mathrm{Call}}_{T_1,K_1}(T,p)$ strictly increases in p and $X^{\mathrm{Put}}_{T_2,K_2}(T,p)$ strictly decreases in p, $B_{Ch} = X^{\mathrm{Call}}_{T_1,K_1}(T)\cdot 1_{\{P_1(T)\ge p^*\}} + X^{\mathrm{Put}}_{T_2,K_2}(T)\cdot 1_{\{P_1(T)<p^*\}}$. Define

$$\tilde{w} := \frac{1}{\sigma}\cdot\left(\ln\left(\frac{p^*}{P_1(t)}\right) - (r - \frac{1}{2}\sigma^2)(T-t)\right).$$

Let $g_1(t)$, $g_2(t)$, $h_1(t)$ and $h_2(t)$, ρ_1 be defined as in Proposition 4.1 and define

$$h_3(t) := \frac{\ln\left(\frac{P_1(t)}{K_2}\right) + (r + \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}},$$

$$h_4(t) := h_3(t) - \sigma\sqrt{T_2 - t}, \quad \lambda_1 := \sqrt{\frac{T - t}{T_2 - t}}.$$

Then for $t < T$, we have that

$$\begin{aligned}
X_{Ch}(t) &= \mathbb{E}_Q(e^{-r(T-t)} B_{Ch} \mid P_1(t)) \\
&= \mathbb{E}_Q(e^{-r(T-t)} X^{\text{Call}}_{T_1,K_1}(T) \cdot 1_{\{P_1(T) \geq p^*\}} \mid P_1(t)) + \mathbb{E}_Q(e^{-r(T-t)} X^{\text{Put}}_{T_2,K_2}(T) \cdot 1_{\{P_1(T) < p^*\}} \mid P_1(t)) \\
&= \frac{1}{\sqrt{2\pi(T-t)}} \int_{\tilde{w}}^{\infty} e^{-\frac{x^2}{2(T-t)}} e^{-r(T-t)} X^{Call}_{T_1,K_1}(T, P_1(t) \cdot e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma x})\, dx \\
&\qquad + \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\tilde{w}} e^{-\frac{x^2}{2(T-t)}} e^{-r(T-t)} X^{Put}_{T_2,K_2}(T, P_1(t) \cdot e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma x})\, dx \\
&= P_1(t)\Phi^{(\rho_1)}(g_1(t), h_1(t)) - K_1 e^{-r(T_1-t)} \Phi^{(\rho_1)}(g_2(t), h_2(t)) \\
&\qquad + \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\tilde{w}}^{\infty} e^{-\frac{x^2}{2(T-t)}} \left(e^{-r(T_2-t)} K_2 \Phi(a) - P_1(t) e^{-\sigma x - \frac{1}{2}\sigma^2(T-t)} \Phi(b)\right) dx,
\end{aligned}$$

where

$$a := \frac{\ln\left(\frac{K_2}{P_1(t)}\right) - (r - \frac{1}{2}\sigma^2)(T_2 - t) + \sigma x}{\sigma\sqrt{T_2 - T}},$$

$$b := \frac{\ln\left(\frac{K_2}{P_1(t)}\right) - (r + \frac{1}{2}\sigma^2)(T_2 - T) - (r - \frac{1}{2}\sigma^2)(T - t) + \sigma x}{\sigma\sqrt{T_2 - T}}.$$

Applying Lemma 4.2 and through a chain of computations much like in Problem 4.1, we have that

$$\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\tilde{w}}^{\infty} e^{-\frac{x^2}{2(T-t)}} e^{-r(T_2-t)} K_2 \Phi(a)\, dx = K_2 e^{-r(T_2-t)} \Phi^{(\lambda_1)}(-g_2(t), -h_4(t)),$$

$$\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\tilde{w}}^{\infty} P_1(t) e^{-\sigma x - \frac{1}{2}\sigma^2(T-t)} \Phi(b)\, dx = P_1(t)\Phi^{(\lambda_1)}(-g_1(t), -h_3(t)).$$

It follows that

$$X_{Ch}(t) = P_1(t)\Phi^{(\rho_1)}(g_1(t), h_1(t)) - K_1 e^{-r(T_1-t)} - P_1(t)\Phi^{(\lambda_1)}(-g_1(t), -h_3(t)) + K_2 e^{-r(T_2-t)} \Phi^{(\lambda_1)}(-g_2(t), -h_4(t)).$$

Exercise 4. Consider the two-dimensional Black-Scholes model. Let Q_1 be the unique equivalent martingale measure for $P_0(t), P_1(t), P_2(t)$, if $P_1(t)$ is used as the numeraire.

(a) Determine the Radon-Nikodym density of Q_1 with respect to P.

Solution. By Theorem 3.51, the Radon-Nikodym density $Y(T) = \frac{dQ_1}{dP}$ is given by

$$\begin{aligned}
Y(T) &= H(T) \cdot P_1(T) \\
&= P_1(T) \exp\left(-\left(r + \frac{1}{2}\|\sigma^{-1}(b - r\underline{1})\|^2\right)T - (b - r\underline{1})'\sigma'^{-1}W(T)\right) \\
&= \exp\left(\left(b_1 - r - \frac{1}{2}\left(\sigma_{11}^2 + \sigma_{12}^2 + \left(\frac{(b_1-r)\sigma_{22} - (b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\right)^2 + \left(\frac{(b_2-r)\sigma_{11} - (b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}}\right)^2\right)\right)T\right. \\
&\qquad \left. + \left(-\frac{(b_1-r)\sigma_{22} - (b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} + \sigma_{11}\right)W_1(T)\left(-\frac{(b_2-r)\sigma_{11} - (b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} + \sigma_{12}\right)W_2(T)\right) \\
&= \exp\left(-\frac{1}{2}\left(\left(\frac{(b_1-r)\sigma_{22} - (b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} - \sigma_{11}\right)^2 + \left(\frac{(b_2-r)\sigma_{11} - (b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} - \sigma_{12}\right)^2\right)T\right. \\
&\qquad \left. - \left(\frac{(b_1-r)\sigma_{22} - (b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} - \sigma_{11}\right)W_1(T) - \left(\frac{(b_2-r)\sigma_{11} - (b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} - \sigma_{12}\right)W_2(T)\right) \\
&= \exp\left(-\sum_{i=1}^{2}\int_0^T X_i\,dW_i(s) - \frac{1}{2}\int_0^T \|X\|^2\,ds\right) \\
&= Z(T, X),
\end{aligned}$$

where we define $X := \begin{pmatrix} \frac{(b_1-r)\sigma_{22}-(b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{11} \\ \frac{(b_2-r)\sigma_{11}-(b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{12} \end{pmatrix}$.

(b) Show that

$$W^{(1)}(t) = W(t) + \begin{pmatrix} \left(\frac{(b_1-r)\sigma_{22}-(b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{11}\right)t \\ \left(\frac{(b_2-r)\sigma_{11}-(b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{12}\right)t \end{pmatrix}$$

is a Q_1-Brownian motion.

Proof. Observe that by Theorem 3.51, $Z(t, X)$ is a P-martingale. The conclusion then follows by Grisanov's Theorem. □

Exercise 5. Use the notation of Proposition 4.4 and prove the following equalities

(a) $X_{\min}^{Put}(0) = X_{\min}^{Call}(0) + Ke^{-rT} - p_1\Phi(d_3(0)) - p_2\Phi(d_4(0))$

Proof. Observe that

$$\begin{aligned}
B_{\min}^{Put} &= (K - \min(P_1(T), P_2(T)))^+ \\
&= (\min(P_1(T), P_2(T)) - K)^+ + K - \min(P_1(T), P_2(T)) \\
&= B_{\min}^{Call} + K - \min(P_1(T), P_2(T)).
\end{aligned}$$

Thus,

$$\begin{aligned}
X_{\min}^{Put}(0) &= X_{\min}^{Call}(0) + e^{-rT}\mathbb{E}_Q(K - \min(P_1(T), P_2(T))) \\
&= X_{\min}^{Put}(0) + e^{-rT}K - \mathbb{E}_Q(e^{-rT}P_1(T) \cdot 1_{\{P_1(T) \le P_2(T)\}}) - \mathbb{E}_Q(e^{-rT}P_2(T) \cdot_{\{P_1(T) > P_2(T)\}}).
\end{aligned}$$

Define

$$W^{(i)}(t) = W(t) + \begin{pmatrix} \left(\frac{(b_1-r)\sigma_{22}-(b_2-r)\sigma_{12}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{i1}\right)t \\ \left(\frac{(b_2-r)\sigma_{11}-(b_1-r)\sigma_{21}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} - \sigma_{i2}\right)t \end{pmatrix}.$$

Applying Problem 4.4, we have that

$$\begin{aligned}
&\mathbb{E}_Q(e^{-rT}P_i(T)\cdot 1_{\{P_i(T)\le P_{i \pmod 2)+1}(T)\}}) = p_i\mathbb{E}_{Q_i}(1_{\{P_i(T)\le P_{i \pmod 2)+1}(T)\}})\\
&= p_iQ_i(P_i(T)\le P_{i \pmod 2)+1}(T))\\
&= p_iQ_i\left((-1)^{i+1}(\sigma_{11}-\sigma_{21})W_1^{(i)}(T)+(-1)^{i+1}(\sigma_{12}-\sigma_{22})W_2^{(i)}(T)\le \frac{\ln\left(\frac{p_{i \pmod 2)+1}}{p_i}\right)+(-1)^i\frac{1}{2}((\sigma_{11}-\sigma_{21})^2+(\sigma_{12}-\sigma_{22})^2)T}{\sqrt{((\sigma_{11}-\sigma_{21})^2+(\sigma_{12}-\sigma_{22})^2)T}}\right)\\
&= p_iQ_i\left(Z^{(i)}\le \frac{\ln\left(\frac{p_{i \pmod 2)+1}}{p_i}\right)+(-1)^i\frac{1}{2}\sigma^2T}{\sqrt{\sigma^2T}}\right)\\
&= p_i\Phi(d_{i+2}).
\end{aligned}$$

Putting everything together, we have that

$$X^{Put}_{\min}(0) = X^{Call}_{\min}(0) - p_1\Phi(d_3) - p_2\Phi(d_4).$$

□

(b) $X^{Call}_{\max}(0) = X^{Call}_{(1)}(0) + X^{Call}_{(2)}(0) - X^{Call}_{\min}(0)$

Proof. Note that

$$\begin{aligned}
B^{Call}_{\max} + B^{Call}_{\min} &= (\max(P_1(T),P_2(T)) - K)^+ + (\min(P_1(T),P_2(T)) - K)^+\\
&= \max((P_1(T)-K)^+,(P_2(T)-K)^+) + \min((P_1(T)-K)^+,(P_2(T)-K)^+)\\
&= B^{Call}_{(1)} + B^{Call}_{(2)}.
\end{aligned}$$

Thus,

$$X^{Call}_{\max}(0) + X^{Call}_{\min}(0) = X^{Call}_{(1)}(0) + X^{Call}_{(2)}(0).$$

□

(c) $X^{Put}_{\max}(0) = X^{Put}_{(1)}(0) + X^{Put}_{(2)}(0) - X^{Put}_{\min}(0)$

Proof. Note that

$$\begin{aligned}
B^{Put}_{\max} + B^{Put}_{\min} &= (K - \max(P_1(T),P_2(T)))^+ + (K - \min(P_1(T),P_2(T)))^+\\
&= \min((K-P_1(T))^+,(K-P_2(T))^+) + \max((K-P_1(T))^+,(K-P_2(T))^+)\\
&= B^{Put}_{(1)} + B^{Put}_{(2)}.
\end{aligned}$$

Thus,

$$X^{Put}_{\max}(0) + X^{Put}_{\min}(0) = X^{Put}_{(1)}(0) + X^{Put}_{(2)}(0).$$

□

Exercise 6. Do the explicit calculations needed for the determination of the price $X^{Call}_{do}(0)$ of a European down-and-out call.

