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C^*-Algebras Generated by Weakly Quasi-Lattice Ordered Groups

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Submitted in partial fulfilment of the requirements for the
Bachelor of Science with Honours in Mathematics.

Abstract

In 1992 Alexandru Nica published a highly influential paper examining the C^*-algebras generated by a semigroup he calls a quasi-lattice ordered group. Of particular interest to Nica were the C^*-algebra generated by the Toeplitz representation of a quasi-lattice order, and a universally defined C^*-algebra he calls $C^*(G, P)$. Due to the concreteness of the first C^*-algebra and the universal properties of the second, Nica was interested in finding sufficient conditions to determine when these two algebras were isomorphic. He called quasi-lattice orders that satisfied this isomorphism condition amenable. Finding techniques that establish amenability is now a topic at the cutting edge of research in the study of C^*-algebras of semigroups. In this dissertation, we follow the outline set out by Nica's paper to study the C^*-algebras generated by weakly quasi-lattice ordered groups. Most of the proofs throughout this dissertation required a substantial amount of original work to fill in details omitted by Nica, as he often only offered a proof outline.

Acknowledgments

I would like to extend my utmost gratitude to my supervisor, Professor Astrid an Huef. Over the last year Astrid has pushed me to meet my full potential, and motivated me to continue writing on more than one occasion while battling through an incredibly challenging third trimester.

I would like to thank my family for their continual love and support, and for the many years they put up with me rambling on about mathematics. I am particularly grateful to my grandpa who first inspired me to study mathematics and from whom I arguably inherited my mathematical abilities.

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Chapter 1

Introduction

There is an extensive literature studying C^*-algebras generated by semigroups of isometries. Recent examples include Xin Li's 2013 paper *Nuclearity of semigroup C^*-algebras and the connection to amenability* [7], Brownlowe, Larsen and Stammeier's 2018 paper *C^*-algebras of algebraic dynamical systems and right LCM semigroups* [4], and an Huef, Nucinkis, Sehnhem and Yang's 2021 paper *Nuclearity of semigroup C^*-algebras* [1]. In 1992, Alexandru Nica published an influential paper titled *C^*-algebras generated by isometries and Wiener-Hopf operators* [8]. In this paper, Nica examines C^*-algebras generated by a class of partially ordered discrete groups which he calls the quasi-lattice ordered groups. Two C^*-algebras of particular interest to Nica were the C^*-algebra generated by the Toeplitz (or left regular) representation of a quasi-lattice ordered group $\mathcal{T}(G, P)$, and a universally defined C^*-algebra $C^*(G, P)$. For a discrete group G, the reduced and full C^*-algebras coincide if and only if G is amenable. In analogy, Nica calls a quasi-lattice ordered group (G, P) amenable when $C^*(G, P)$ is isomorphic to $\mathcal{T}(G, P)$. Throughout the remaining sections of Nica's paper, he examines a range of techniques for proving that a given quasi-lattice ordered group is amenable.

This dissertation will examine the C^*-algebras generated by weakly quasi-lattice ordered groups. We closely follow the outline of Nica's original paper. Although we often go into much more detail than Nica does, and fill in many of the details that he omitted. In Chapter 2, we explicitly construct the Toeplitz algebra $\mathcal{T}(G, P)$ associated to any weak quasi-lattice (G, P) by proving the existence of the Toeplitz representation $T : P \to B(\ell^2(P))$. We then turn to the task of constructing the universally defined algebra $C^*(G, P)$ in Chapter 3. The proofs in the chapter represent a significant amount of original work, filling in the details briefly outlined by Nica. Then in Chapter 4, we study a technique for proving that a weak quasi-lattice is amenable which uses a conditional expectation map. Many of the proofs in Chapter 4 also contain substantial original work as Nica often only gave proof outlines. In particular, our proofs of Proposition 4.1.3 and Lemma 4.2.4 fill in many details omitted by Nica. Finally, we conclude with an example of an amenable weakly quasi-lattice ordered group using the non-amenable free group on n generators.

In the final chapter of this dissertation, we prove a theorem which establishes amenability for a weakly quasi-lattice ordered group. Finding techniques that establish amenability is an active area of research in the field of operator algebras and is a topic which is at the cutting edge of research in the study of C^*-algebras of semigroups. Currently, there are theorems which give sufficient conditions for a quasi-lattice to be amenable (e.g. Laca-Raeburn [6]). These theorems use 'controlled maps' and finding necessary conditions using controlled maps is an interesting problem. Possible direction of future research would be to extend the constructions and results presented in this dissertation to study C^*-algebras generated by right LCM monoids, a class of algebraic structures that generalise weak quasi-lattice orders.

Chapter 2

The Toeplitz Algebra $\mathcal{T}(G, P)$

2.1 Weakly Quasi-Lattice Ordered Groups

Consider the usual ordering on the integers $\mathbb{Z}$ under addition. One can completely characterise the ordering on $\mathbb{Z}$ as follows: for any integers n, m we have that $n \leq m$ if and only if $m - n \in \mathbb{N}$. Thus the usual ordering on $\mathbb{Z}$ is completely determined by the group operations of $\mathbb{Z}$ and containment in its submonoid $\mathbb{N}$. A natural question to consider is how and when one can generalise this characterisation of the ordering on $\mathbb{Z}$ to define a partial ordering on an arbitrary group G with submonoid P.

2.1.1 Quasi-lattice Ordered Groups

Let G be a group endowed with the discrete topology and suppose that P is a submonoid of G such that $P \cap P^{-1} = \{e\}$ where we define $P^{-1} := \{q \in G : q^{-1} \in P\}$.

Proposition 2.1.1. *One may define a partial ordering $\leq$ on G by $g \leq h$ if and only if $g^{-1}h \in P$.*

Proof. It suffices to prove that the binary partial ordering $\leq$ is reflexive, antisymmetric and transitive. For reflexivity, note that since $e \in P \cap P^{-1} \subset P$, for any $g \in G$, we have that $g^{-1}g = e \in P$ so that $g \leq g$. Towards proving antisymmetry, suppose that for some $g, h \in G$, we have that $g \leq h$ and $h \leq g$. Then $g^{-1}h$ and $h^{-1}g$ are both elements of P, so that

$$h^{-1}g = (g^{-1}h)^{-1} \in P \cap P^{-1}.$$

It follows that $g = h$, which proves that for any $g \neq h \in G$ such that $g \leq h$, it is not the case that $h \leq g$. Finally, $\leq$ is transitive since for any elements $g, h, k \in G$ such that $g \leq h$ and $h \leq k$, we have that $g^{-1}k = (g^{-1}h)(h^{-1}k) \in P$ so that $g \leq k$ as required. $\square$

Such a pair (G, P) is called a *partially ordered group* with *positive cone P*.

Definition 2.1.2. *Let (G, P) be a partially ordered group. In [8], Nica defined (G, P) to be a quasi-lattice ordered group if the following condition is satisfied:*

> **(QL)** *For any finite collection $F \subset G$ such that F is bounded above by elements in P, F has a least upper bound in P.*

Note that in any partially ordered group (G, P), the set P is just the set $\{g \in G : e \leq g\}$ and, similarly, the set P^{-1} is the set $\{g \in G : g \leq e\}$.

In [8], Nica asserted that the following are examples of quasi-lattice ordered groups. Here we prove each of Nica's examples are indeed quasi-lattice orders.

2.1.2 Examples

Example 2.1 *Any totally ordered group* (G, P) is a quasi-lattice ordered group since any finite collection $F \subset G$ must have a maximum element m which is comparable to e. Thus, either $m \leq e$ so that e is the least upper bound in P of F, or $e \leq m$ so that $m \in P$ and m is the least upper bound in P of F. In particular, any subgroup G of $\mathbb{R}$ under addition is a quasi-lattice ordered group with P taken to be $G \cap [0, \infty)$.

Example 2.2 *Direct Products*. Let $(G_i, P_i)_{i=1}^n$ be a sequence of quasi-lattice ordered groups. Since the finite product of a sequence of discrete topological spaces is a discrete topological space and $\prod_{i=1}^n P_i \cap \left(\prod_{i=1}^n P_i\right)^{-1} = \prod_{i=1}^n P_i \cap \prod_{i=1}^n P_i^{-1} = \prod_{i=1}^n P_i \cap P_i^{-1} = \{(e_{G_i})_{i=1}^n\}$, it follows that $(\prod_{i=1}^n G_i, \prod_{i=1}^n P_i)$ is a partially ordered group. Note that for any $(g_i)_{i=1}^n, (h_i)_{i=1}^n \in (\prod_{i=1}^n G_i, \prod_{i=1}^n P_i)$,

$$\begin{aligned}
&(g_i)_{i=1}^n \leq (h_i)_{i=1}^n \\
\iff &(g_i^{-1}h_i)_{i=1}^n = [(g_i)_{i=1}^n]^{-1}(h_i)_{i=1}^n \in \prod_{i=1}^n P_i \\
\iff &\forall i \in \{1, \ldots, n\} : g_i^{-1}h_i \in P_i \\
\iff &\forall i \in \{1, \ldots, n\} : g_i \leq h_i.
\end{aligned} \tag{2.1}$$

Suppose that F is a finite subset of $(\prod_{i=1}^n G_i, \prod_{i=1}^n P_i)$ such that F has upper bounds in $\prod_{i=1}^n P_i$. Then there exists some $(p_i)_{i=1}^n$ in $\prod_{i=1}^n P_i$ such that for each $(g_i)_{i=1}^n$ in F, $(g_i)_{i=1}^n \leq (p_i)_{i=1}^n$. For each i in $\{1, \ldots, n\}$, define $F_i := \pi_i(F)$. Each F_i is a finite subset of G_i bounded above by $p_i \in P_i$. Since (G_i, P_i) is quasi-lattice ordered, F_i has a least upper bound, $\rho_i \in P_i$. Two applications of (2.1) verify that $(\rho_i)_{i=1}^n$ is the least upper bound for F in $\prod_{i=1}^n P_i$, proving that $(\prod_{i=1}^n G_i, \prod_{i=1}^n P_i)$ is quasi-lattice ordered.