Solution. The task is to fill in the details of the explicit calculation for the down-and-out call in the section on one-sided barrier options, where it assumed that the barrier $b < p_1$ and $K < b$. Fix $\mu \in \mathbb{R}$ and define $\tilde{W}(t) := W(t) + \mu t$, $\tilde{M}(t) = \min_{0\le s\le t}\tilde{W}(s)$. Observe that since the distributions of $W(t)$ and $-W(t)$ are identical, applying Lemma 4.5 we have that for any $\mu \in \mathbb{R}$ and $x < \min(w,0)$,

$$\begin{aligned}
\mathbb{P}(W(t)+\mu t \ge w, \min_{0\le s\le t}(W(s)+\mu s) > x) &= \mathbb{P}(-W(t)-\mu t \le -w, \max_{0\le s\le t}(-W(s)-\mu s) < -x)\\
&= \Phi\left(\frac{-w+\mu t}{\sqrt{t}}\right) - e^{2\mu x}\Phi\left(\frac{-w+2x+\mu t}{\sqrt{t}}\right).
\end{aligned}$$

It follows that the joint density function $\varphi_{\tilde{W},\tilde{M}}(w,x)$ is given by

$$\begin{aligned}\varphi_{\tilde{W},\tilde{M}}(w,x) &= \frac{\partial^2}{\partial w \partial x}\left(\Phi\left(\frac{-w+\mu t}{\sqrt{t}}\right) - e^{2\mu x}\Phi\left(\frac{-w+2x+\mu t}{\sqrt{t}}\right)\right)1_{\{x\le\min(w,0)\}}\\ &= -\left(2e^{2\mu x}\frac{1}{\sqrt{2\pi t}}\frac{\partial}{\partial w}\exp\left(-\frac{(2x+\mu t-w)^2}{2t}\right) - 2\mu e^{2\mu x}\frac{\partial}{\partial w}\Phi\left(\frac{-w+2x+\mu t}{\sqrt{t}}\right)\right)1_{\{x\le\min(w,0)\}}\\ &= \left(-(2x+\mu t-w)\frac{1}{t}\sqrt{\frac{2}{\pi t}}e^{2\mu x-\frac{(2x+\mu t-w)^2}{2t}} + \sqrt{\frac{2}{\pi t}}\mu e^{2\mu x-\frac{(2x+\mu t-w)^2}{2t}}\right)1_{\{x\le\min(w,0)\}}\\ &= \frac{1}{t}\sqrt{\frac{2}{\pi t}}(w-2x)e^{-\mu^2 t/2+\mu w-(2x-w)^2/(2t)}1_{\{x\le\min(w,0)\}}.\end{aligned}$$

Set $\mu := \frac{r-\frac{1}{2}\sigma^2}{\sigma}$. Since $P_1(T) > K$ if and only if

$$W(T) + \mu T > \frac{1}{\sigma}\ln\left(\frac{K}{p_1}\right) =: \hat{w},$$

and, assuming $\sigma > 0$, $\min_{0\le s\le T} P_1(s) > b$ if and only if

$$\min_{0\le s\le T}(W(s)+\mu s) > \frac{1}{\sigma}\ln\left(\frac{b}{p_1}\right) =: \hat{x},$$

it follows that

$$\begin{aligned}X_{do}^{Call}(0) &= \mathbb{E}_Q(e^{-rT}(P_1(T)-K)^+ \cdot 1_{\{P_1(t)>b\,\forall t\in[0,T]\}})\\ &= \int_{\hat{x}}^{w}\int_{\hat{x}}^{\infty} e^{-rT}(P_1(T)-K)\frac{1}{T}\sqrt{\frac{2}{\pi T}}(w-2x)e^{-\mu^2 T/2+\mu w-(2x-w)^2/(2T)}\,dw\,dx.\end{aligned}$$

Since the computation of this integral is rather long and similar to the computation of $X_{do}^{Put}(0)$, I will just summarize the steps here: Substituting $u = w-2x$, completing the square, using the identity $\int_a^\infty ue^{-(u-m)^2/(2T)}\,du = mT\Phi\left(\frac{m-a}{\sqrt{T}}\right)+T\frac{1}{\sqrt{2T}}e^{-\frac{(a-m)^2}{2T}}$, integrating a number of terms by parts, and completing the square again on the exponents of these terms, we get

$$X_{do}^{Call}(0) = p_1\Phi(d_1) - be^{-rT}\Phi(d_1-\sigma\sqrt{T}) + e^{-rT}(b-K)\Phi(d_1-\sigma\sqrt{T}) - p_1\left(\frac{b}{p_1}\right)^{2\frac{r}{\sigma^2}+1}\Phi(d_2) + e^{-rT}\left(\frac{b}{p_1}\right)^{2\frac{r}{\sigma^2}-1}\Phi(d_2-\sigma\sqrt{T}),$$

where

$$d_1 := \frac{\ln\left(\frac{p_1}{b}\right)+(r+\frac{1}{2}\sigma^2)T}{\sigma T},\quad d2 := \frac{\ln\left(\frac{b}{p_1}\right)+(r+\frac{1}{2}\sigma^2)T}{\sigma T}.$$

Exercise 7. Compute the price $X_{do}^{Put}(0)$ of a European down-and-out put.

Solution. Observe that if $K \le b$, then the option is worthless. Hence, we may assume that $K > b$. Note that $B_{do}^{Put} > 0$ if and only if

$$W(T) + \mu T < \frac{1}{\sigma}\ln\left(\frac{K}{p_1}\right) =: \hat{w},$$

and

$$\min_{0\le s\le T}(W(s)+\mu s) > \frac{1}{\sigma}\ln\left(\frac{b}{p_1}\right) =: \hat{x}.$$

Thus, using the joint density function computed in Problem 6, we find that

$$\begin{aligned}X_{do}^{Put}(0) &= \frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{\hat{x}}^{0}\int_{x}^{\hat{w}}(w-2x)e^{-\mu^2 T/2+\mu w-(2x-w)^2/(2T)}(e^{-rT}K - p_1e^{\sigma w})\,dw\,dx\\ &= \frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{\hat{x}}^{0}\int_{-x}^{\hat{w}-2x} ue^{-\mu^2 T/2+\mu(u+2x)-u^2/(2T)}(e^{-rT}K - p_1e^{\sigma(u+2x)})\,du\,dx\\ &= \frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{\hat{x}}^{0}e^{2\mu x}\int_{-x}^{\hat{w}-2x} ue^{-\frac{(u-\mu T)^2}{2T}}(e^{-rT}K - p_1e^{\sigma(u+2x)})\,du\,dx\end{aligned}$$

Focusing first on the inner integral, we have

$$\int_{-x}^{\hat{w}-2x} ue^{-\frac{(u-\mu T)^2}{2T}} (e^{-rT}K - p_1 e^{\sigma(u+2x)})\, du = e^{-rT}KI_1 - p_1 e^{2\sigma x} I_2,$$

with

$$\begin{aligned} I_1 &= \int_{-x}^{\hat{w}-2x} ue^{-\frac{(u-\mu T)^2}{2T}} du \\ &= -T\left(e^{-\frac{(\hat{w}-2x-\mu T)^2}{2T}} - e^{-\frac{(x+\mu T)^2}{2T}} - \mu T\sqrt{2\pi T}\left(\Phi\left(\frac{\hat{w}-2x-\mu T}{\sqrt{T}}\right) - \Phi\left(\frac{-x-\mu T}{\sqrt{T}}\right)\right)\right) \\ &= -T(J_1 - J_2) + \mu T^2\sqrt{2\pi T}(J_3 - J_4), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_{-x}^{\hat{w}-2x} ue^{-\frac{(u-\mu T)^2}{2T}+\sigma u}\, du \\ &= \int_{\sqrt{(x+\mu T)^2+2\sigma xT}}^{\sqrt{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}} e^{-\frac{y^2}{2T}} (y + (\mu+\sigma)T)\, dy \\ &= -T\left(e^{-\frac{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}{2T}} - e^{-\frac{(x+\mu T)^2+2\sigma xT}{2T}}\right) \\ &\quad + (\mu+\sigma)T\sqrt{2\pi T}\left(\Phi\left(\sqrt{\frac{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}{T}}\right) - \Phi\left(\sqrt{\frac{(x+\mu T)^2+2\sigma xT}{T}}\right)\right) \\ &= -T(J_5 - J_6) + (\mu+\sigma)T\sqrt{2\pi T}(J_7 - J_8). \end{aligned}$$

It follows that,

$$\begin{aligned} X_{do}^{Put}(0) = \frac{1}{T}\sqrt{\frac{2}{\pi T}} \int_{\hat{x}}^{0} e^{2\mu x} (\mu T^2\sqrt{2\pi T}e^{-rT}K(J_3 - J_4) - e^{-rT}KT(J_1 - J_2) \\ + p_1 e^{2\sigma x}T(J_5 - J_6) - p_1 e^{2\sigma x}(\mu+\sigma)T\sqrt{2\pi T}(J_7 - J_8))\, dx. \end{aligned}$$

We compute,

$$\begin{aligned} \int_{\hat{x}}^{0} e^{2\mu x} J_1\, dx &= \int_{\hat{x}}^{0} e^{-\frac{(\hat{w}-2x-\mu T)^2}{2T}+2\mu x}\, dx \\ &= e^{-\frac{(\hat{w}-\mu T)^2+4\hat{w}^2}{2T}} \int_{\hat{x}}^{0} e^{-\frac{2(x+\hat{w})^2}{T}}\, dx \\ &= \sqrt{\frac{\pi T}{2}} e^{-\frac{(\hat{w}-\mu T)^2+4\hat{w}^2}{2T}} \left(\Phi\left(\frac{2}{\sqrt{T}}\hat{w}\right) - \Phi\left(\frac{2(\hat{x}+\hat{w})}{\sqrt{T}}\right)\right). \end{aligned}$$

$$\begin{aligned} \int_{\hat{x}}^{0} e^{2\mu x} J_2\, dx &= \int_{\hat{x}}^{0} e^{-\frac{(x+\mu T)^2}{2T}+2\mu x}\, dx \\ &= \int_{\hat{x}}^{0} e^{-\frac{(x-\mu T)^2}{2T}}\, dx \\ &= \sqrt{2\pi T}\left(\Phi(-\mu\sqrt{T}) - \Phi\left(\frac{\hat{x}-\mu T}{\sqrt{T}}\right)\right). \end{aligned}$$

$$\begin{aligned} \int_{\hat{x}}^{0} e^{2\mu x}\Phi\left(\frac{\hat{w}-2x-\mu T}{\sqrt{T}}\right) dx &= \frac{1}{2\mu}\left(\Phi\left(\frac{\hat{w}-\mu T}{\sqrt{T}}\right) - e^{2\mu\hat{x}}\Phi\left(\frac{\hat{w}-2\hat{x}-\mu T}{\sqrt{T}}\right) - \int_{\hat{x}}^{0} e^{2\mu x - \frac{(\hat{w}-2x-\mu T)^2}{2T}}\, dx\right) \\ &= \frac{1}{2\mu}\left(\Phi\left(\frac{\hat{w}-\mu T}{\sqrt{T}}\right) - e^{2\mu\hat{x}}\Phi\left(\frac{\hat{w}-2\hat{x}-\mu T}{\sqrt{T}}\right)\right. \\ &\qquad \left. - \sqrt{\frac{\pi T}{2}} e^{-\frac{(\hat{w}-\mu T)^2+4\hat{w}^2}{2T}} \left(\Phi\left(\frac{2}{\sqrt{T}}\hat{w}\right) - \Phi\left(\frac{2(\hat{x}+\hat{w})}{\sqrt{T}}\right)\right)\right). \end{aligned}$$