Example 2.3 *Outer Semidirect Products*. Let (G, P) and (H, R) be quasi-lattice ordered groups and suppose that $\varphi : H \to \mathrm{Aut}(G)$ is a group homomorphism such that for any $h \in H$, $\varphi_h(P) \subset P$. The outer semidirect product of G by H with respect to φ, denoted $G \rtimes_\varphi H$, is defined to be the group with underlying set $G \times H$ whose binary operation is defined by $(g_1, h_1) \bullet (g_2, h_2) = (g_1\varphi_{h_1}(g_2), h_1h_2)$. Note that the identity of $G \rtimes_\varphi H$ is just (e_G, e_H) and the inverse of any element (g, h) is $(\varphi_{h^{-1}}(g^{-1}), h^{-1})$. By the above proof that direct products of of quasi-lattice ordered groups are quasi-lattice ordered, to prove that $(G \rtimes_\varphi H, P \times R)$ is a quasi-lattice ordered group, it suffices to show that $(G \rtimes_\varphi H, P \times R)$ has the product order given by (2.1). Because $(g_1, h_1) \leq (g_2, h_2)$ holds if and only if $(\varphi_{h_1^{-1}}(g_1^{-1}g_2), h_1^{-1}h_2)$ is an element of $P \times R$, it suffices to prove that for any pair (g, h) in $G \times H$, $\varphi_h(g) \in P$ if and only if $g \in P$. One direction is simply the assumption that φ is P-invariant so it only remains to prove that if, for some $g \in G$ and $h \in H$, $\varphi_h(g)$ is an element of P then $g \in P$. To that end, fix $g \in G$ and $h \in H$ such that $\varphi_h(g) \in P$ and note that $g = \varphi_{h^{-1}h}(g) = \varphi_{h^{-1}}(\varphi_h(g)) \in P$. Hence, $(G \rtimes_\varphi H, P \times R)$ is a quasi-lattice ordered group.

2.1.3 Generalising to Weak Quasi-Lattices

Definition 2.1.3. *Let (G, P) be a partially ordered group. Call (G, P) a weakly quasi-lattice ordered group or simply a weak quasi-lattice if the following condition is satisfied:*

(WQL) *Any $p, q \in P$ with a common upper bound have a least common upper bound.*

As one would expect, the notion of a weakly quasi-lattice ordered group generalises condition (QL) in the definition of a quasi-lattice ordered group.

Proposition 2.1.4. *Let (G, P) be a quasi-lattice ordered group. Then (G, P) is a weak quasi-lattice.*

Proof. Let (G, P) be a quasi-lattice ordered group and suppose $p, q \in P$ have a common upper bound. Let $b \in G$ be any common upper bound of p and q. Since $p^{-1}b \in P$, there exists an element $g \in P$ such that $g = p^{-1}b$. It follows that $b = pg \in P$ so that every common upper bound of p, q is an element of P. By condition (QL), the elements p, q have a least common upper bound s in P. Because all of the common upper bounds of p, q lie in P, s is a least common upper bound for p, q in G. □

2.2 Defining the Toeplitz Algebra

In this section, we construct the Toeplitz algebra $\mathcal{T}(G, P)$ of a weakly quasi-lattice ordered group (G, P). The construction is generated by an isometric representation $T : P \to B(\ell^2(P))$, called the Toeplitz representation of the positive cone P.

2.2.1 The Toeplitz Representation

Often in mathematics, one seeks to find a realisation of a particular mathematical structure - such as a group or a partially ordered set - within another mathematical structure. Such realisations allow one to use the tools of the second mathematical structure when studying the first. For example, because the theory of linear transformations is well-understood, representation theorists study realisations of algebraic structures as a set of linear transformations of vector spaces. Such representations often simplify calculations and help to highlight certain properties of the original structure that were not immediately apparent.

Definition 2.2.1. *Let P be a monoid. A* ***representation*** *of P is a monoid homomorphism $\pi : P \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Note that because every C^*-algebra is isometrically isomorphic to a subspace of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, one could equivalently define a representation of P to be a monoid homomorphism $\pi : P \to B$ for some C^*-algebra, B. Working with representations of P into a general C^*-algebra is often useful and so we will freely switch between these two equivalent definitions throughout the remainder of this text.

Throughout this section, fix a weakly quasi-lattice ordered group (G, P). In defining a representation of P, we must first choose a suitable Hilbert space. It is well known that every Hilbert space $\mathcal{H}$ with orthonormal basis $\{u_a\}_{a\in\mathcal{A}}$ is isometrically isomorphic to the Hilbert space,

$$\ell^2(\mathcal{A}) := \{f : \mathcal{A} \to \mathbb{C} : \sum_{a\in\mathcal{A}} |f(a)|^2 < \infty\},$$

with inner product,

$$\langle f, g\rangle = \sum_{a\in\mathcal{A}} f(a)\overline{g(a)} \qquad (f, g \in \ell^2(\mathcal{A})).$$

Hence, a natural Hilbert space to consider when defining a representation of P is $\ell^2(P)$, which has as an orthonormal basis the set $\{\delta_q : q \in P\}$, where $\delta_q(p) = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{otherwise} \end{cases}$.

Define $\mathcal{B} := \operatorname{span}(\{\delta_q : q \in P\})$ and note that $\mathcal{B}$ is a dense subspace of $\ell^2(P)$.

Lemma 2.2.2. *For any nonzero vector $a \in \mathcal{B}$, there exists a unique finite set $F \subset P$ and a unique tuple $(c_q)_{q\in F} \subset \mathbb{C}$ with the property that $c_q \neq 0$ for all $q \in F$ such that $a = \sum_{q\in F} c_q\delta_q$.*

Proof. Fix a nonzero vector $a \in \mathcal{B} = \operatorname{span}(\{\delta_q : q \in P\})$. By definition, there exists a finite set $F_0 \subset P$ and a tuple $(c_q)_{q\in F_0}$ such that $a = \sum_{q\in F} c_q\delta_q$. Let $F := \{q \in F_0 : c_q \neq 0\}$ and note that $a = \sum_{q\in F} c_q\delta_q$. Suppose that $F' \subset P$ is another finite set and $(c'_q)_{q\in F'} \subset \mathbb{C}$ another tuple with the property that $c'_q \neq 0$ for all $q \in F'$ such that $a = \sum_{q\in F'} c'_q\delta_q$. Then for any $p \in P$, it follows that $p \in F$ if and only if

$$0 \neq c_p = \Big(\sum_{q\in F} c_q\delta_q\Big)(p) = \sum_{q\in F'} c'_q\delta_q(p) = \begin{cases} c'_p, & \text{if } p \in F' \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $F = F'$ and $(c_q)_{q\in F} = (c'_q)_{q\in F'}$. $\square$

Throughout the remainder of this section, when writing $\sum_{q\in F} c_q\delta_q$ we refer either to the zero vector or to some nonzero vector $a \in \mathcal{B}$, where it is understood that $F \subset P$ is the unique finite set and $(c_q)_{q\in F} \subset \mathbb{C}$ the unique tuple with the property that $c_q \neq 0$ for all $q \in F$ such that $a = \sum_{q\in F} c_q\delta_q$.

Theorem 2.2.3. *Fix an element $p \in P$ and define $\tilde{T}_p : \mathcal{B} \to \mathcal{B}$ by $\tilde{T}_p(\sum_{q\in F} c_q\delta_q) = \sum_{q\in F} c_q\delta_{pq}$. The function $\tilde{T}_p$ extends uniquely to an isometric linear operator $T_p \in B(\ell^2(P))$.*

Proof. Note that $\tilde{T}_p$ is well defined by Lemma 2.2.2 and is clearly a linear operator on $\mathcal{B}$. Further, $\tilde{T}_p$ is isometric since for any $\sum_{q\in F} c_q\delta_q \in \mathcal{B}$,

$$\Big\|\tilde{T}_p\Big(\sum_{q\in F} c_q\delta_q\Big)\Big\|^2 = \Big\|\sum_{q\in F} c_q\delta_{pq}\Big\|^2 = \sum_{q\in F} |c_q|^2 = \Big\|\sum_{q\in F} c_q\delta_q\Big\|^2,$$

where the second and third equality are just applications of the Pythagorean theorem.

Define $T_p : \ell^2(P) \to \ell^2(P)$ as follows: Fix $a \in \ell^2(P)$. Since $\mathcal{B}$ is dense in the metric space $\ell^2(P)$, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $a_n \to a$ as $n \to \infty$. Because $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $\tilde{T}_p$ is a linear isometric mapping, the sequence $(\tilde{T}_p(a_n))_{n\in\mathbb{N}}$ is Cauchy and therefore converges to a unique point, $\lim_{n\to\infty} \tilde{T}_p(a_n) \in \ell^2(P)$. Define $T_p(a) = \lim_{n\to\infty} \tilde{T}_p(a_n)$ and note that T_p is well defined since, for any $a \in \ell^2(P)$ and for any pair of sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ that converge to a,

$$\|\lim_{n\to\infty} \tilde{T}_p(a_n) - \lim_{n\to\infty} \tilde{T}_p(b_n)\| = \lim_{n\to\infty} \|\tilde{T}_p(a_n - b_n)\| = \lim_{n\to\infty} \|a_n - b_n\| = 0.$$

To see that T_p is indeed an extension of $\tilde{T}_p$ to $\ell^2(P)$, note that because for any $a \in \mathcal{B}$ the constant sequence $(a)_{n\in\mathbb{N}}$ converges to itself, we must have that $T_p(a) = \lim_{n\to\infty} \tilde{T}_p(a) = \tilde{T}_p(a)$.

Towards proving that T_p is linear, fix vectors $a, b \in \ell^2(P)$, scalars $z, w \in \mathbb{C}$ and sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ that converge to a and b, respectively. By the continuity of addition and scalar multiplication in $\ell^2(P)$,

$$T_p(za + wb) = \lim_{n\to\infty} \tilde{T}_p(za_n + wb_n) = z \lim_{n\to\infty} \tilde{T}_p(a_n) + w \lim_{n\to\infty} \tilde{T}_p(b_n) = zT_p(a) + wT_p(b).$$

Furthermore, by the continuity of the norm on $\ell^2(P)$ and the fact the $\tilde{T}_p$ is an isometric mapping on $\mathcal{B}$, $\|T_p(a)\| = \|\lim_{n\to\infty} \tilde{T}_p(a_n)\| = \lim_{n\to\infty} \|\tilde{T}_p(a_n)\| = \lim_{n\to\infty} \|a_n\| = \|a\|$. Hence, T_p is an isometric linear operator. To see that T_p is the unique extension of $\tilde{T}_p$ to $B(\ell^2(P))$, observe that any other extension $S \in B(\ell^2(P))$ is continuous and necessarily agrees with T_p on $\mathcal{B}$, which is dense in $B(\ell^2(P))$. $\square$

To summarise, we have shown that for every element $p \in P$, there exists a unique isometric linear operator $T_p \in B(\ell^2(P))$ with the property that for any $\sum_{q\in F} c_q\delta_q \in \mathcal{B}$, $T_p(\sum_{q\in F} c_q\delta_q) = \sum_{q\in F} c_q\delta_{pq}$. For any $p \in P$, we shall refer to the operator T_p as the *Toeplitz operator associated to* p.

Definition 2.2.4. *Let $T : P \to B(\ell^2(P))$ be the map which sends an element $p \in P$ to the Toeplitz operator T_p associated to p. This map is called the* ***Toeplitz representation*** *or sometimes the* ***left-regular representation*** *of P.*

Proposition 2.2.5. *The Toeplitz representation of P is a monoid homomorphism of P into $B(\ell^2(P))$ and, therefore, is a representation of P in the sense of Definition 2.2.1.*

Proof. Let $T : P \to B(\ell^2(P))$ be the Toeplitz representation of P. To see that $T(e) = I$ where I is the identity of $B(\ell^2(P))$, note that for any $\delta_q \in \mathcal{B}$, $T(e)(\delta_q) = \delta_{eq} = \delta_q = I(\delta_q)$. Because $T(e)$ and I are linear, they must agree over the dense set, $\mathcal{B}$. Hence, since both operators are continuous, $T(e) = I$. Now fix $p, t \in P$ and note that for any $\delta_q \in \mathcal{B}$, $T(pt)(\delta_q) = \delta_{ptq} = T(p)(\delta_{tq}) = T(p) \circ T(t)(\delta_q)$. Again by the linearity of these operators, they must agree over the dense set $\mathcal{B}$. Since both operators are continuous, they agree over all of $B(\ell^2(P))$. $\square$

We are now ready to define the Toeplitz algebra of the weakly quasi-lattice ordered group (G, P)

Definition 2.2.6. *Let $T : P \to B(\ell^2(P))$ be the Toeplitz representation of P. The* ***Toeplitz algebra*** *$\mathcal{T}(G, P)$ is defined to be the smallest C^*-subalgebra of $B(\ell^2(P))$ that includes the image of the Toeplitz representation. That is,*

$$\mathcal{T}(G, P) := C^*(\{T_p : p \in P\}) \subset B(\ell^2(P)).$$

2.2.2 Nica Covariance

Till this point, the Toeplitz algebra and Toeplitz representation that generates it could have been defined for any monoid P without reference to the fact that P is the positive cone of the weakly quasi-lattice ordered group (G, P). In this subsection, we examine an important identity that the algebra $\mathcal{T}(G, P)$ inherits from the weak quasi-lattice order condition on (G, P).

Definition 2.2.7. *Suppose that elements $p_1, \ldots, p_n \in G$ have a least common upper bound $b \in P$. Then we write $p_1 \vee \cdots \vee p_n < \infty$ and we let $p_1 \vee \cdots \vee p_n$ denote the element b. When the collection of elements $p_1, \ldots, p_n$ have no least common upper bound in P we write $p_1 \vee \cdots \vee p_n = \infty$.*

Proposition 2.2.8. *Fix elements $p \in P$ and $\sum_{q\in F} c_q\delta_q \in \mathcal{B}$, and let $T : P \to B(\ell^2(P))$ be the Toeplitz representation of P. Then,*

$$T_p^*\Big(\sum_{q\in F} c_q\delta_q\Big) = \sum_{\substack{q\in F\\ p\leq q}} c_q\delta_{p^{-1}q}. \tag{2.2}$$

Proof. First observe that because $T_p \in B(\ell^2(P))$, the adjoint T_p^* exists. Fix elements $\sum_{q\in F_1} c_q\delta_q$, $\sum_{t\in F_2} z_t\delta_t \in \mathcal{B}$. We have that:

$$\begin{aligned}\Big\langle T_p\Big(\sum_{q\in F_1} c_q\delta_q\Big), \sum_{t\in F_2} z_t\delta_t\Big\rangle &= \Big\langle \sum_{q\in F_1} c_q\delta_{pq}, \sum_{t\in F_2} z_t\delta_t\Big\rangle \\ &= \sum_{q\in F_1}\sum_{\substack{t\in F_2\\ q=p^{-1}t}} c_{p^{-1}t}\overline{z_t} \\ &= \sum_{q\in F_1}\sum_{\substack{t\in F_2\\ p\le t}} \langle c_q\delta_q, z_t\delta_{p^{-1}t}\rangle \\ &= \Big\langle \sum_{q\in F_2} c_q\delta_q, \sum_{\substack{t\in F_2\\ p\le t}} z_t\delta_{p^{-1}t}\Big\rangle\end{aligned}$$

Hence, because $\mathcal{B}$ is dense in $\ell^2(P)$, it follows that the adjoint of T_p is uniquely defined to be the bounded linear operator that sends an element $\sum_{q\in F} c_q\delta_q \in \mathcal{B}$ to the element $\sum_{\substack{q\in F\\ p\le q}} c_q\delta_{p^{-1}q}$. □

Proposition 2.2.9. *As Nica observed in [8], for any $x, y \in P$, the following identity holds in $\mathcal{T}(G, P)$:*

$$T_xT_x^*T_yT_y^* = \begin{cases} T_{x\vee y}T_{x\vee y}^* & \text{if } x\vee y<\infty \\ 0 & \text{if } x\vee y=\infty. \end{cases} \tag{2.3}$$

Proof. Fix elements $x, y \in P$. Because $\mathcal{B}$ is dense in $\ell^2(P)$ and all operators being considered are linear and continuous, it suffices to prove that the identity holds for a basis element δ_q. To that end, fix $\delta_q \in \{\delta_q : q \in P\}$. Suppose first that $x \vee y < \infty$. If either $q \not\geq x$ or $q \not\geq y$ then either $T_x^*(\delta_q) = 0$ or $T_y^*(\delta_q) = 0$. In either case, $T_xT_x^*T_yT_y^*(\delta_q) = 0$. Furthermore, it must be the case that $q \not\geq x \vee y$ so that $T_{x\vee y}^*(\delta_q) = 0$. Hence, the identity holds when $x \vee y < \infty$ and either $q \not\geq x$ or $q \not\geq y$. Now consider the case where $q \geq x$ and $q \geq y$. Since $q \in P$ we must have that $q \geq x \vee y$. Thus, $T_xT_x^*T_yT_y^*(\delta_q) = \delta_{xx^{-1}yy-1q} = \delta_{(x\vee y)(x\vee y)^{-1}q} = T_{x\vee y}T_{x\vee y}^*(\delta_q)$. Finally, if $x \vee y = \infty$ then since x and y are both elements of P, the condition (WQL) implies that x and y share no common upper bounds. Hence, either $q \not\geq x$ or $q \not\geq y$ so that $T_xT_x^*T_yT_y^*(\delta_q) = 0 = 0(\delta_q)$. □

Definition 2.2.10. *An isometric representation of P is called* ***Nica-covariant*** *or simply* ***covariant*** *if it satisfies equation (2.3).*

Just as we constructed the Toeplitz algebra $\mathcal{T}(G, P)$ by considering the smallest C^*-subalgebra of $B(\ell^2(P))$ containing the image space of the Toeplitz representation, we may associate to any Nica-covariant representation $S : P \to B(\mathcal{H})$ the C^*-algebra $C^*(\{S_p : p \in P\})$, the smallest C^*-subalgebra of $B(\mathcal{H})$ containing the image space of S.

The following lemma will be used to construct a useful characterisation of the C^*-algebra $C^*(\{S_p : p \in P\})$ associated to a Nica-covariant representation $S : P \to B(\mathcal{H})$.