$$
\begin{aligned}
\int_{\hat{x}}^{0} e^{2\mu x} J_4\, dx &= \int_{\hat{x}}^{0} e^{2\mu x} \Phi\left(\frac{-x-\mu T}{\sqrt{T}}\right) dx \\
&= \frac{1}{2\mu}\left(\Phi\left(-\mu\sqrt{T}\right) - e^{2\mu\hat{x}}\Phi\left(\frac{-\hat{x}-\mu T}{\sqrt{T}}\right) - \int_{\hat{x}}^{0} e^{-\frac{(x+\mu T)^2}{2T}+2\mu x}\, dx\right) \\
&= \frac{1}{2\mu}\left(\Phi\left(-\mu\sqrt{T}\right) - e^{2\mu\hat{x}}\Phi\left(\frac{-\hat{x}-\mu T}{\sqrt{T}}\right) - \sqrt{2\pi T}\left(\Phi(-\mu\sqrt{T}) - \Phi\left(\frac{\hat{x}-\mu T}{\sqrt{T}}\right)\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} J_5\, dx &= \int_{\hat{x}}^{0} e^{-\frac{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}{2T}+2(\mu+\sigma)x}\, dx \\
&= e^{-\frac{\hat{w}^2-2\sigma\hat{w}T}{2T}+\frac{(\mu\hat{w}-2\sigma-\mu)^2}{2\mu^2 T}} \int_{\hat{x}}^{0} e^{-\frac{\left(x+\frac{\mu\hat{w}-2\sigma-\mu}{2\mu^2 T}\right)^2}{2\left(\frac{1}{4\mu^2 T}\right)}}\, dx \\
&= \sqrt{\frac{\pi}{2\mu^2 T}} e^{-\frac{\hat{w}^2-2\sigma\hat{w}T}{2T}+\frac{(\mu\hat{w}-2\sigma-\mu)^2}{2\mu^2 T}} \left(\Phi\left(\frac{2\sigma+\mu(1-\hat{w})}{\mu\sqrt{T}}\right) - \Phi\left(\frac{2\mu^2 T\hat{x}+2\sigma+\mu(1-\hat{w})}{\mu\sqrt{T}}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} J_6\, dx &= \int_{\hat{x}}^{0} e^{-\frac{(x+\mu T)^2+2\sigma T x}{2T}+2(\mu+\sigma)x}\, dx \\
&= e^{-\frac{\mu^2 T+(\sigma+\mu)^2 T}{2}} \int_{\hat{x}}^{0} e^{-\frac{(x-\frac{\sigma+\mu}{4T})^2}{2T}}\, dx \\
&= \sqrt{2\pi T} e^{-\frac{\mu^2 T+(\sigma+\mu)^2 T}{2}} \left(\Phi\left(-\frac{\sigma+\mu}{4T^{3/2}}\right) - \Phi\left(\frac{4T\hat{x}-\sigma-\mu}{4T^{3/2}}\right)\right).
\end{aligned}
$$

$$
\begin{aligned}
\int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} J_7\, dx &= \int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} \Phi\left(\sqrt{\frac{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}{T}}\right) dx \\
&= \frac{1}{2(\mu+\sigma)}\Bigg(\Phi\left(\sqrt{\frac{(\hat{w}-\mu T)^2-2\sigma\hat{w}T}{T}}\right) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\sqrt{\frac{(\hat{w}-2\hat{x}-\mu T)^2-2\sigma(\hat{w}-2\hat{x})T}{T}}\right) \\
&\qquad - \int_{\hat{x}}^{0} e^{-\frac{(\hat{w}-2x-\mu T)^2-2\sigma(\hat{w}-2x)T}{2T}+2(\mu+\sigma)x}\, dx\Bigg) \\
&= \frac{1}{2(\mu+\sigma)}\Bigg(\Phi\left(\sqrt{\frac{(\hat{w}-\mu T)^2-2\sigma\hat{w}T}{T}}\right) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\sqrt{\frac{(\hat{w}-2\hat{x}-\mu T)^2-2\sigma(\hat{w}-2\hat{x})T}{T}}\right) \\
&\qquad - \sqrt{\frac{\pi T}{2}} e^{-\frac{(\hat{w}-\mu T)^2-2\sigma\hat{w}T+\hat{w}^2}{2T}} \left(\Phi\left(-\frac{\hat{w}}{\sqrt{T}}\right) - \Phi\left(\frac{2\hat{x}-\hat{w}}{\sqrt{T}}\right)\right)\Bigg).
\end{aligned}
$$

$$
\begin{aligned}
\int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} J_8\, dx &= \int_{\hat{x}}^{0} e^{2(\mu+\sigma)x} \Phi\left(\sqrt{\frac{(x+\mu T)^2+2\sigma x T}{T}}\right) dx \\
&= \frac{1}{2(\mu+\sigma)}\left(\Phi(\mu^2 T) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\frac{(\hat{x}+\mu T)^2+2\sigma\hat{x}T}{T}\right) - \int_{\hat{x}}^{0} e^{-\frac{(x+\mu T)^2+2\sigma x T}{2T}+2(\mu+\sigma)x}\, dx\right) \\
&= \frac{1}{2(\mu+\sigma)}\Bigg(\Phi(\mu^2 T) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\frac{(\hat{x}+\mu T)^2+2\sigma\hat{x}T}{T}\right) \\
&\qquad - \sqrt{2\pi T} e^{\frac{(\sigma^2+2\sigma\mu)T}{2}} \left(\Phi\left(-(\mu+\sigma)\sqrt{T}\right) - \Phi\left(\frac{\hat{x}-(\mu+\sigma)T}{\sqrt{T}}\right)\right)\Bigg).
\end{aligned}
$$

Putting everything together, we have that

$$
\begin{aligned}
X_{do}^{Put}(0) = e^{-rT}TK\Bigg\{ & \Phi\left(\frac{\hat{w}-\mu T}{\sqrt{T}}\right) - e^{2\mu\hat{x}}\Phi\left(\frac{\hat{w}-2\hat{x}-\mu T}{\sqrt{T}}\right) \\
& - \sqrt{\frac{\pi T}{2}} e^{-\frac{(\hat{w}-\mu T)^2+4\hat{w}^2}{2T}} \left(\Phi\left(\frac{2}{\sqrt{T}}\hat{w}\right) - \Phi\left(\frac{2(\hat{x}+\hat{w})}{\sqrt{T}}\right)\right) \\
& - \Phi\left(-\mu\sqrt{T}\right) + e^{2\mu\hat{x}}\Phi\left(\frac{-\hat{x}-\mu T}{\sqrt{T}}\right) - \sqrt{2\pi T}\left(\Phi(-\mu\sqrt{T}) + \Phi\left(\frac{\hat{x}-\mu T}{\sqrt{T}}\right)\right)\Bigg\} \\
- e^{-rT}K\Bigg\{ & e^{-\frac{(\hat{w}-\mu T)^2+4\hat{w}^2}{2T}} \left(\Phi\left(\frac{2}{\sqrt{T}}\hat{w}\right) - \Phi\left(\frac{2(\hat{x}+\hat{w})}{\sqrt{T}}\right)\right) - 2\left(\Phi(-\mu\sqrt{T}) - \Phi\left(\frac{\hat{x}-\mu T}{\sqrt{T}}\right)\right)\Bigg\} \\
+ p_1\Bigg\{ & \frac{1}{\mu T} e^{-\frac{\hat{w}^2-2\sigma\hat{w}T}{2T} + \frac{(\mu\hat{w}-2\sigma-\mu)^2}{2\mu^2 T}} \left(\Phi\left(\frac{2\sigma+\mu(1-\hat{w})}{\mu\sqrt{T}}\right) - \Phi\left(\frac{2\mu^2 T\hat{x}+2\sigma+\mu(1-\hat{w})}{\mu\sqrt{T}}\right)\right) \\
& - 2e^{-\frac{\mu^2 T+(\sigma+\mu)^2 T}{2}} \left(\Phi\left(-\frac{\sigma+\mu}{4T^{3/2}}\right) - \Phi\left(\frac{4T\hat{x}-\sigma-\mu}{4T^{3/2}}\right)\right)\Bigg\} \\
- p_1\Bigg\{ & \Phi\left(\sqrt{\frac{(\hat{w}-\mu T)^2-2\sigma\hat{w}T}{T}}\right) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\sqrt{\frac{(\hat{w}-2\hat{x}-\mu T)^2-2\sigma(\hat{w}-2\hat{x})T}{T}}\right) \\
& - \sqrt{\frac{\pi T}{2}} e^{-\frac{(\hat{w}-\mu T)^2-2\sigma\hat{w}T+\hat{w}^2}{2T}} \left(\Phi\left(-\frac{\hat{w}}{\sqrt{T}}\right) - \Phi\left(\frac{2\hat{x}-\hat{w}}{\sqrt{T}}\right)\right) - \Phi(\mu^2 T) - e^{2(\mu+\sigma)\hat{x}}\Phi\left(\frac{(\hat{x}+\mu T)^2+2\sigma\hat{x}T}{T}\right) \\
& - \sqrt{2\pi T} e^{\frac{(\sigma^2+2\sigma\mu)T}{2}} \left(\Phi\left(-(\mu+\sigma)\sqrt{T}\right) - \Phi\left(\frac{\hat{x}-(\mu+\sigma)T}{\sqrt{T}}\right)\right)\Bigg\}.
\end{aligned}
$$

Exercise 8. (a) Show that the binomial model consisting of a stock and a bond is complete. Compute the corresponding equivalent martingale measure Q_n.

Proof. Fix a binomial model with parameters $0 < d < e^{r\frac{T}{n}} < u, n, q$ (using the same notation as in section 4.3). Towards computing Q_n, let $\hat{q}_i(P_1^{(n)}(i)) := Q_n(P_1^{(n)}(i+1) = uP_1^{(n)}(i) \mid P_1^{(n)}(i))$ and observe that the martingale requirement gives

$$
\begin{aligned}
0 &= \mathbb{E}_{Q_n}\left(\frac{P_1^{(n)}(i)}{P_0(i\frac{T}{n})} - \frac{P_1^{(n)}(i-1)}{P_0((i-1)\frac{T}{n})} \mid \mathcal{F}_{i-1}^{(n)}\right) \\
&= \frac{P_1^{(n)}(i-1)}{P_0((i-1)\frac{T}{n})}\left((\hat{q}_i(P_1^{(n)}(i-1))u + (1-\hat{q}_i(P_1^{(n)}(i-1)))d)e^{-r\frac{T}{n}} - 1\right) \\
\implies \hat{q}_i(P_1^{(n)}(i-1)) &\equiv \frac{e^{r\frac{T}{n}} - d}{u-d} =: \hat{q}.
\end{aligned}
$$

Observe that since $d < e^{r\frac{T}{n}} < u$, it follows that $\hat{q} \in (0,1)$. Hence, $Q_n \sim B(n, \hat{q})$ defines a valid martingale probability measure on the binomial model. Moreover, Q_n is clearly equivalent to P, and has Radon-Nikodym derivative $\left(\frac{\hat{q}}{q}\right)^U \left(\frac{1-\hat{q}}{1-q}\right)^{n-U} = \frac{dQ_n}{dP}$, where U is the defined to be the number of "ups" for a given path.