Lemma 2.2.11. *The completion of a normed $*$-algebra satisfying the Banach and C^*-identities is a C^*-algebra.*

Proof. Suppose that $\mathcal{A}$ is a normed $*$-algebra satisfying the Banach and C^*-identities:

$$\begin{aligned}\|ab\| &\le \|a\|\|b\| \quad (a, b \in \mathcal{A}) \\ \|a^*a\| &= \|a\|^2 \quad (a \in \mathcal{A}).\end{aligned}$$

Let $J : \mathcal{A} \to \overline{\mathcal{A}}$ be the completion of $\mathcal{A}$. That is, J is an isometric embedding of $\mathcal{A}$ into the complete space $\overline{\mathcal{A}}$ such that $J(\mathcal{A})$ is dense in $\overline{\mathcal{A}}$. Define a $*$-algebra structure on the image space $J(\mathcal{A})$ as follows: For any $J(x), J(y) \in J(\mathcal{A})$, define addition on $J(\mathcal{A})$ by $J(x) + J(y) := J(x+y)$, multiplication by $J(x)J(y) := J(xy)$, and involution by $J(x)^* := J(x^*)$. Note that the continuity of the operations on $J(\mathcal{A})$ directly follows from the continuity of J and the operations on $\mathcal{A}$. Checking that $J(\mathcal{A})$ satisfies the $*$-algebra axioms is simply a matter of repeatedly applying the definition of each operation and then referencing the fact that the operations on $\mathcal{A}$ satisfy the $*$-algebra axioms. Hence, $J(\mathcal{A})$ is a normed $*$-algebra, whose norm is defined by restricting the metric on $\overline{\mathcal{A}}$ to the image space of J.

Towards extending the operations of addition, multiplication and involution to $\overline{\mathcal{A}}$, note that because the operations on $J(\mathcal{A})$ are continuous and $J(\mathcal{A})$ is dense in $\overline{\mathcal{A}}$, it suffices to prove that a continuous operator f on a dense subset S of a complete space C extends to a continuous operator $\tilde{f}$ on C. To this end, fix an element x of C and define $\tilde{f}(x) : C \to C$ as follows: pick a sequence $(x_n) \subset S$ converging to x in C and define $\tilde{f}(x) := \lim_{n\to\infty} f(x_n)$. The existence of $\lim_{n\to\infty} f(x_n)$ follows by observing that, because f is a continuous function and (x_n) is a Cauchy sequence in S, it follows that $\{f(x_n)\}$ is a Cauchy sequence in the complete space C. To see that $\tilde{f}$ is well-defined, suppose that (y_n) is another sequence in S converging to x. Then,

$$\begin{aligned}&\|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)\| \\ &\qquad \le \|\lim_{n\to\infty} f(x_n) - f(x_N)\| + \|f(x_N) - f(y_N)\| + \|f(y_N) - \lim_{n\to\infty} f(y_n)\|,\end{aligned}$$

for all natural numbers $N \in \mathbb{N}$. Fixing $\varepsilon > 0$, we may choose $N \in \mathbb{N}$ large enough such that $f(x_N), f(y_N)$ are within $\varepsilon/3$ of $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$, respectively. Moreover, for any $N \in \mathbb{N}$, we may pick a $\delta > 0$ such that $\|f(x_N) - f(y)\| < \varepsilon/3$ whenever $\|x_N - y\| < \delta$. By requiring N large enough so that in addition to the above inequalities holding, we also have that $\|x_N - x\| < \delta/2$ and $\|y_N - x\| < \delta/2$, it follows that $\|f(x_N) - f(y_N)\| < \varepsilon/3$. Hence, for every $\varepsilon > 0$, $\|\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n)\| < \varepsilon$, proving that these limits coincide. It follows that $\tilde{f}$ is well-defined. Note that $\tilde{f}$ extends f since for any element x in S, the constant sequence (x) converges to x in C so that $\tilde{f}(x) = \lim_{n\to\infty} f(x) = f(x)$. To show that the extension $\tilde{f}$ is continuous over C, fix $\varepsilon > 0$, an element x in C and a sequence $(x_n) \subset C$ converging to x. For each $k \in \mathbb{N}$, there exist sequences $(y_n), (z_n^k) \subset S$ converging to x and x_k, respectively. We may choose $N, K(N) \in \mathbb{N}$ large enough such that $\|\tilde{f}(x) - f(y_n)\|, \|\tilde{f}(x_n) - f(z_k^n)\| < \varepsilon/3$ for all $n \ge N$ and $k \ge K(N)$. Pick $\delta > 0$ such that $\|f(y_N) - f(y)\| < \varepsilon/3$ for any element $y \in S$ such that $\|y_N - y\| < \delta$. Then by choosing $N, K(N) \in \mathbb{N}$ large enough so that in addition to the above inequalities holding, we also have that $\|y_N - x\|, \|x - x_N\|, \|x_N - z^N_{K(N)}\| < \delta/3$, it follows that $\|f(y_N) - f(z^N_{K(N)})\| < \varepsilon/3$. Hence,

$$\|\tilde{f}(x) - \tilde{f}(x_n)\| \le \|\tilde{f}(x) - f(y_n)\| + \|f(y_n) - f(z_k^n)\| + \|f(z_k^n) - \tilde{f}(x_n)\| < \varepsilon,$$

for all $n \ge N$ and $k \ge K(N)$. It follows that $\tilde{f}$ is continuous. Note that $\tilde{f}$ is the unique continuous function on C that extends f since any two continuous functions that agree on a dense subset of their domain must agree over the entire domain.

Thus, all of the operations on $J(\mathcal{A})$ must extend uniquely to continuous operations on $\overline{\mathcal{A}}$. Checking that the continuous extensions of the operations on $J(\mathcal{A})$ satisfy the $*$-algebra axioms is a trivial but tedious exercise. To complete the proof that $\overline{\mathcal{A}}$ is a C^*-algebra, it only remains to prove that $\overline{\mathcal{A}}$ satisfies the Banach and C^* identities. To this end, fix elements $x, y \in$

$\overline{\mathcal{A}}$ and sequences $\{J(x_n)\}, \{J(y_n)\} \subset J(\mathcal{A})$ converging to the elements x and y, respectively. By the continuity of multiplication in $\overline{\mathcal{A}}$, the sequence $\{J(x_n)J(y_n)\}$ converges to xy. Hence, for any $\varepsilon > 0$, we may pick $N \in \mathbb{N}$ large enough such that,

$$\begin{aligned}\|xy\| &\leq \|J(x_N)J(y_N)\| + \varepsilon = \|J(x_N y_N)\| + \varepsilon \\ &= \|x_N y_N\| + \varepsilon \leq \|x_N\|\|y_N\| + \varepsilon = \|J(x_N)\|\|J(y_N)\| + \varepsilon \\ &\leq (\|x\| + \varepsilon)(\|y\| + \varepsilon) + \varepsilon.\end{aligned}$$

By taking the limit as ε goes to zero, it follows that $\|xy\| \leq \|x\|\|y\|$, proving the Banach identity. Moreover, by the continuity of multiplication and involution in $\overline{\mathcal{A}}$, the sequence $\{J(x_n)^* J(x_n)\}$ converges to x^*x. Hence, for any $\varepsilon > 0$, we may also pick $N \in \mathbb{N}$ large enough such that,

$$\begin{aligned}\|x^*x\| &\leq \|J(x_N)^* J(x_N)\| + \varepsilon = \|J(x_N^* x_N)\| + \varepsilon \\ &= \|x_N^* x_N\| + \varepsilon = \|x_N\|^2 + \varepsilon \\ &= \|J(x_N)\|^2 + \varepsilon \leq (\|x\| + \varepsilon)^2 + \varepsilon.\end{aligned}$$

By taking the limit as ε approaches zero, the inequality $\|x^*x\| \leq \|x\|^2$ holds in $\overline{\mathcal{A}}$. For the inequality in the other direction, we may choose $N \in \mathbb{N}$ large enough such that,

$$\begin{aligned}\|x\|^2 &\leq (\|J(x_N)\| + \varepsilon)^2 = \|J(x_N)\|^2 + o(\varepsilon) \\ &= \|x_N\|^2 + o(\varepsilon) = \|x_N^* x_N\| + o(\varepsilon) \\ &= \|J(x_N)^* J(x_N)\| + o(\varepsilon) \leq \|x^*x\| + o(\varepsilon).\end{aligned}$$

Thus, the C^*-identity holds in $\overline{\mathcal{A}}$, completing the proof that $\overline{\mathcal{A}}$ is a C^*-algebra. □

Theorem 2.2.12. *Let $S : P \to B(\mathcal{H})$ be a Nica-covariant representation of (G, P) and define $\mathcal{D} := span(\{S_p S_q^* : p, q \in P\})$. Then $\mathcal{D}$ is a dense unital $*$-subalgebra of the C^*-algebra $C^*(\{S_p : p \in P\})$.*

Proof. Towards proving that $\mathcal{D}$ is a unital $*$-subalgebra of the C^*-algebra $B(\mathcal{H})$, first note that $1_{B(\mathcal{H})} \in \mathcal{D}$ because, by definition, S is a monoid homomorphism so that $1 = S(e) = S_e$. It follows that $1 = 1 \circ 1^* = S_e S_e^* \in \mathcal{D}$. Hence, $\mathcal{D}$ is indeed unital. It remains to prove that $\mathcal{D}$ is closed under involution and multiplication. By the standard properties of the adjoints, for any $\sum_{j=1}^n \alpha_j S_{p_j} S_{q_j}^* \in \mathcal{D}$, we have that,

$$\left(\sum_{j=1}^n \alpha_j S_{p_j} S_{q_j}^* \right)^* = \sum_{j=1}^n \overline{\alpha_j} S_{q_j} S_{p_j}^* \in \mathcal{D}, \tag{2.4}$$

proving that $\mathcal{D}$ is closed under involution. To show that $\mathcal{D}$ is closed under multiplication, it suffices to prove that for any elements $p, q, t, r \in P$, there exist elements $u, v \in P$ such that $S_p S_q^* S_t S_r^* = S_u S_v^*$. To that end, fix $p, q, t, r \in P$ and observe that, because Nica-covariant representations are by definition isometric, $S : P \to B(\mathcal{H})$ is an isometric representation so that $S_q^* S_q = S_t^* S_t = 1$. Hence,

$$\begin{aligned}S_p S_q^* S_t S_r^* &= S_p (S_q^* S_q) S_q^* S_t (S_t^* S_t) S_r^* \\ &= \begin{cases} S_p S_q^* S_{q \vee t} S_{q \vee t}^* S_t S_r^*, & \text{if } q \vee t < \infty \\ 0, & \text{if } q \vee t = \infty \end{cases} \\ &= \begin{cases} S_{pq^{-1}q \vee t} (S_r S_t^* S_{q \vee t})^*, & \text{if } q \vee t < \infty \\ 0, & \text{if } q \vee t = \infty \end{cases} \\ &= \begin{cases} S_{pq^{-1}q \vee t} S_{rt^{-1}q \vee t}^*, & \text{if } q \vee t < \infty \\ 0, & \text{if } q \vee t = \infty. \end{cases}\end{aligned} \tag{2.5}$$

Because the final expression is an element of $\mathcal{D}$, the set is closed under multiplication and is therefore a unital $*$-subalgebra of $B(\mathcal{H})$.

To complete the proof, it remains to show that $\mathcal{D}$ is a dense $*$-subalgebra of $C^*(\{S_p : p \in P\})$. By definition, $C^*(\{S_p : p \in P\})$ is the smallest C^*-subalgebra of $B(\mathcal{H})$ that includes both 1 and the set $\{S_p : p \in P\}$ and so clearly $\mathcal{D} \subset C^*(\{S_p : p \in P\})$. It follows that $\mathcal{D}$ is a $*$-subalgebra of $C^*(\{S_p : p \in P\})$. Moreover, because $\mathcal{D}$ is a $*$-subalgebra of the bounded operators on $\mathcal{H}$, $\mathcal{D}$ must satisfy the Banach and C^*-identities. By lemma 2.2.11, the closure of $\mathcal{D}$ is a C^*-algebra. Because $C^*(\{S_p : p \in P\})$ is closed and includes $\mathcal{D}$, it follows that $\overline{\mathcal{D}} \subset C^*(\{S_p : p \in P\})$. On the other hand, if p is an element of P, then $S_p = S_p S_e^* \in \mathcal{D} \subset \overline{\mathcal{D}}$ so that $\overline{\mathcal{D}}$ is a C^*-algebra which includes both 1 and the set $\{S_p : p \in P\}$. It follows that $C^*(\{S_p : p \in P\}) \subset \overline{\mathcal{D}}$, proving that $\mathcal{D}$ is a dense unital $*$-subalgebra of $C^*(\{S_p : p \in P\})$. $\square$

The following lemma will be used in Corollary 2.2.14 to prove that the image space of the Toeplitz representation is linearly independent in $B(\ell^2(P))$.

Lemma 2.2.13. *Every non-empty finite subset F of P must contain an element q such that for all elements $p \in F$, if $p \leq q$ then $p = q$.*

Proof. Towards a contradiction, suppose that for some non-empty finite set $F \subset P$, there exists no element $p \in F$ with the property that every other element in F is not strictly less than p. Fix an element $p_1 \in F$. Suppose that for some integer n, there exists distinct elements $p_1, \ldots, p_n \in F$ such that $p_n \leq \cdots \leq p_1$. By assumption, there must exist an element $p_{n+1} \in F$ distinct from p_n such that $p_{n+1} \leq p_n$. Note that if p_{n+1} were equal to p_k for some $k \in \{1, \ldots, n-1\}$, then $p_{n+1} \leq p_n \leq p_k = p_{n+1}$ would contradict p_{n+1} being distinct from p_n. Hence, p_{n+1} must be distinct from the elements $p_1, \ldots, p_n$. By induction, F must contain an infinite sequence of distinct elements $(p_n)_{n \in \mathbb{N}}$ such that $p_1 \geq p_2 \geq \cdots$. However, this contradicts F being a finite set. Hence, by contradiction, every finite set F must contain an element p such that no other element of F is strictly less than p. $\square$

Corollary 2.2.14. *Let $T : P \to B(\ell^2(P))$ be the Toeplitz representation of (G, P). Then $\{T_p T_q^* : p, q \in P\}$ is a linearly independent set whose span is a dense unital $*$-subalgebra of the Toeplitz algebra $\mathcal{T}(G, P)$.*

Proof. Note that by Theorem 2.2.12, $\operatorname{span}\{T_p T_q^* : p, q \in P\}$ is a dense $*$-subalgebra of $C^*(\{T_p : p \in P\} = \mathcal{T}(G, P)$. Hence, it only remains to prove that $\{T_p T_q^* : p, q \in P\}$ is a linearly independent set. Towards a contradiction, suppose that there exist a finite set of nonzero scalars $\{\alpha_j : j \in \{1, \ldots, n\}\} \subset \mathbb{C}$ and distinct pairs of elements $(p_1, q_1), \ldots, (p_n, q_n)$ such that $\sum_{j=1}^n \alpha_j T_{p_j} T_{q_j}^* = 0$. Note that by Lemma 2.2.13, the set of elements $\{q_j : j \in \{1, \ldots, n\}\}$ must contain at least one element q_m such that for any $j \in \{1, \ldots, n\}$, if $q_j \leq q_m$ then $q_j = q_m$. Moreover because $T_{p_m} T_{q_m}^* (\delta_{q_m}) = \delta_{p_m}$, it follows that,

$$\alpha_m \delta_{p_m} + \sum_{\substack{j \neq m \\ q_j \leq q_m}} \alpha_j \delta_{p_j q_j^{-1} q_m} = 0.$$

Since $\alpha_m \delta_{p_m} \neq 0$ and $\{\delta_q : q \in P\}$ is a basis of $\ell^2(P)$, there must exist at least one $j \neq m$ such that $q_j \leq q_m$ and $p_j q_j^{-1} q_m = p_m$. Note that, $q_j \leq q_m$ implies that $q_j = q_m$, and so $p_j q_j^{-1} q_m = p_m$ implies that $p_j = p_m$. However, this contradicts the fact that the pairs $(p_1, q_1), \ldots, (p_n, q_n)$ are distinct. Therefore $\{T_p T_q^* : p, q \in P\}$ must be a linearly independent set. $\square$

Chapter 3

The Universal Algebra $C^*(G, P)$

3.1 Constructing the Universal Nica-Covariant Representation

After defining the Toeplitz algebra $\mathcal{T}(G, P)$ in [8], Nica defines a second C^*-algebra $C^*(G, P)$ associated to a quasi-lattice ordered group (G, P). He claims that this algebra is universal with respect to Nica-covariant representations. In this chapter, we explicitly construct $C^*(G, P)$ and prove Nica's claim that this algebra is universal for Nica-covariant representations.

3.1.1 Defining the $*$-Algebra $\mathcal{A}$

Throughout the remainder of this chapter, fix a weakly quasi-lattice ordered group (G, P). The goal for this section is to construct a C^*-algebra $C^*(G, P)$ and an associated Nica-covariant representation $U : P \to C^*(G, P)$ that have the following universal property: for any Nica-covariant representation $\pi_{\mathcal{H}} : P \to B(\mathcal{H})$, there exists a unique unital $*$-representation $\overline{\pi}_{\mathcal{H}} : C^*(G, P) \to B(\mathcal{H})$ such that $\overline{\pi}_{\mathcal{H}} \circ U = \pi_{\mathcal{H}}$. We then prove that $C^*(G, P)$ and its associated Nica-covariant representation U are unique up to isomorphism with respect to these properties.

Definition 3.1.1. *Define $C_c(P \times P)$ to be the set of continuous compactly supported functions $f : P \times P \to \mathbb{C}$. For any $(p, q) \in P \times P$, let $\chi_{p,q}$ be the characteristic function on $\{(p, q)\}$.*

We aim to define a $*$-algebra $\mathcal{A}$ by defining a product and involution on $C_c(P \times P)$. Note that $C_c(P \times P)$ is already a complex vector space under point-wise addition and scalar multiplication. Because the positive cone P inherits the discrete topology from G, every function $f : P \times P \to \mathbb{C}$ is continuous. Furthermore, because only finite sets are compact in the discrete topology, the set $C_c(P \times P)$ may be characterised as the set of all functions $f : P \times P \to \mathbb{C}$ such that f is nonzero for finitely many $(p, q) \in P \times P$. It follows that $C_c(P \times P) = \operatorname{span}(\{\chi_{p,q} : (p, q) \in P \times P\})$. Hence, to define the $*$-algebra $\mathcal{A}$, it suffices to define the product and involution for basis elements $\chi_{p,q}$. In defining the multiplication and involution operations on the basis elements $\chi_{p,q}$, we are guided by the multiplication and involution for an arbitrary Nica-covariant representation $S : P \to B(\mathcal{H})$ determined in equations (2.4) and (2.5) above.