Fix a contingent claim B in the binomial model. We need to prove that there exists an admissible trading strategy $\varphi(k)$ with corresponding wealth process $X(k)$ such that $B = X(T)$ a.s. P, such that $\hat{X}(k) = X(k)/P_0(k)$ is a martingale with respect to Q_n. Observe that for a given price $P_1^{(n)}(n-1)$, the system of equations

$$
\begin{aligned}
\varphi_1(n-1)uP_1^{(n)}(n-1) + \varphi_0(n-1)e^{r\frac{T}{n}} &= B(uP_1^{(n)}(n-1)) \\
\varphi_1(n-1)dP_1^{(n)}(n-1) + \varphi_0(n-1)e^{r\frac{T}{n}} &= B(dP_1^{(n)}(n-1))
\end{aligned}
$$

has the unique solution given by

$$\varphi_1(n-1) = \frac{B(uP_1^{(n)}(n-1)) - B(dP_1^{(n)}(n-1))}{P_1^{(n)}(n-1)(u-d)}$$
$$\varphi_0(n-1) = e^{-r\frac{T}{n}}\frac{dB(uP_1^{(n)}(n-1)) - uB(dP_1^{(n)}(n-1))}{d-u}.$$

Thus, since by Theorem 3.45 there exist no arbitrage opportunities in our model, the price of the option at time $n-1$ must be given by $0 \le X(n-1, P_1^{(n)}(n-1)) = P_1^{(n)}(n-1)\varphi_1(n-1) + \varphi_0(n-1)$. Now suppose that the option has been priced at time $n-k$ for some $k \in \{1, \dots, n-1\}$ by replicating the price $X(n-k+1, P_1^{(n)}(n-k+1))$ via a time $n-k$ strategy $\varphi_1(n-k), \varphi_0(n-k)$. Observe that the system of equations

$$\varphi_1(n-k-1)uP_1^{(n)}(n-k-1) + \varphi_1(n-k-1)e^{r\frac{T}{n}} = X(n-k, uP_1^{(n)}(n-k-1))$$
$$\varphi_1(n-k-1)dP_1^{(n)}(n-k-1) + \varphi_1(n-k-1)e^{r\frac{T}{n}} = X(n-k, dP_1^{(n)}(n-k-1))$$

has the unique solution given by

$$\varphi_1(n-k-1) = \frac{X(n-k, uP_1^{(n)}(n-k-1)) - X(n-k, dP_1^{(n)}(n-k-1))}{P_1^{(n)}(n-k-1)(u-d)}$$
$$\varphi_0(n-k-1) = e^{-r\frac{T}{n}}\frac{dX(n-k, uP_1^{(n)}(n-k-1)) - uX(n-k, dP_1^{(n)}(n-k-1))}{d-u}.$$

Again, due to the lack of arbitrage opportunities, the time $n-k-1$ price of B must be given by $0 \le X(n-k-1, P_1^{(n)}(n-k-1)) = \varphi_1(n-k-1)P_1^{(n)}(n-k-1) + \varphi_0(n-k-1)$. Thus, we inductively obtain a unique trading strategy $\varphi(i)$ whose wealth process has the property that $X(n) = B$ and $X(i) \ge 0$ for all $i \in \{0, \dots, n\}$. Moreover, due to the equation defining $\varphi(n-k-1)$ above, we see that

$$\begin{aligned}\varphi_1(n-k-1)P_1^{(n)}(n-k) + \varphi_1(n-k-1)e^{r\frac{T}{n}} &= X(n-k, P_1^{(n)}(n-k)) \\ &= \varphi_1(n-k)P_1^{(n)}(n-k) + \varphi_0(n-k),\end{aligned}$$

and so φ is admissible. Finally, towards verifying that $\hat{X}$ is a Q_n-martingale, observe that

$$\begin{aligned}\mathbb{E}_{Q_n}(\hat{X}(k) \mid \mathcal{F}_{k-1}^{(n)}) &= \mathbb{E}_{Q_n}(\hat{P}_1^{(n)}(k)\varphi_n(k-1) + \varphi_0(k-1)e^{r\frac{T}{n}(1-k)} \mid \mathcal{F}_{k-1}^{(n)}) \\ &= \hat{P}_1^{(n)}(k-1)\varphi_n(k-1) + \varphi_0(k-1)e^{-r\frac{T}{n}(k-1)} \\ &= \hat{X}(k-1).\end{aligned}$$

□

(b) Show that the price of an option B in the binomial model is given as $\mathbb{E}_{Q_n}(e^{-rT}B)$.

Proof. Observe that

$$\begin{aligned}\mathbb{E}_{Q_n}(e^{-r\frac{nT-(n-1)T}{n}}B \mid P_1^{(n)}(n-1)) &= \hat{q}e^{-r\frac{T}{n}}B(uP_1^{(n)}(n-1)) + (1-\hat{q})e^{-r\frac{T}{n}}B(dP_1^{(n)}(n-1)) \\ &= \frac{1-de^{-r\frac{T}{n}}}{u-d}B(uP_1^{(n)}(n-1)) + \frac{ue^{-r\frac{T}{n}}-1}{u-d}B(dP_1^{(n)}(n-1)) \\ &= \varphi_1(n-1)P_1^{(n)}(n-1) + \varphi_0(n-1) \\ &= X(n-1).\end{aligned}$$

Now suppose that $\mathbb{E}_{Q_n}(e^{-r\frac{nT-(n-k)T}{n}}B \mid \mathcal{F}_{n-k}^{(n)}) = X(n-k)$ for some $k \in \{1, \dots, n-1\}$ and observe that

$$\begin{aligned}\mathbb{E}_{Q_n}(e^{-r\frac{nT-(n-k-1)T}{n}}B \mid \mathcal{F}_{n-k-1}^{(n)}) &= \mathbb{E}_{Q_n}(e^{-r\frac{T}{n}}\mathbb{E}_{Q_n}(e^{-r\frac{nT-(n-k)T}{n}}B \mid \mathcal{F}_{n-k}^{(n)}) \mid \mathcal{F}_{n-k-1}^{(n)}) \\ &= \mathbb{E}_{Q_n}(e^{-r\frac{T}{n}}X(n-k) \mid \mathcal{F}_{n-k-1}^{(n)}) \\ &= e^{r\frac{T}{n}(n-k-1)}\mathbb{E}_{Q_n}(\hat{X}(n-k) \mid \mathcal{F}_{n-k-1}^{(n)}) \\ &= e^{r\frac{T}{n}(n-k-1)}\hat{X}(n-k-1) \\ &= X(n-k-1).\end{aligned}$$

The statement follows by induction. □

Exercise 9. Show by an example that in the trinomial model a European call cannot always be replicated by a trading strategy in bond and stock.

Proof. Take the one-period trinomial model with up parameter $u = 2$ and let $B(up_1) = 3, B(p_1) = 2, B(\frac{1}{u}p_1) = 1$. Suppose for a contradiction that some trading strategy φ replicates B. Then φ must satisfy the system of equations:

$$\begin{aligned} 2\varphi_1 p_1 + \varphi_0 &= 3 \\ \varphi_1 p_1 + \varphi_0 &= 2 \\ \varphi_1 \frac{p_1}{2} + \varphi_0 &= 1. \end{aligned}$$

But then we must have $\varphi_1 = \frac{1}{p_1} = \frac{2}{p_1}$, a contradiction. Hence, B cannot be replicated by a trading strategy. □

Exercise 10. In the one-period trinomial model compute two different equivalent martingale measures.

Solution. Let u, q_1, q_2 be the parameters for the one-period trinomial model. Observe that any $\hat{q}_1, \hat{q}_2 \in (0,1)$ such that $\hat{q}_1 + \hat{q}_2 < 1$ and

$$\begin{aligned} p_1 &= \mathbb{E}_{Q_n}\left(e^{-rT} P_1(1)\right) \\ &= e^{-rT} p_1 \left(u\hat{q}_1 + \frac{\hat{q}_2}{u} + 1 - \hat{q}_1 - \hat{q}_2\right), \end{aligned}$$

defines an equivalent martingale measure for our model. Solving, we have that

$$\hat{q}_1 = \frac{e^{rT} - 1 + (u-1)\hat{q}_2}{u(u-1)}.$$

Since $1 \le e^{rT} < u$ (assuming $r \ge 0$), any two choices of $\hat{q}_2 \in \left(0, \min\left(1, \frac{u}{u+1} - \frac{e^{rT}-1}{(u-1)(u+1)}\right)\right)$ will do.

Exercise 11. Give the proof of assertions (1) and (2) in Theorem 4.18:

(1) The random variables $\{\tau_{n+1} - \tau_n\}_{n\in\mathbb{N}}$ are independent and identically distributed. Their Laplace transform $\varphi(\lambda)$ is given by

$$\varphi(\lambda) = \mathbb{E}\left(e^{-\lambda\tau_1}\right) = \frac{\cosh(\mu\sigma^{-2}\Delta y)}{\cosh(\gamma\Delta y)}$$

$$\text{with } \mu := r - \frac{1}{2}\sigma^2, \; \gamma := \frac{\sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}, \; \lambda > 0.$$

Proof. Observe that

$$(\tau_{n+1} - \tau_n)(\omega) = \inf\{s > 0 : |\sigma(W_{s+\tau_n}(\omega) - W_{\tau_n}(\omega)) + s(r - 1/2\sigma^2)| > \Delta y\}.$$

By the strong Markov property of Brownian motion, $B_t = W_{t+\tau_n} - W_{\tau_n}$ is a Brownian motion, independent of $\mathcal{F}_{\tau_n}$. Since (τ_k) is an increasing sequence of stopping times, $(\mathcal{F}_{\tau_k})$ is an increasing sequence of σ-algebras. It follows that $\sigma(W_{s+\tau_n} - W_{\tau_n}) + s(r - 1/2\sigma^2)$ is independent of $\mathcal{F}_{\tau_k}$ for all $0 \le k \le n$. Since measurable functions preserve independence, it follows that $\tau_{n+1} - \tau_n$ is independent from $\mathcal{F}_{\tau_k} \supset \sigma(\tau_k)$ for all $0 \le k \le n$, proving that $(\tau_{n+1} - \tau_n)_{n\in\mathbb{N}}$ are independent. Moreover, since for all $n \in \mathbb{N}$, $B_t = W_{t+\tau_n} - W_{\tau_n} \sim \mathcal{N}(0,t)$, it follows that $\sigma(W_{t+\tau_n} - W_{\tau_n}) + t(r - 1/2\sigma^2) \overset{d}{=} \sigma(W_{t+\tau_m} - W_{\tau_m}) + t(r - 1/2\sigma^2)$ for all $n, m \in \mathbb{N}$, proving that $(\tau_{n+1} - \tau_n)_{n\in\mathbb{N}}$ are also identically distributed.

Towards computing their Laplace transform $\varphi(\lambda) = \mathbb{E}(e^{-\lambda\tau_1})$, let $\tau_1^{(n)} := \tau_1 \wedge n$ and observe that for a twice continuously differentiable function $g \in C^2(y - \Delta y, y + \Delta y)$, the Itô formula yields

$$\begin{aligned} g(Y(\tau_1^{(n)}))e^{-\lambda\tau_1^{(n)}} = g(y) &+ \int_0^{\tau_1^{(n)}} -\lambda g(Y(s))e^{-\lambda s} + \mu g'(Y(s))e^{-\lambda s} + \frac{1}{2}\sigma^2 g''(Y(s))e^{-\lambda s}\, ds \\ &+ \int_0^{\tau_1^{(n)}} \sigma g'(Y(s))e^{-\lambda s}\, dW(s), \end{aligned}$$

where $\mu := r - \frac{1}{2}\sigma^2$. By definition, $Y(s)$ is bounded in $\left[0, \tau_1^{(n)}\right]$ and so $\sigma g'(Y(s))e^{-\lambda s}$ is also bounded on this interval. Hence,

$$\mathbb{E}\left(\int_0^{\tau_1^{(n)}} \sigma g'(Y(s))e^{-\lambda s}\, dW(s)\right) = 0.$$

This implies that

$$\mathbb{E}\left[g(Y(\tau_1^{(n)}))e^{-\lambda\tau_1^{(n)}}\right] = g(y) + \mathbb{E}\left[\int_0^{\tau_1^{(n)}} -\lambda g(Y(s))e^{-\lambda s} + \mu g'(Y(s))e^{-\lambda s} + \frac{1}{2}\sigma^2 g''(Y(s))e^{-\lambda s}\, ds\right]. \tag{1}$$

Now to determine $\mathbb{E}(e^{-\lambda\tau_1})$, we look for a $g \in C^2$ with

$$\frac{1}{2}\sigma^2 g''(x) + \mu g'(x) - \lambda g(x) \equiv 0, \quad \text{for all } x \in (y - \Delta y, y + \Delta y) \tag{2}$$

$$g(y - \Delta y) = 1 \tag{3}$$

$$g(y + \Delta y) = 1. \tag{4}$$

Applying dominated convergence and boundary conditions (3) and (4), we see that

$$\lim_{n\to\infty} \mathbb{E}\left(g(Y(\tau_1^{(n)}))e^{-\lambda\tau_1^{(n)}}\right) = \mathbb{E}\left(g(Y(\tau_1))e^{-\lambda\tau_1}\right) = \varphi(\lambda).$$

Thus, for such $g \in C^2$, (1) and (2) imply that

$$g(y) = \varphi(\lambda).$$

Solving the given two-point boundary value problem for g, we get

$$g(x) = e^{-\frac{\mu}{\sigma^2}x}\left(C_1 e^{\gamma x} + C_2 e^{-\gamma x}\right)$$

$$\begin{cases} C_1 e^{\gamma(y+\Delta y)} + C_2 e^{-\gamma(y+\Delta y)} = e^{\mu\sigma^{-2}(y+\Delta y)} \\ C_1 e^{\gamma(y-\Delta y)} + C_2 e^{-\gamma(y-\Delta y)} = e^{\mu\sigma^{-2}(y-\Delta y)} \end{cases}$$

$$\implies \cosh(\gamma\Delta y)(C_1 e^{\gamma y} + C_2 e^{-\gamma y}) = e^{\mu\sigma^{-2}y}\cosh(\mu\sigma^{-2}\Delta y)$$

$$\implies \frac{\cosh(\mu\sigma^{-2}\Delta y)}{\cosh(\gamma\Delta y)} = g(y) = \varphi(\lambda).$$

□

(2) $\mathbb{E}(\tau_1) = \frac{\Delta y}{\mu} \cdot \tanh\left(\frac{\mu}{\sigma^2} \cdot \Delta y\right)$ for $\mu \neq 0$,
$\mathbb{E}(\tau_1^2) = 2(\mathbb{E}(\tau_1))^2 + \frac{\sigma^2 \Delta y}{\mu^3} \cdot \tanh\left(\frac{\mu}{\sigma^2}\right)\Delta y - \left(\frac{\Delta y}{\mu}\right)^2$ for $\mu \neq 0$.

Proof. See Problem 14 below. □

Exercise 12. Derive part (2) of Lemma 4.5 from part (1) with the help of Grisanov's Theorem 3.11:

For $\mu \in \mathbb{R}$, let $\tilde{W}(t) := W(t) + \mu \cdot t$ and $\tilde{M}(t) := \max_{0\le s\le t} \tilde{W}(s)$.Then the following relation is valid:

$$P\left(\tilde{W}(t) \le w, \tilde{M}(t) < x\right) = \Phi\left(\frac{w - \mu t}{\sqrt{t}}\right) - e^{2\mu x}\Phi\left(\frac{w - 2x - \mu t}{\sqrt{t}}\right).$$

Proof. By Grisanov's Theorem, for any $T \ge 0$, $\tilde{W}(t)$ is a Brownian motion with respect to the probability measure Q_T defined by the Radon-Nikodym density $Z(T, \mu) = e^{-\mu W(T) - 1/2\mu^2 T}$. Fix $t \ge 0$. By part (1) of Lemma 4.5, for any $x \ge \max(w, 0)$,

$$Q_T(\tilde{W}(t) \le w, \tilde{M}(t) < x) = \Phi\left(\frac{w}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2x - w}{\sqrt{t}}\right).$$

It follows that the joint Q_t-density function $\varphi_{\tilde{W},\tilde{M},Q_t}(w,x)$ is given by

$$\begin{aligned}\varphi_{\tilde{W},\tilde{M},Q_T}(w,x) &= \frac{\partial^2}{\partial w \partial x}\left(\Phi\left(\frac{w}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2x-w}{\sqrt{t}}\right)\right) 1_{\{x \geq \max(w,0)\}} \\ &= \frac{4x-2w}{t\sqrt{2\pi t}} e^{-\frac{(2x-w)^2}{2t}} 1_{\{x\geq \max(w,0)\}}.\end{aligned}$$

Using this Q_t-density function, we compute

$$\begin{aligned}P(\tilde{W}(t) \leq w, \tilde{M}(t) < x) &= \mathbb{E}_P\left(1_{\{\tilde{W}(t)\leq w, \tilde{M}(t)<x\}}\right) \\ &= \mathbb{E}_{Q_t}\left(1_{\{\tilde{W}(t)\leq w, \tilde{M}(t)<x\}} e^{\mu W(t) + 1/2\mu^2 t}\right) \\ &= \mathbb{E}_{Q_t}\left(1_{\{\tilde{W}(t)\leq w, \tilde{M}(t)<x\}} e^{\mu \tilde{W}(t) - 1/2\mu^2 t}\right) \\ &= \int_{-\infty}^{x}\int_{0\vee\tilde{w}}^{x} e^{\mu\tilde{w} - 1/2\mu^2 t}\frac{4\tilde{x}-2\tilde{w}}{t\sqrt{2\pi t}} e^{-\frac{(2\tilde{x}-\tilde{w})^2}{2t}}\, d\tilde{x}\, d\tilde{w} \\ &= \sqrt{\frac{2}{\pi t}} e^{-1/2\mu^2 t}\int_{-\infty}^{x} e^{\mu\tilde{w}}\int_{0\vee\tilde{w}}^{x}\frac{2\tilde{x}-\tilde{w}}{t} e^{-\frac{(2\tilde{x}-\tilde{w})^2}{2t}}\, d\tilde{x}\, d\tilde{w} \\ &= \frac{1}{\sqrt{2\pi t}} e^{-1/2\mu^2 t}\int_{-\infty}^{x} e^{\mu\tilde{w}}\left(e^{-\frac{(2(0\vee\tilde{w})-\tilde{w})^2}{2t}} - e^{-\frac{(2x-\tilde{w})^2}{2t}}\right) d\tilde{w} \\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{x} e^{\mu\tilde{w} - \frac{\tilde{w}^2}{2t} - 1/2\mu^2 t} - e^{\mu\tilde{w} - \frac{(2x-\tilde{w})^2}{2t} - 1/2\mu^2 t}\, d\tilde{w} \\ &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{x} e^{-\frac{(\tilde{w}-\mu t)^2}{2t}} - e^{2x\mu - \frac{(\tilde{w}-(\mu t+2x))^2}{2t}}\, d\tilde{w} \\ &= \Phi\left(\frac{w-\mu t}{\sqrt{t}}\right) - e^{2\mu x}\Phi\left(\frac{w-2x-\mu t}{\sqrt{t}}\right).\end{aligned}$$

□

Exercise 13. (a) In the binomial model, determine the parameters u, d, q if additionally to the moment conditions (4.21) and (4.22) we require $u = 1/d$.

Solution. To first dispense with the case $q = 1/2$, observe that condition (4.21) forces $r = 1/2\sigma^2$ and then (4.22) implies that $u = e^{\sigma\sqrt{\Delta t}}$. Now suppose that $q \neq 1/2$. From (4.21), we have that

$$\begin{aligned}(r - 1/2\sigma^2)\Delta t &= \ln(u)q + \ln(d)(1-q) \\ &= \ln(u)(2q-1).\end{aligned}$$

It follows that $u = e^{\frac{r-1/2\sigma^2}{2q-1}\Delta t}$. From (4.22), we also have that

$$\begin{aligned}(r-1/2\sigma^2)^2(\Delta t)^2 + \sigma^2\Delta t &= \ln(u)^2 q + \ln(d)^2(1-q) \\ &= \ln(u)^2 \\ &= \frac{(r-1/2\sigma^2)^2(\Delta t)^2}{(2q-1)^2}.\end{aligned}$$

Solving for q, we find that

$$q = \frac{(r-1/2\sigma^2)\sqrt{\Delta t}}{2\sqrt{(r-1/2\sigma^2)^2(\Delta t)+\sigma^2}} + \frac{1}{2},$$

and

$$u = e^{\sqrt{(r-1/2\sigma^2)^2(\Delta t)^2 + \sigma^2\Delta t}}.$$

(b) Cox, Ross, Rubinstein suggest the choice of

$$u = e^{\sigma\sqrt{\Delta t}},\ d = e^{-\sigma\sqrt{\Delta t}}.$$

Show that with this requirement, (4.21) is satisfied but not requirement (4.22). How do we have to choose the left-hand side of (4.22) such that with the above choice of u, d (4.22) is also satisfied? How do we have to interpret this left-hand side?

Solution. Observe that in order for (4.21) to be satisfied, we must pick

$$q = \frac{(r - 1/2\sigma^2)\sqrt{\Delta t} + \sigma}{2\sigma}.$$

However, we than have

$$\begin{aligned}\ln(u)^2 q + \ln(d)^2(1-q) &= \sigma^2 \Delta t \\ &\neq (r - 1/2\sigma^2)^2(\Delta t)^2 + \sigma^2 \Delta t,\end{aligned}$$

and so (4.22) would be satisfied if and only if the riskless interest rate is given by $r = 1/2\sigma^2$. That is, if and only if the price process has no drift component.

Exercise 14. Let τ_1 be defined as in Section 4.5. Determine $\mathbb{E}(\tau_1)$ and $\mathbb{E}(\tau_1^2)$.

Solution. From Exercise 11, we computed

$$\varphi(\lambda) = \mathbb{E}(e^{-\lambda\tau_1}) = \frac{\cosh(\mu\sigma^{-2}\Delta y)}{\cosh(\gamma\Delta y)}.$$

Observe that if $\mu \neq 0$, then this expression is smooth in some neighborhood of 0, and

$$\begin{aligned}\mathbb{E}(\tau_1) &= -\mathbb{E}\left(\frac{d}{d\lambda}\bigg|_{\lambda=0} e^{-\lambda\tau_1}\right) \\ &= -\varphi'(0) \\ &= \frac{\cosh(\mu\sigma^{-2}\Delta y)}{\cosh^2(\gamma\Delta y)}\sinh(\gamma\Delta y)\frac{\Delta y}{\sqrt{\mu^2 + 2\lambda\sigma^2}}\Bigg|_{\lambda=0} \\ &= \frac{\Delta y}{|\mu|}\tanh(\mu\sigma^{-2}\Delta y).\end{aligned}$$

We also have that

$$\begin{aligned}\mathbb{E}(\tau_1^2) &= \mathbb{E}\left(\frac{d^2}{d\lambda^2}\bigg|_{\lambda=0} e^{-\lambda\tau_1}\right) \\ &= \varphi''(0) \\ &= \frac{d}{d\lambda}\bigg|_{\lambda=0}\left(\varphi(\lambda)\tanh(\gamma\Delta y)\frac{\Delta y}{\sqrt{\mu^2 + 2\lambda\sigma^2}}\right) \\ &= \varphi'(0)\tanh(\mu\sigma^{-2}\Delta y)\frac{\Delta y}{|\mu|} - \varphi(0)\frac{1}{\cosh^2(\mu\sigma^{-2}\Delta y)}\left(\frac{\Delta y}{\mu}\right)^2 + \varphi(0)\tanh(\mu\sigma^{-2}\Delta y)\frac{\sigma^2\Delta y}{|\mu|^3} \\ &= \mathbb{E}(\tau_1)^2 - \frac{1}{\cosh^2(\mu\sigma^{-2}\Delta y)}\left(\frac{\Delta y}{\mu}\right)^2 + \tanh(\mu\sigma^{-2}\Delta y)\frac{\sigma^2\Delta y}{|\mu|^3} \\ &= 2\mathbb{E}(\tau_1)^2 + \tanh(\mu\sigma^{-2})\frac{\sigma^2\Delta y}{|\mu|^3} - \left(\frac{\Delta y}{\mu}\right)^2.\end{aligned}$$