Definition 3.1.2. *For any basis elements $\chi_{p,q}, \chi_{t,r}$ in $\mathcal{A}$, we define the product and involution operations as follows:*

$$\chi_{p,q}\chi_{t,r} := \begin{cases} \chi_{pq^{-1}(q \vee t),\, rt^{-1}(q \vee t)} & q \vee t < \infty \\ 0 & q \vee t = \infty \end{cases}$$

$$\chi^*_{p,q} := \chi_{q,p}$$

Proposition 3.1.3. *By extending the product and involution operations defined above to any element in the vector space $\mathcal{A} = span\{\chi_{p,q} : (p,q) \in P \times P\}$ in the obvious way, $\mathcal{A}$ becomes a unital $*$-algebra.*

Proof. To prove that $\mathcal{A}$ is an algebra under the product defined above, we only need to check that the product is associative. It clearly suffices to prove that the product is associative on basis elements. To this end, fix basis elements $\chi_{p,q}, \chi_{t,r}, \chi_{s,v} \in \mathcal{A}$. We have that:

$$\chi_{p,q}(\chi_{t,r}\chi_{s,v}) = \begin{cases} \chi_{p,q}\chi_{tr^{-1}r\vee s,\, vs^{-1}r\vee s} & r \vee s < \infty \\ 0 & r \vee s = \infty \end{cases}$$

$$= \begin{cases} \chi_{pq^{-1}(q\vee(tr^{-1}r\vee s)),\, vs^{-1}r\vee s(tr^{-1}r\vee s)^{-1}(q\vee(tr^{-1}r\vee s))} & q \vee (tr^{-1}r \vee s) < \infty \text{ and } r \vee s < \infty \\ 0 & \text{otherwise} \end{cases}$$

On the other hand,

$$(\chi_{p,q}\chi_{t,r})\chi_{s,v} = \begin{cases} \chi_{pq^{-1}q\vee t,\, rt^{-1}q\vee t}\chi_{s,v} & q \vee t < \infty \\ 0 & q \vee t = \infty \end{cases}$$

$$= \begin{cases} \chi_{pq^{-1}q\vee t(rt^{-1}q\vee t)^{-1}((rt^{-1}q\vee t)\vee s),\, vs^{-1}((rt^{-1}q\vee t)\vee s)} & (rt^{-1}q \vee t) \vee s < \infty \text{ and } q \vee t < \infty \\ 0 & \text{otherwise} \end{cases}$$

Thus, to prove associativity of the product, we must show that the following relations in P hold:

(a) $q \vee (tr^{-1}r \vee s) < \infty$ and $r \vee s < \infty$ if and only if $(rt^{-1}q \vee t) \vee s < \infty$ and $q \vee t < \infty$;

(b) When $r \vee s < \infty$ and $q \vee (tr^{-1}r \vee s) < \infty$,

$$\begin{aligned} pq^{-1}(q \vee (tr^{-1}r \vee s)) &= pq^{-1}q \vee t(rt^{-1}q \vee t)^{-1}((rt^{-1}q \vee t) \vee s); \text{ and} \\ vs^{-1}r \vee s(tr^{-1}r \vee s)^{-1}(q \vee (tr^{-1}r \vee s)) &= vs^{-1}((rt^{-1}q \vee t) \vee s) \end{aligned} \tag{3.1}$$

(a) Suppose that $q \vee (tr^{-1}r \vee s) < \infty$ and $r \vee s < \infty$. By definition, $q \leq q \vee (tr^{-1}r \vee s)$ and, because $r \vee s < \infty$, it follows that $r^{-1}r \vee s$ is an element of P. Hence, $t \leq tr^{-1}r \vee s \leq q \vee (tr^{-1}r \vee s)$, proving that q and t share a common upper bound in P. By the weak quasi-lattice condition (WQL), $q \vee t < \infty$. Now, towards proving that $(rt^{-1}q \vee t) \vee s < \infty$, note that because $tr^{-1}r \vee s \leq q \vee (tr^{-1}r \vee s)$, it follows that

$$s \leq r \vee s \leq rt^{-1}(q \vee (tr^{-1}r \vee s)). \tag{3.2}$$

Moreover, because $t \leq q \vee t$ and $q \vee t \leq q \vee (tr^{-1}r \vee s)$, we also have that

$$rt^{-1}q \vee t \leq rt^{-1}(q \vee (tr^{-1}r \vee s)). \tag{3.3}$$

Inequalities (3.2) and (3.3), together with the weak quasi-lattice condition (WQL), imply

$$(rt^{-1}q \vee t) \vee s \leq rt^{-1}(q \vee (tr^{-1}r \vee s)) < \infty. \tag{3.4}$$

Towards proving the "only if" direction, suppose that $q \vee t < \infty$ and $(rt^{-1}q \vee t) \vee s < \infty$. By definition we have that $s \leq (rt^{-1}q \vee t) \vee s$. Because $t^{-1}q \vee t$ is an element of P, it follows that $r \leq rt^{-1}q \vee t \leq (rt^{-1}q \vee t) \vee s$. Piecing the above two sentences together and applying (WQL) gives $r \vee s \leq (rt^{-1}q \vee t) \vee s < \infty$. Now, towards showing that $q \vee (tr^{-1}r \vee s) < \infty$, note that because $rt^{-1}q \vee t \leq (rt^{-1}q \vee t) \vee s$, it follows that

$$q \leq q \vee t \leq tr^{-1}((rt^{-1}q \vee t) \vee s). \tag{3.5}$$

Moreover, because we showed above that $r \vee s \leq (rt^{-1}q \vee t) \vee s$, we have that

$$tr^{-1}r \vee s \leq tr^{-1}((rt^{-1}q \vee t) \vee s). \tag{3.6}$$

Inequalities (3.5) and (3.6), together with (WQL) imply

$$q \vee (tr^{-1}r \vee s) \leq tr^{-1}((rt^{-1}q \vee t) \vee s) < \infty. \tag{3.7}$$

(b) Suppose that $r \vee s < \infty$ and $q \vee (tr^{-1}r \vee s) < \infty$. By (a), it follows that $q \vee t < \infty$ and $(rt^{-1}q \vee t) \vee s < \infty$. By rearranging inequality (3.4) above we have that, $tr^{-1}((rt^{-1}q \vee t) \vee s) \leq q \vee (tr^{-1}r \vee s)$, and by inequality (3.7), we actually have equality. That is,

$$tr^{-1}((rt^{-1}q \vee t) \vee s) = q \vee (tr^{-1}r \vee s).$$

Hence,

$$\begin{aligned} pq^{-1}(q \vee (tr^{-1}r \vee s) &= pq^{-1}tr^{-1}((rt^{-1}q \vee t) \vee s) \\ &= pq^{-1}q \vee t(rt^{-1}q \vee t)^{-1}((rt^{-1}q \vee t) \vee s) \end{aligned}$$

and

$$\begin{aligned} vs^{-1}((rt^{-1}q \vee t) \vee s) &= vs^{-1}rt^{-1}(q \vee (tr^{-1}q \vee s)) \\ &= vs^{-1}r \vee s(tr^{-1}r \vee s)^{-1}(q \vee (tr^{-1}q \vee s)). \end{aligned}$$

It follows that the product operation on $\mathcal{A}$ is associative, which proves that $\mathcal{A}$ is an algebra.

We claim that $\chi_{e,e}$ is the identity element of $\mathcal{A}$. To this end, fix a basis element $\chi_{p,q}$ in $\mathcal{A}$. Note that for any $t \in P$, we have that $e \leq t$ and $t \leq t$ so that $e \vee t = t \vee e$ exists and is at most t. Because t must also be bounded above by $e \vee t$, it follows that $e \vee t = t \vee e = t$. Hence, we have that $\chi_{e,e}\chi_{p,q} = \chi_{ee^{-1}e\vee p,\, qp^{-1}e \vee p} = \chi_{p,q}$ and $\chi_{p,q}\chi_{e,e} = \chi_{pq^{-1}q\vee e,\, ee^{-1}q \vee e} = \chi_{p,q}$, proving $\chi_{e,e}$ is indeed the identity of $\mathcal{A}$.

Finally, to prove that $\mathcal{A}$ is a $*$-algebra under the involution defined above, it suffices to prove that $\chi_{e,e}^* = \chi_{e,e}$ and for any basis elements $\chi_{p,q}, \chi_{t,r}$ in $\mathcal{A}$, $(\chi_{p,q}\chi_{t,r})^* = \chi_{t,r}^*\chi_{p,q}^*$. Note that the former relation simply holds by the definition of $*$. Fix basis elements $\chi_{p,q}, \chi_{t,r} \in \mathcal{A}$. We have that,

$$\begin{aligned} (\chi_{p,q}\chi_{t,r})^* &= \begin{cases} \chi^*_{pq^{-1}q\vee t,\, rt^{-1}q \vee t} & q \vee t < \infty \\ 0 & q \vee t = \infty \end{cases} \\ &= \begin{cases} \chi_{rt^{-1}q\vee t,\, pq^{-1}q \vee t} & q \vee t < \infty \\ 0 & q \vee t = \infty \end{cases} \\ &= \chi_{r,t}\chi_{q,p} = \chi_{t,r}^*\chi_{p,q}^* \end{aligned}$$

Thus, $\mathcal{A}$ is indeed a unital $*$-algebra under the product and involution operations defined in Definition 3.1.2. □

The following proposition proves that the algebraic structure of the unital $*$-algebra $\mathcal{A}$ internalises the necessary underlying algebraic structure of all Nica-covariant representations of P.