Chapter 5: Optimal Portfolios

Exercise 1. Use the martingale method to solve the portfolio problem (5.2) in the case of constant market coefficients and with the utility functions

$$U_1(t,x) = U_2(x) = \frac{1}{\gamma}x^\gamma \quad \text{for } \gamma \in (0,1) \text{ fixed.}$$

Solution. Using the same notation as in Section 5.2, we have that

$$I_1(t,y) = I_2(y) = y^{1/(\gamma-1)}.$$

Setting $p := \frac{\gamma}{\gamma-1}$, we compute

$$\begin{aligned}
\chi(y) &= \mathbb{E}\left(\int_0^T H(t)I_1(t,yH(t))\,dt + H(T)I_2(yH(t))\right) \\
&= \mathbb{E}\left(\int_0^T y^{1/(\gamma-1)}H(t)^{\frac{\gamma}{\gamma-1}}\,dt + y^{1/(\gamma-1)}H(T)^{\frac{\gamma}{\gamma-1}}\right) \\
&= y^{1/(\gamma-1)}\left(\int_0^T \mathbb{E}\left(e^{-rpt-p\theta W(t)-\frac{1}{2}p\theta^2 t}\right)dt + \mathbb{E}\left(e^{-rpT-p\theta W(T)-\frac{1}{2}p\theta^2 T}\right)\right) \\
&= y^{1/(\gamma-1)}\left(\int_0^T e^{-tp(r+\frac{1}{2}\theta^2(1-p))}\,dt + e^{-Tp(r+\frac{1}{2}\theta^2(1-p)))}\right) \\
&= y^{1/(\gamma-1)}\left(\left(\frac{1}{\kappa}+1\right)e^{\kappa T} - \frac{1}{\kappa}\right),
\end{aligned}$$

where we define $\kappa := p\left(\frac{1}{2}\theta^2\frac{1}{\gamma-1} - r\right)$ (and assume for now that $\kappa \neq 0$). Given initial wealth $x > 0$, Theorem 5.8 tells us that the optimal terminal wealth is given by

$$\begin{aligned}
B^* &= I_2(\chi^{-1}(x)H(T)) \\
&= \frac{xH(T)^{1/(\gamma-1)}}{\left(\frac{1}{\kappa}+1\right)e^{\kappa T} - \frac{1}{\kappa}},
\end{aligned}$$

and the optimal consumption is

$$\begin{aligned}
c^*(t) &= I_1(t,\chi^{-1}(x)H(t)) \\
&= \frac{xH(t)^{1/(\gamma-1)}}{\left(\frac{1}{\kappa}+1\right)e^{\kappa T} - \frac{1}{\kappa}}.
\end{aligned}$$

Towards applying Theorem 5.9, we compute

$$\begin{aligned}
\frac{1}{H(t)}\mathbb{E}\left(\int_t^T H(s)c^*(s)\,ds + H(T)B^* \mid \mathcal{F}_t\right) &= \frac{1}{H(t)^{1-p}}\frac{x}{\left(\frac{1}{\kappa}+1\right)e^{\kappa T} - \frac{1}{\kappa}}\left(\int_t^T \mathbb{E}\left(\frac{H(s)}{H(t)}\right)^p ds + \mathbb{E}\left(\frac{H(T)}{H(t)}\right)^p\right) \\
&= \frac{x}{H(t)^{1-p}}\frac{\left(\frac{1}{\kappa}+1\right)e^{\kappa(T-t)} - \frac{1}{\kappa}}{\left(\frac{1}{\kappa}+1\right)e^{\kappa T} - \frac{1}{\kappa}} \\
&=: f(t,W(t)).
\end{aligned}$$

Moreover, there exists a portfolio π^* with corresponding wealth process X^{x,π^*,c^*} such that $X^{x,\pi^*,c^*}(T) = B^*$ a.s. Since $\chi(y) < \infty$ for all $y > 0$, $f(0,0) = x$, and one can easily check that $f \in C^{1,2}([0,T]\times\mathbb{R})$, Theorem 5.9 implies that the optimal portfolio is given by

$$\begin{aligned}
\pi^*(t) &= \frac{1}{X^{x,\pi^*,c^*}(t)}\sigma^{-1}f_x(t,W(t)) \\
&= (p-1)\frac{H(t)^{1-p}}{\sigma H(t)^{2-p}}(-\theta)H(t) \\
&= \frac{\theta}{\sigma(1-\gamma)}.
\end{aligned}$$

Now, dealing with the edge case $\kappa = 0$, we see from above that in this case $\chi(y) = y^{1/(\gamma-1)}(T+1)$, and the optimal terminal wealth and consumption are given by $B^* = \frac{xH(T)^{1/(\gamma-1)}}{T+1}$, and $c^*(t) = \frac{xH(t)^{1/(\gamma-1)}}{T+1}$. The optimal wealth process is then $X^{x,\pi^*,c^*}(t) = \frac{x(T-t+1)}{H(t)^{1-p}(T+1)}$, and the optimal portfolio is again $\pi^*(t) = \frac{\theta}{\sigma(1-\gamma)}$.

Exercise 2. Use the martingale method to solve the consumption problem (5.8) with the utility functions

$$U_1(t,x) = \frac{1}{\gamma}e^{-\beta t}x^{\gamma}, \quad \gamma \in (0,1), \quad \beta > 0 \text{ fixed.}$$

How do the optimal strategies (π^*, c^*) depend on β?

Solution. Again using the notation from Section 5.2, we have that

$$I_1(t,y) = e^{\frac{\beta}{\gamma-1}t}y^{1/(\gamma-1)} = e^{\beta(p-1)t}y^{p-1},$$

with $p := \frac{\gamma}{\gamma-1}$. Thus, using $\kappa := p\left(\frac{1}{2}\theta^2(p-1) - r\right)$ (and assuming for now that $\beta(p-1) + \kappa \neq 0$),

$$\begin{aligned}
\chi(y) &= \mathbb{E}\left(\int_0^T H(t)I_1(t, yH(t))\, dt\right) \\
&= y^{p-1}\int_0^T e^{\beta(p-1)t}\mathbb{E}\left(H(t)^p\right)\, dt \\
&= y^{p-1}\int_0^T e^{(\beta(p-1)+\kappa)t}\, dt \\
&= y^{p-1}\frac{e^{(\beta(p-1)+\kappa)T} - 1}{\beta(p-1)+\kappa} \\
&= y^{p-1}C_T.
\end{aligned}$$

By Corollary 5.10, the optimal consumption is given by

$$\begin{aligned}
c^*(t) &= I_1(t, \chi^{-1}(x)H(t)) \\
&= \frac{xe^{\beta(p-1)t}}{C_T}H(t)^{p-1} \\
&= \frac{xH(t)^{p-1}(\beta(p-1)+\kappa)e^{\beta(p-1)t}}{e^{(\beta(p-1)+\kappa)T} - 1} \\
&= \frac{x(\beta(1-p)-\kappa)}{1-e^{(\beta(p-1)+\kappa)T}}e^{(1-p)((r+\theta^2-\beta)t+\theta W(t))}
\end{aligned}$$

Now solving the edge case $\beta(p-1)+\kappa = 0$, we get that $\chi(y) = y^{p-1}T$, so that $c^*(t) = \frac{x}{T}e^{(1-p)((r+\theta^2-\beta)t+\theta W(t))}$. Observe that, in both cases,

$$\mathbb{E}(c^*(t)) = C\mathbb{E}\left(e^{(1-p)((r+\theta^2-\beta)t+\theta W(t))}\right) = Ce^{(1-p)(r+\theta^2-\beta-\frac{1}{2}(1-p)^2\theta^2)t},$$

for some constant C. Thus, if $\beta < r + \theta^2 - \frac{1}{2}(1-p)^2\theta^2$ then expected optimal consumption increases with time. If these two quantities are equal, expected optimal consumption remains constant in time, and if $\beta > r+\theta^2-\frac{1}{2}(1-p)^2\theta^2$, then expected optimal consumption decreases with time.

Exercise 3. Consider the example "logarithmic utility" of Section 5.3 with an option with the final payoff

$$B = |P_1(T) - K|.$$

(a) Determine the price of B and the corresponding replicating trading strategy $\Psi(t) = (\Psi_0(t), \Psi_1(t))$.

Solution. Observe that $B = B_K^{Call} + B_K^{Put}$, and so the price process f corresponding to payoff B is

$$f(t) = X_K^{Call}(t) + X_K^{Put}(t),$$

where X_K^{Call} and X_K^{Put} are defined to be the price processes for a European call and put option with strike price K, respectively. Thus, the replicating trading strategy is given by

$$(\Psi_0(t), \Psi_1(t)) = (Ke^{-rT}(1-2\Phi(d_2(t))), 2\Phi(d_1(t)) - 1).$$

(b) Show that with the above option Theorem 5.11 remains valid if (with the usual notations) we set

$$\varphi_1(t) := \begin{cases} \frac{\xi_1(t)}{\Psi_1(t)} & \text{if } \Psi_1(t) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let ξ be the optimal stock-bond trading strategy with corresponding wealth process X^ξ, take c to be the optimal consumption process, and let $X^\varphi(t) = \varphi_0(t)P_0(t) + \varphi_1(t)f(t)$ be the wealth process corresponding to the option-bond trading strategy

$$\varphi(t) = \left(\frac{X^\xi(t) - \varphi_1(t)f(t)}{P_0(t)}, \varphi_1(t) \right).$$

Let π_φ be the option portfolio process associated to φ. Observe that, from the definition of φ, we have that

$$X^\varphi(t) = \frac{X^\xi(t) - \varphi_1(t)f(t)}{P_0(t)} P_0(t) + \varphi_1(t)f(t) = X^\xi(t), \quad \text{for all } t \in [0,T].$$

It follows that $X^\varphi(0) = X^\xi(0) = x$ and for all $t \in [0,T]$, $X^\varphi(t) = X^\xi(t) \geq 0$. Moreover, assuming φ is a valid trading strategy, we immediately get that $J(x;\pi_\varphi,c) = J(x;\pi_\xi,c)$. Hence, to complete the proof, it suffices to verify that φ is a self financing trading strategy. Towards verifying that φ satisfies stochastic integrability, note that by the Itô formula,

$$df(t,P_1(t)) = \left(f_t(t) + P_1(t)\Psi_1(t)r + \frac{1}{2}\sigma^2 P_1(t) f_{pp}(t) \right) dt + \sigma P_1(t)\Psi_1(t)\, dW(t),$$

and so

$$\begin{aligned} \int_0^T \varphi_1(t)^2 \, d\langle f \rangle_t &= \int_0^T \sigma^2 \frac{\xi_1(t)^2}{\Psi_1(t)^2} 1_{\{\Psi_1 = 0\}}(t)\Psi_1(t)^2 P_1(t)^2 \, dt \\ &\leq \sigma^2 \int_0^T \xi_1(t)^2 P_1(t)^2 \, dt < \infty \quad \text{a.s.} \end{aligned}$$

Towards proving that φ is self financing, we use the fact that f solves the Black-Scholes PDE,

$$f_t + rpf_p + \frac{1}{2}\sigma^2 p^2 f_{pp} - rf = 0, \quad f(T,p) = |p - K|,$$

to compute that

$$\begin{aligned} \varphi_0(t)\, dP_0(t) + \varphi_1(t)\, df(t) - c(t)\, dt &= \frac{X^\xi(t) - \varphi_1(t)f(t)}{P_0(t)} rP_0(t)\, dt \\ &+ \frac{\xi_1(t)}{\Psi_1(t)} 1_{\{\Psi_1 \neq 0\}}(t) \left(\left(f_t(t) + P_1(t)\Psi_1(t)r + \frac{1}{2}\sigma^2 P_1(t)^2 f_{pp}(t) \right) dt + \sigma P_1(t)\Psi_1(t)\, dW(t) \right) - c(t)\, dt \\ &= rX^\xi(t)\, dt + \varphi_1(t) \left(f_t(t) + P_1(t) r \Psi_1(t) + \frac{1}{2}\sigma^2 P_1(t)^2 f_p p(t) - rf(t) \right) dt + \sigma\xi_1(t)P_1(t)\, dW(t) - c(t)\, dt \\ &= r\xi_0(t)P_0(t)\, dt + r\xi_1(t)P_1(t)\, dt + \sigma\xi_1(t)P_1(t)\, dW(t) - c(t)\, dt \\ &= \xi_0(t)\, dP_0(t) + \xi_1(t)\, dP_1(t) - c(t)\, dt \\ &= dX^\xi(t) \\ &= dX^\varphi(t). \end{aligned}$$

□

(c) For fixed $t \in [0,T]$ regard the optimal portfolio process $\pi_{opt}(t)$ as a function of $P_1(t)$. What happens at that value of $P_1(t)$ for which $\Psi_1(t)$ vanishes?

Solution. Observe that $\Psi_1(t) = 0$ if and only if $d_1(t) = 0$, if and only if

$$P_1(t) = Ke^{-(r+\frac{1}{2}\sigma^2)(T-t)}.$$

Hence, by monotone dependence of $\Psi_1(t)$ on $P_1(t)$ and the continuity of $\Phi(d_2(t))$ with respect to $P_1(t)$, as $\Psi_1(t) \to 0$,

$$f(t) = P_1(t)\Psi_1(t) + Ke^{-r(T-t)}(1 - 2\Phi(d_2(t))) \to Ke^{-r(T-t)}(1 - 2\Phi(-\sigma\sqrt{T-t})) > 0.$$

It follows that, for the case $b \neq r$,

$$\begin{aligned}
\pi_{opt}(t) &= \frac{\varphi_1(t)f(t)}{X^{\varphi}(t)} \\
&= \frac{\xi_1(t)f(t)1_{\{\Psi\neq 0\}}}{\Psi_1(t)\varphi_0(t)P_0(t) + \xi_1(t)f(t)1_{\{\Psi\neq 0\}}} \\
&= \frac{\xi_1(t)f(t)}{\Psi_1(t)X^{\varphi}(t)}1_{\{\Psi_1\neq 0\}} \\
&= \frac{b-r}{\sigma^2}\frac{f(t)}{\Psi_1(t)P_1(t)}1_{\{\Psi_1\neq 0\}} \\
&\to \begin{cases} \infty, & \text{as } \Psi_1(t) \downarrow 0 \\ -\infty, & \text{as } \Psi_1(t) \uparrow 0. \end{cases}
\end{aligned}$$

In the case where $b = r$, $\xi_1(t) \equiv 0$ and so $\pi_{opt}(t) \equiv 0$.

Exercise 4. For $T > 0$ solve the following stochastic control problem

$$\min_{u(\cdot)} \mathbb{E}^{0,x}\left(\int_0^T \left(MX(s)^2 + Nu(s)^2\right) ds + DX(T)^2\right)$$

with

$$\begin{aligned}
dX(s) &= (AX(s) + Bu(s))\,ds + \sigma\,dW(s), \\
X(0) &= x \in \mathbb{R},
\end{aligned}$$

and $M, N, D > 0$, $A, B, \sigma \in \mathbb{R}$, and $U = \mathbb{R}$.

Solution. The HJB-equation corresponding to this stochastic control problem admits the form

$$\min_{u\in\mathbb{R}}\left\{v_t + \frac{1}{2}\sigma^2 v_{xx} + (Ax + Bu)v_x + Mx^2 + Nu^2\right\} = 0, \quad (t,x) \in [0,T] \times \mathbb{R}$$

$$v(T,x) = Dx^2, \quad x \in \mathbb{R}.$$

Formal minimization yields the following candidate for the optimal control:

$$u^*(t) = -\frac{Bv_x(t, X(t))}{2N}.$$

Inserting this candidate into the HJB-equation results in the PDE

$$v_t + \frac{1}{2}\sigma^2 v_{xx} + Axv_x - \frac{B^2}{4N}v_x^2 + Mx^2 = 0, \quad (t,x) \in [0,T] \times \mathbb{R}$$

$$v(T,x) = Dx^2, \quad x \in \mathbb{R}.$$

To solve this PDE, we use the ansatz $v(t,x) = f(t)x^2 + g(t)$. This transforms the PDE into the ordinary differential equation

$$\left(f'(t) + 2Af(t) - \frac{B^2}{N}f(t)^2 + M\right)x^2 + g'(t) + \sigma^2 f(t) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}$$

$$f(T)x^2 + g(T) = Dx^2, \quad x \in \mathbb{R}.$$

Since this equation has to hold for all $x \in \mathbb{R}$, this differential equation breaks into the following system of differential equations:

$$\begin{cases} f'(t) + 2Af(t) - \frac{B^2}{N}f(t)^2 + M = 0, & t \in [0,T] \\ f(T) = D. \end{cases}$$

$$\begin{cases} g'(t) + \sigma^2 f(t) = 0, & t \in [0,T] \\ g(T) = 0. \end{cases}$$

Thus, given f, we obtain g via $g(t) = \sigma^2 \int_t^T f(s)\,ds$. Towards solving the first of these equations, define $a := -\frac{B^2}{N}$, $b := 2A$ and $c := M$, and let r_1, r_2 be the real roots of the polynomial $ax^2 + bx + c$ and $\Delta \geq 0$ the discriminant. Solving via partial fractions gives

$$\begin{aligned} -\int dt &= \int \frac{df}{af^2 + bf + c} \\ &= \int \frac{df}{a(f - r_1)(f - r_2)} \\ &= \int \frac{1}{\sqrt{\Delta}} \frac{df}{f - r_1} - \int \frac{1}{\sqrt{\Delta}} \frac{df}{f - r_2} \\ &= \frac{1}{\sqrt{\Delta}} \ln \frac{f - r_1}{f - r_2}. \end{aligned}$$

This suggests the solution

$$f(t) = \frac{r_1 - r_2 \frac{D - r_1}{D - r_2} e^{-\sqrt{\Delta}(t-T)}}{1 - \frac{D - r_1}{D - r_2} e^{-\sqrt{\Delta}(t-T)}}.$$

and plugging this back into u^* gives

$$u^*(t) = -\frac{B(r_1 - r_2 \frac{D - r_1}{D - r_2} e^{-\sqrt{\Delta}(t-T)})}{N(1 - \frac{D - r_1}{D - r_2} e^{-\sqrt{\Delta}(t-T)})} x.$$

Observe that since $e^{-\sqrt{\Delta}(t-T)} \geq 1$ and $\frac{D-r_1}{D-r_2} > 1$, it follows that $u^*(t)$ is a smooth function on the bounded interval $[0,T]$. Thus, the SDE for X is linear with bounded coefficients and so by the variation of constants theorem, this equation has a unique solution X^* with respect to the control u^*. The moment condition (5.13) for u^* is satisfied by the boundedness of u^*, and the moment condition (5.14) for X^* follows by Lemma 3.23. Finally, since f is smooth, g is also smooth and so $v \in C^{1,2}$ and clearly satisfies the polynomial growth condition by the boundedness of $f(t)$. Thus, by Theorem 5.17, u^* is an optimal control and v coincides with the value function.

Exercise 5. For $T > 0$ solve the stochastic control problem

$$\max_{u(\cdot)} \mathbb{E}(X(T)^\gamma)$$

with

$$\begin{aligned} dX(t) &= au(t)\,dt + u(t)\,dW(t) \\ X(0) &= x > 0 \end{aligned}$$

and $a \in \mathbb{R}$, $0 < \gamma < 1$, $U = \mathbb{R}$, $\mathcal{O} = (0, \infty)$. In particular, show that the optimal strategy $u^*(t)$ and the value function $V(t,x)$ have the forms

$$u^*(t) = \frac{a}{1-\gamma} X(t),$$

$$V(t,x) = \exp\left(a^2 \frac{\gamma}{2(1-\gamma)}(T-t)\right) x^\gamma.$$

Solution. The HJB-equation corresponding to this stochastic control problem admits the form

$$\min_{u \in \mathbb{R}} \left\{ v_t + \frac{1}{2} u^2 v_{xx} + auv_x \right\} = 0, \quad (t,x) \in Q,$$

$$v(T,x) = x^\gamma, \quad x > 0$$

for $Q := [0,\tau) \times \mathcal{O}$. Formal minimization yields the following candidate for the optimal control:

$$u^*(t) = -\frac{av_x(t, X(t))}{v_{xx}(t, X(t))}.$$

Inserting $u^*(t)$ into the HJB-equation results in the partial differential equation

$$v_t - \frac{a^2 v_x^2}{2v_{xx}} = 0, \quad (t,x) \in Q$$

$$v(T,x) = x^\gamma, \quad x > 0.$$

To solve this PDE, we choose the ansatz $v(t,x) = f(t)x^\gamma$. This transforms the partial differential equation into the ordinary differential equation for $f(t)$

$$f'(t)x^\gamma + a^2 \frac{\gamma}{2(1-\gamma)} f(t)x^\gamma = 0, \quad (t,x) \in Q$$

$$f(T) = 1.$$

Solving, we obtain $f(t) = e^{a^2 \frac{\gamma}{2(1-\gamma)}(T-t)}$ and $u^*(t) = \frac{1}{1-\gamma}X(t)$. Thus, if we can show that the conditions of Theorem 5.17 are satisfied, then it will follow that u^* is an optimal control and $V(t,x) = v(t,x) = \exp\left(a^2 \frac{\gamma}{2(1-\gamma)}(T-t)\right)x^\gamma$, as required. To this end, observe that the SDE corresponding to the control u^* for X, given by $dX(t) = X(t)\left(\frac{a^2}{1-\gamma}\,dt + \frac{1}{1-\gamma}\,dW(t)\right)$ is linear with constant coefficients, and therefore has a unique solution X^* by the variation of constants theorem. Moreover, this solution X^* satisfies moment condition (5.14) by Lemma 3.23, and moment condition (5.13) immediately follows as

$$\mathbb{E}\left(\int_0^{t_1} |u^*(s)|^k\,ds\right) \le \frac{t_1 a}{1-\gamma}\mathbb{E}^{0,x}\left(\sup_{s\in[0,t_1]} |X(s)|^k\right) < \infty, \quad \forall k \in \mathbb{N}.$$

Finally, since $v \in C^{1,2}(Q)$ and obviously satisfies the polynomial growth condition on Q, all the conditions of Theorem 5.17 hold, and the conclusion follows.

Exercise 6. Show that in the market with constant coefficients and an infinite horizon the problem

$$\max_{(\pi,c)\in\mathcal{A}'(x)} \mathbb{E}^x\left(\int_0^\infty e^{-\beta t}U(c(t))\,dt\right), \quad \beta > 0,$$

admits the optimal solution pair of the form

$$\pi^*(t) \equiv \pi \in \mathbb{R}^d,$$

$c^*(t) = \delta X(t)$, for suitable constants $\pi \in \mathbb{R}^d$, $\delta > 0$ if and only if we have

$$U(x) = \alpha x^\gamma + d$$

for suitable $\gamma \in (0,1), \alpha, d > 0$.