Proposition 3.1.4. *For any Nica-covariant representation $S : P \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, there exists a unique unital $*$-homomorphism $\pi_S : \mathcal{A} \to B(\mathcal{H})$ such that $\pi_S(\chi_{p,q}) = S_pS_q^*$ for all basis elements $\chi_{p,q} \in \mathcal{A}$.*

Proof. Define $\pi_S : \mathcal{A} \to B(\mathcal{H})$ as follows: For any finite subset $F \subset P \times P$ and element $\sum_{(p,q)\in F} z_{(p,q)}\chi_{p,q}$ in $\mathcal{A}$, define

$$\pi_S(\sum_{(p,q)\in F} z_{(p,q)}\chi_{p,q}) := \sum_{(p,q)\in F} z_{(p,q)}S_pS_q^*.$$

Clearly π_S is a linear map. Note that $\pi_S(\chi_{e,e}) = S_eS_e^* = 1$ where 1 is the identity operator in $B(\mathcal{H})$, which proves that π_S is unital. Moreover, for any basis elements $\chi_{p,q}, \chi_{t,r} \in \mathcal{A}$,

$$\begin{aligned}\pi_S(\chi_{p,q}\chi_{t,r}) &= \begin{cases}\pi_S(\chi_{pq^{-1}q\vee t,\, rt^{-1}q\vee t}) & q\vee t < \infty \\ \pi_S(0) & q \vee t = \infty\end{cases} \\ &= \begin{cases}S_{pq^{-1}q\vee t}S^*_{rt^{-1}q\vee t} & q\vee t < \infty \\ 0 & q \vee t = \infty\end{cases} \\ &= S_pS_q^*S_tS_r^* = \pi_S(\chi_{p,q}) \circ \pi_S(\chi_{t,r}),\end{aligned} \tag{3.8}$$

where the third equality follows from equation (2.5). By the linearity of π_S and (3.8), it follows that π_S preserves the product structure of $\mathcal{A}$. Finally, observe that $\pi_S(\chi^*_{p,q}) = \pi_S(\chi_{q,p}) = S_qS_p^* = \pi_S(\chi_{p,q})^*$, which proves by the linearity of π_S that π_S preserves the involution operation. It follows that π_S is indeed a unital $*$-representation of $\mathcal{A}$ such that $\pi_S(\chi_{p,q}) = S_pS_q^*$ for all basis elements $\chi_{p,q} \in \mathcal{A}$. Note that by the linearity of unital $*$-homomorphism of $\mathcal{A}$ into $B(\mathcal{H})$, π_S is the unique such unital $*$-representation of $\mathcal{A}$ which sends basis elements $\chi_{p,q}$ to the operator $S_pS_q^* \in B(\mathcal{H})$. □

3.1.2 The Universal Nica-Covariant Representation

Now that we have successfully defined a $*$-algebra $\mathcal{A}$ that captures the underlying algebraic structure of all Nica-covariant representations, we seek to extend $\mathcal{A}$ to a C^*-algebra by defining an appropriate norm on $\mathcal{A}$ and then taking the completion.

Definition 3.1.5. *Define* $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to [0,\infty)$ *as follows:*

$$\|f\|_{\mathcal{A}} := \sup\{\|\pi(f)\| : \pi \textit{ is a unital } *\textit{-homomorphism of } \mathcal{A} \textit{ on some Hilbert space } \mathcal{H}\}.$$

Theorem 3.1.6. *The mapping* $\|\cdot\|_{\mathcal{A}} : \mathcal{A} \to [0,\infty)$ *is well-defined and, in particular, defines a norm on* $\mathcal{A}$ *that satisfies both the Banach and* C^**-identities.*

Proof. Towards proving that $\|\cdot\|_{\mathcal{A}}$ is well-defined, fix an element $f \in \mathcal{A}$. Note that although the class of all unital $*$-homomorphisms of $\mathcal{A}$ may not be a set, the norm defined in definition 3.1.5 takes a supremum over a subset of real numbers. Hence, to prove that $\|\cdot\|_{\mathcal{A}}$ is well-defined, it suffices to prove that the set

$$\Pi_f := \{\|\pi(f)\| : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\}$$

is non-empty and bounded above. Since the Toeplitz representation is Nica-covariant, by Proposition 3.1.4 there exists a unique unital $*$-homomorphism $\pi : \mathcal{A} \to B(\ell^2(P))$ such that $\pi(\chi_{p,q}) = T_pT_q^*$ for all $p, q \in P$. Hence, $\|\pi(f)\| \in \Pi_f$ which proves that Π_f is non-empty.

To show that Π_f is bounded above, fix a unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Observe that because $\mathcal{A} = \text{span}\{\chi_{p,q} : (p,q) \in P \times P\}$, there must exist a finite subset $F \subset P \times P$ and a set $\{z_{(p,q)}\}_{(p,q)\in F} \subset \mathbb{C}$ such that $f = \sum_{(p,q)\in F} z_{(p,q)}\chi_{p,q}$. In fact, we can further simplify this expression by noting that because each basis element $\chi_{p,q}$ is the characteristic function on $\{(p,q)\}$, it follows that $f = \sum_{(p,q)\in F} f(p,q)\chi_{p,q}$. Thus,

$\pi(f) = \sum_{(p,q)\in F} f(p,q)\pi(\chi_{p,q})$. We claim that for any basis element $\chi_{p,q} \in \mathcal{A}$, $\|\pi(\chi_{p,q})\| = 1$ which would prove that $\|\pi(f)\| \leq \sum_{(p,q)\in F} |f(p,q)|$. To this end, fix a basis element $\chi_{p,q} \in \mathcal{A}$ and note that,

$$\pi(\chi_{p,q})^* \circ \pi(\chi_{p,q}) = \pi(\chi_{p,q}^* \chi_{p,q}) = \pi(\chi_{q,p}\chi_{p,q}) = \pi(\chi_{qp^{-1}p\vee p,\, qp^{-1}p\vee p}) = \pi(\chi_{q,q}).$$

Hence, $\|\pi(\chi_{p,q})\|^2 = \|\pi(\chi_{p,q})^*\pi(\chi_{p,q})\| = \|\pi(\chi_{q,q})\|$. Because $\pi(\chi_{q,q})^* = \pi(\chi_{q,q})$ and $\pi(\chi_{q,q})\pi(\chi_{q,q}) = \pi(\chi_{q,q}\chi_{q,q}) = \pi(\chi_{q,q})$, it follows that $\pi(\chi_{q,q})$ is a projection in $B(\mathcal{H})$. Therefore, $\|\pi(\chi_{p,q})\| = \sqrt{\|\pi(\chi_{q,q})\|} = 1$, proving that $\|\pi(f)\| \leq \sum_{(p,q)\in F} |f(p,q)|$. Because $\pi : \mathcal{A} \to B(\mathcal{H})$ was any unital $*$-homomorphism of $\mathcal{A}$, the set Π_f is bounded above by $\sum_{(p,q)\in F} |f(p,q)|$ and so $\|\cdot\|_{\mathcal{A}}$ is indeed well-defined.

It remains to prove that $\|\cdot\|_{\mathcal{A}}$ defines a norm on $\mathcal{A}$ satisfying the Banach and C^*-identities. Towards proving that $\|\cdot\|_{\mathcal{A}}$ is subadditive, fix $f, g \in \mathcal{A}$ and note that, for any unital $*$-homomorphism π of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, $\|\pi(f+g)\| = \|\pi(f) + \pi(g)\| \leq \|\pi(f)\| + \|\pi(g)\| \leq \|f\|_{\mathcal{A}} + \|g\|_{\mathcal{A}}$. It follows that $\|f+g\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} + \|g\|_{\mathcal{A}}$, proving subadditivity. Note that $\|\cdot\|_{\mathcal{A}}$ is also absolutely homogeneous because for any $\lambda \in \mathbb{C}$ and $f \in \mathcal{A}$,

$$\|\lambda f\|_{\mathcal{A}} = \sup \Pi_{\lambda f} = \sup |\lambda| \Pi_f = |\lambda| \sup \Pi_f = |\lambda| \|f\|_{\mathcal{A}}.$$

Towards proving positive definiteness, fix an element f in $\mathcal{A}$. Note that by the absolute homogeneity of $\|\cdot\|_{\mathcal{A}}$, it suffices to prove that if $\|f\|_{\mathcal{A}} = 0$, then $f = 0$. To that end, suppose that $\|f\|_{\mathcal{A}} = 0$. Then for all unital $*$-homomorphisms $\pi : \mathcal{A} \to B(\mathcal{H})$, it must be the case that $\|\pi(f)\| = 0$ which further implies that $\pi(f) = 0_{\mathcal{H}}$. By Proposition 3.1.4, there exists a (unique) unital $*$-homomorphism $\pi_T : \mathcal{A} \to B(\ell^2(P))$ such that for any basis element $\chi_{p,q}$ of $\mathcal{A}$, $\pi_T(\chi_{p,q}) = T_p T_q^*$, where $T : P \to B(\ell^2(P))$ is the Toeplitz representation of (G, P). Let $F \subset P \times P$ be a finite support for f so that $f = \sum_{(p,q)\in F} f(p,q)\chi_{p,q}$. Then,

$$0 = \pi_T(f) = \sum_{(p,q)\in F} f(p,q) T_p T_q^*.$$

By Corollary 2.2.14, the set $\{T_p T_q^* : p, q \in P\}$ is linearly independent. It follows that $f(p,q) = 0$ for all $(p,q) \in F$. Because F is a support for f, it follows that for all $(p,q) \in P \times P$, it must be the case that $f(p,q) = 0$. Hence, $f = 0$, proving that $\|\cdot\|_{\mathcal{A}}$ is positive definite and therefore a norm on $\mathcal{A}$.