Proof. Towards proving the "if" direction, fix $\alpha, d > 0$, $\gamma \in (0,1)$ and set $U(x) = \alpha x^\gamma + d$. Using the notation of Section 5.4, the HJB-equation corresponding to this choice of U admits the form

$$\max_{(u_1,u_2)\in[\alpha_1,\alpha_2]^d\times[0,\infty)} \left\{\frac{1}{2}u_1'\sigma\sigma' u_1 v''(x) + ((r + u_1'(b - r\underline{1}))x - u_2)v'(x) + \alpha u_2^\gamma + d - \beta v(x)\right\} = 0, \quad x > 0.$$

Formal maximization suggests the following choices for u_1 and u_2:

$$u_1^*(t) = -(\sigma\sigma')^{-1}(b - r\underline{1})\frac{v'(x)}{xv''(x)},$$

$$u_2^*(t) = \left(\frac{1}{\alpha\gamma}v'(x)\right)^{\frac{1}{\gamma-1}}.$$

Inserting this choice of u_1^* and u_2^* into the HJB-equation results in the differential equation in $v(x)$

$$\left(\alpha\left(\frac{1}{\alpha\gamma}\right)^{\frac{\gamma}{\gamma-1}} - \left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{1-\gamma}}\right)v'(x)^{\frac{\gamma}{\gamma-1}} - \frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{v'(x)^2}{v''(x)} + rv'(x)x - \beta v(x) + d = 0, \quad x > 0.$$

The requirement of the polynomially bounded solution in the verification theorem suggests the ansatz

$$v(x) = \frac{1}{\gamma}Kx^\gamma + K_0$$

for some choice of constants $K > 0, K_0 \in \mathbb{R}$. Inserting this ansatz into the above differential equation results in the equation

$$\left(\left(\alpha\left(\frac{1}{\alpha\gamma}\right)^{\frac{\gamma}{\gamma-1}} - \left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{1-\gamma}}\right)K^{\frac{1}{\gamma-1}} - \frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{1}{\gamma-1} + r - \beta\frac{1}{\gamma}\right)Kx^\gamma + (d - \beta K_0) = 0, \quad x > 0.$$

Since this equation must hold for all $x > 0$, we must choose $K_0 = \frac{d}{\beta}$. Inserting this choice of K_0 and then dividing out by Kx^γ yields the solution

$$K = \left(\alpha\left(\frac{1}{\alpha\gamma}\right)^{\frac{\gamma}{\gamma-1}} - \left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}\left(\frac{1}{2(\gamma-1)}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1}) - r + \frac{\beta}{\gamma}\right)^{\gamma-1}.$$

Plugging this finding back into u_1^* and u_2^*, we find that

$$u_1^*(t) \equiv \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(b - r\underline{1}) \in \mathbb{R}^d$$

$$u_2^*(t) = \left(\frac{K}{\alpha\gamma}\right)^{1/(\gamma-1)} X(t).$$

Observe that $\left(\frac{K}{\alpha\gamma}\right)^{1/(\gamma-1)} > 0$ for suitable β. It is clear that constant u_1^* and linear u_2^* satisfies all the conditions of the verification theorem, and the "if" direction follows.

Towards proving the "only if" direction, suppose that $\pi^*(t) \equiv \pi \in \mathbb{R}^d$ and $c^*(t) = \delta X(t)$ for some $\delta > 0$. Then (π^*, c^*) maximizes the HJB-equation

$$\max_{(u_1,u_2)\in[\alpha_1,\alpha_2]^d\times[0,\infty)}\left\{\frac{1}{2}u_1'\sigma\sigma' u_1 x^2 v''(x) + ((r + \pi'(b - r\underline{1}))x - u_2)v'(x) + U(u_2) - \beta v(x)\right\} = 0, \quad x > 0.$$

It follows that the partial derivatives evaluated at (π^*, c^*) are zero for all $x > 0$. Thus, $0 = -v'(x) + U'(c^*)$ for all $x > 0$, and so

$$U'(\delta x) = v'(x), \quad x > 0.$$

For convenience, set $s^2 := \pi'\sigma\sigma'\pi$ and $\mu := r + \pi'(b - r\underline{1}) - \delta$. Then substituting in the optimal controls, the HJB-equation collapses to

$$\frac{1}{2}s^2x^2v''(x) + \mu x v'(x) + U(\delta x) - \beta v(x) = 0, \quad x > 0.$$

Differentiating with respect to x, substituting $U'(\delta x) = v'(x)$, and setting $w(x) := v'(x)$, we arrive at the following ODE in $w(x)$

$$\frac{1}{2}s^2x^2w''(x) + (s^2 + \mu)xw'(x) + (\mu + \delta - \beta)w(x) = 0.$$

Using the ansatz $w(x) = Cx^{\gamma-1}$, the equation is transformed into

$$\left(\frac{1}{2}s^2(\gamma-1)(\gamma-2) + (s^2 + \mu)(\gamma-1) + (\mu + \delta - \beta)\right)Cx^{\gamma-1} = 0, \quad x > 0.$$

Dividing out by $Cx^{\gamma-1}$, we can solve for γ, subject to suitable δ. Thus, we have that

$$\begin{aligned} U'(x) &= v'(x/\delta) \\ &= w(x/\delta) \\ &= C(x/\delta)^{\gamma-1}. \end{aligned}$$

The conclusion follows after integrating this expression. □

Exercise 7. Solve the terminal wealth maximization problem (5.7) via the stochastic control approach in the case of constant coefficients for $d = m = 1$,

$$U_2(x) = \frac{1}{\gamma}x^\gamma,$$

if instead of the bond a stock with price

$$P_0(t) = p_0 \exp\left(\left(b_0 t - \frac{1}{2}\sigma_0^2\right)t + \sigma_0 W(t)\right)$$

is traded.

Solution. Note that P_0 satisfies the following SDE

$$dP_0(t) = P_0\left(b_0\,dt + \sigma_0\,dW(t)\right).$$

Let the "original" risky asset have constant drift term μ and constant volatility term σ. Then if we define $\pi(t)$ to be the time t proportion of wealth invested in the "original" risky asset, then we arrive at the following family of wealth SDEs controlled by π

$$dX^\pi(t) = X^\pi(t)\Bigg(\Big(b_0 + \pi(t)(\mu - b_0)\Big)\,dt + \Big(\sigma_0 + \pi(t)(\sigma - \sigma_0)\Big)\,dW(t)\Bigg).$$

Using this controlled wealth SDE and the valuation function

$$J(0,x;\pi) = \mathbb{E}^{0,x}\left(\frac{1}{\gamma}X^\pi(T)^\gamma\right)$$

yields the HJB-equation

$$\max_\pi\left\{\frac{1}{2}(\sigma_0 + \pi(\sigma - \sigma_0))^2 x^2 v_{xx} + (b_0 + \pi(\mu - b_0))x v_x + v_t\right\} = 0, \quad x, t > 0,$$

$$v(T,x) = \frac{1}{\gamma}x^\gamma, \quad x > 0.$$

Formally maximizing and using the standard separable ansatz $v(t,x) = \frac{1}{\gamma}x^\gamma e^{C(T-t)}$ yields the candidate

$$\pi^* \equiv \frac{\sigma_0}{\sigma - \sigma_0} + \frac{\mu - b_0}{(\sigma - \sigma_0)^2(1-\gamma)}.$$

Inserting this choice for π^* and the given ansatz, and then dividing out the common term $x^\gamma e^{C(T-t)}$, results in the following equation

$$-\frac{1}{2}\left(2\sigma_0 + \frac{\mu - b_0}{(\sigma - \sigma_0)(1-\gamma)}\right)^2(1-\gamma) + b_0 + \frac{\sigma_0(\mu - b_0)}{\sigma - \sigma_0} + \frac{(\mu - b_0)^2}{(\sigma - \sigma_0)^2(1-\gamma)} - \frac{1}{\gamma}C = 0.$$

Simplifying and solving for C, we find that

$$C = -2\gamma(1-\gamma)\sigma_0^2 + b_0 - \frac{\gamma\sigma_0(\mu - b_0)}{(\sigma - \sigma_0))} + \frac{\gamma(\mu - b_0)^2}{2(\sigma - \sigma_0)^2(1-\gamma)}.$$

The standard arguments show that all the conditions for the verification theorem are satisfied, and so the optimal portfolio is given by

$$\pi^* \equiv \frac{\sigma_0}{\sigma - \sigma_0} + \frac{\mu - b_0}{(\sigma - \sigma_0)^2(1-\gamma)}.$$

Exercise 8. Show that the market model of Exercise 7 is complete (without using Theorem 3.47).

Proof. Fix a contingent claim B in the market model of Exercise 7. Define a new asset $\tilde{\pi}$ corresponding to the portfolio

$$\tilde{\pi} \equiv -\frac{\sigma_0}{\sigma - \sigma_0}.$$

Observe that the price process for this asset is determined by the SDE

$$\begin{aligned} dP_{\tilde{\pi}}(t) &= P_{\tilde{\pi}}(t)\Bigg(\Big(b_0 + \tilde{\pi}(\mu - b_0)\Big)\,dt + \Big(\sigma_0 + \tilde{\pi}(\sigma - \sigma_0)\Big)\,dW(t)\Bigg) \\ &= P_{\tilde{\pi}}(t)(b_0 + \tilde{\pi}(\mu - b_0))\,dt. \end{aligned}$$

Hence $\tilde{\pi}$ replicates a riskless bond with interest rate

$$r = b_0 + \tilde{\pi}(\mu - b_0).$$

By Theorem 3.7, the market consisting of a bond with interest rate r and a stock with price process $P_1(t)$ is complete. Since $P_{\tilde{\pi}}(t)$ is equal to a measurable function of $P_0(t)$ and $P_1(t)$, and similarly $P_0(t)$ is equal to a measurable function of $P_{\tilde{\pi}}(t)$ and $P_1(t)$, it follows that $\sigma(P_{\tilde{\pi}}(T), P_1(T)) = \sigma(P_0(T), P_1(T))$. Thus, B is a contingent claim in the market consisting of the riskless bond with price process $P_{\tilde{\pi}}(t)$ and the stock with price process $P_1(t)$. By completeness, there exists a unique replication strategy $\varphi = (\varphi_{\tilde{\pi}}, \varphi_1)$. Pick the trading strategy

$$\theta = (\theta_0(t), \theta_1(t)) = \left(\frac{(1-\tilde{\pi})\varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t)}{P_0(t)}, \varphi_1(t) + \frac{\tilde{\pi}\varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t)}{P_1(t)} \right).$$

Then

$$\begin{aligned} \theta_0(t)P_0(t) + \theta_1(t)P_1(t) &= (1-\tilde{\pi})\varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t) + \varphi_1(t)P_1(t) + \tilde{\pi}\varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t) \\ &= \varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t) + \varphi_1(t)P_1(t). \end{aligned}$$

Moreover,

$$\begin{aligned} \theta_1(t)\, dP_0(t) + \theta_1(t)\, dP_1(t) &= (1-\tilde{\pi})\varphi_{\tilde{\pi}(t)}(t)P_{\tilde{\pi}}(t)(b_0\, dt + \sigma_0\, dW(t)) + \varphi_1(t)\, dP_1(t) + \tilde{\pi}\varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t)(\mu\, dt + \sigma\, dW(t)) \\ &= \varphi_{\tilde{\pi}}(t)P_{\tilde{\pi}}(t)\Big((b_0 + \tilde{\pi}(\mu - b_0))\, dt + (\sigma_0 + \tilde{\pi}(\sigma - \sigma_0))\, dW(t)\Big) + \varphi_1(t)\, dP_1(t) \\ &= \varphi_{\tilde{\pi}}(t)\, dP_{\tilde{\pi}}(t) + \varphi_1(t)\, dP_1(t) \\ &= X(t). \end{aligned}$$

It follows that θ is a self-financing replication strategy for B, proving the the market model of Exercise 7 is complete.

$\square$