Finally, with the goal of proving that $\|\cdot\|_{\mathcal{A}}$ satisfies the Banach and C^*-identities, fix elements f, g of $\mathcal{A}$ and a unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Observe that,

$$\|\pi(fg)\| = \|\pi(f) \circ \pi(g)\| \leq \|\pi(f)\|\|\pi(g)\| \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}.$$

Because the final inequality is independent of the choice of unital $*$-homomorphism, it follows that $\|fg\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}\|g\|_{\mathcal{A}}$ for all $f, g \in \mathcal{A}$, proving that $\|\cdot\|_{\mathcal{A}}$ satisfies the Banach identity. Moreover, because $B(\mathcal{H})$ is a C^*-algebra for any Hilbert space $\mathcal{H}$,

$$\begin{aligned}
\|f^*f\|_{\mathcal{A}} &= \sup\{\|\pi(f^*f)\| : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\} \\
&= \sup\{\|\pi(f)^*\pi(f)\| : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\} \\
&= \sup\{\|\pi(f)\|^2 : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\} \\
&= \sup\{\|\pi(f)\| : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\}^2 \\
&= \|f\|_{\mathcal{A}}^2.
\end{aligned}$$

Hence, $\|\cdot\|_{\mathcal{A}}$ also satisfies the C^*-identity. □

Above we proved that there exists a unique unital $*$-homomorphism of $\mathcal{A}$ into $B(\mathcal{H})$ associated to any Nica-covariant representation $S : P \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Next, we prove that the unique unital $*$-homomorphism associated to any Nica-covariant representation is continuous and, therefore, a unital $*$-representation of $\mathcal{A}$.

Corollary 3.1.7. *For any Nica-covariant representation $S : P \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, there exists a unique unital $*$-representation $\pi_S : \mathcal{A} \to B(\mathcal{H})$ such that $\pi_S(\chi_{p,q}) = S_p S_q^*$ for all basis elements $\chi_{p,q} \in \mathcal{A}$.*

Proof. Fix a Nica-covariant representation $S : P \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. By Proposition 3.1.4, there exists a unique unital $*$-homomorphism $\pi_S : \mathcal{A} \to B(\mathcal{H})$ such that $\pi_S(\chi_{p,q}) = S_p S_q^*$ for all basis elements $\chi_{p,q} \in \mathcal{A}$. Thus, to prove existence, it suffices to prove that π_S is bounded. To this end, fix an element $f \in \mathcal{A}$ and note that,

$$\begin{aligned}\|\pi_S(f)\| &\leq \sup\{\|\pi(f)\| : \pi \text{ is a unital } *\text{-homomorphism of } \mathcal{A} \text{ on some Hilbert space } \mathcal{H}\} \\ &= \|f\|_{\mathcal{A}}.\end{aligned}$$

It follows that π_S is bounded with operator norm $\|\pi_S\| \leq 1$ and, therefore, that π_S is a unital $*$-representation of $\mathcal{A}$. Moreover, because any unital $*$-representation $\pi' : P \to B(\mathcal{H})$ is also a unital $*$-homomorphism, by Proposition 3.1.4, π_S must be the unique unital $*$-representation of $\mathcal{A}$ that sends basis elements $\chi_{p,q} \in \mathcal{A}$ to $S_p S_q^*$. ☐

Theorem 3.1.8. *There exists a unique unital C^*-algebra $\mathcal{C}$ generated by a Nica-covariant representation $U : P \to \mathcal{C}$ such that whenever $\mathcal{B}$ is a C^*-algebra and $S : P \to \mathcal{B}$ is a Nica-covariant representation, there exists a unique unital $*$-representation $\pi_S : \mathcal{C} \to \mathcal{B}$ such that $\pi_S \circ U = S$. Furthermore, the C^*-algebra $\mathcal{C}$ is unique up to isomorphism with respect to this property.*

Proof. Let $\mathcal{C}$ be the completion of the normed $*$-algebra $\mathcal{A}$. By Theorem 3.1.6, the norm on $\mathcal{A}$ satisfies the Banach and C^*-identities. Hence, by Lemma 2.2.11, $\mathcal{C}$ is a C^*-algebra. Now define a map $U : P \to \mathcal{C}$ by,

$$U : p \mapsto \chi_{p,e}.$$

We claim that U defines a Nica-covariant representation of P. Note that U preserves multiplication since for any elements $p, q \in P$,

$$U(p)U(q) = \chi_{p,e}\chi_{q,e} = \chi_{pe^{-1}(e\vee q),\, eq^{-1}(e\vee q)} = \chi_{pq,e} = U(pq).$$

Moreover, as $\chi_{e,e} = U(e)$ is the multiplicative identity in $\mathcal{C}$, it follows that $U : P \to \mathcal{C}$ is monoid homomorphism and, thus, a representation of P. Finally, towards proving that U is a Nica-covariant representation, fix $p, q \in P$ and note that,

$$\begin{aligned}
U(p)U(p)^*U(q)U(q)^* &= \chi_{p,e}\chi_{p,e}^*\chi_{q,e}\chi_{q,e}^* \\
&= \chi_{p,e}\chi_{e,p}\chi_{q,e}\chi_{e,q} \\
&= \chi_{pe^{-1}(e\vee e)\,,\,pe^{-1}(e\vee e)}\chi_{qe^{-1}(e\vee e)\,,\,qe^{-1}(e\vee e)} \\
&= \chi_{p,p}\chi_{q,q} \\
&= \begin{cases} \chi_{pp^{-1}(p\vee q)\,,\,qq^{-1}(p\vee q)} & p\vee q < \infty \\ 0 & p \vee q = \infty \end{cases} \\
&= \begin{cases} \chi_{(p\vee q)\,,\,(p\vee q)} & p\vee q < \infty \\ 0 & p \vee q = \infty \end{cases} \\
&= \begin{cases} \chi_{(p\vee q)\,,\,e}\,\chi_{(p\vee q)\,,\,e}^* & p\vee q < \infty \\ 0 & p \vee q = \infty \end{cases} \\
&= \begin{cases} U(p\vee q)U(p\vee q)^* & p\vee q < \infty \\ 0 & p \vee q = \infty. \end{cases}
\end{aligned}$$

It follows that U satisfies equation (2.3), proving that U is indeed a Nica-covariant representation.

Towards proving that $\mathcal{C}$ is equal to the C^*-algebra $C^*(U) := C^*(\{U(p) : p \in P\})$ generated by U, note that any basis element $\chi_{p,q}$ of $\mathcal{C}$ is equal to $\chi_{p,e}\chi_{e,q} = U(p)U(q)^*$, which is an element $C^*(U)$. It follows that $C^*(U)$ must include all linear combinations and finite products of basis elements of $\mathcal{C}$. That is, $C^*(U)$ must include $\mathcal{A}$. Because $C^*(U)$ is a closed subset of $\mathcal{C}$, the set must also include the closure $\overline{\mathcal{A}} = \mathcal{C}$. Thus, the C^*-algebra $C^*(U)$ generated by U is equal to $\mathcal{C}$.

It remains to prove that whenever $\mathcal{B}$ is a C^*-algebra and $S : P \to \mathcal{B}$ is a Nica-covariant representation, there exists a unital $*$-representation $\pi_S : \mathcal{C} \to \mathcal{B}$ such that $\pi_S \circ U = S$. To this end, fix a C^*-algebra $\mathcal{B}$ and Nica-covariant representation $S : P \to \mathcal{B}$. By Corollary 3.1.7, there exists a unique unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}$ such that $\pi(\chi_{p,q}) = S_pS_q^*$ for all basis elements $\chi_{p,q} \in \mathcal{A}$. Because π is bounded and $\mathcal{A}$ is dense in $\mathcal{C}$, π extends uniquely to a continuous function $\pi_S : \mathcal{C} \to \mathcal{A}$. Towards proving that π_S is a unital $*$-representation, note that it suffices to prove that π_S is a unital homomorphism. Because $\pi_S(\chi_{e,e}) = \pi(\chi_{e,e}) = e_{\mathcal{B}}$, π_S is unital. Fix elements $x, y \in \mathcal{C}$ and $a, b \in \mathbb{C}$. Because $\mathcal{A}$ is dense in $\mathcal{C}$, there must exist sequences $(x_n), (y_n) \subset \mathcal{A}$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then by the continuity of the operations in $\mathcal{C}$, $ax_n + by_n \to ax + by$, $x_ny_n \to xy$, and $x_n^* \to x^*$. Hence, by the continuity of π_S, it follows that,

$$\begin{aligned}
&\pi_S(ax+by) = \lim_{n\to\infty} \pi_S(ax_n + by_n) = \lim_{n\to\infty} \pi(ax_n + by_n) = \lim_{n\to\infty} (a\pi_S(x_n) + b\pi_S(y_n)) \\
&= a\pi_S(x) + b\pi_S(y), \\
&\pi_S(xy) = \lim_{n\to\infty} \pi_S(x_ny_n) = \lim_{n\to\infty} \pi(x_ny_n) = \lim_{n\to\infty} \pi_S(x_n)\pi_S(y_n) = \pi_S(x)\pi_S(y), \text{ and} \\
&\pi_S(x^*) = \lim_{n\to\infty} \pi_S(x_n^*) = \lim_{n\to\infty} \pi(x_n^*) = \lim_{n\to\infty} \pi(x_n)^* = \lim_{n\to\infty} \pi_S(x_n)^* = \pi_S(x)^*,
\end{aligned}$$

proving that π_S is a homomorphism and, thus, a unital $*$-representation. Note that for any $p \in P$, $\pi_S \circ U(p) = \pi_S(\chi_{p,e}) = \pi(\chi_{p,e}) = S_pS_e^* = S_p$. It follows that $\pi_S \circ U = S$. To see that π_S is the unique unital $*$-representation of P such that $\pi_S \circ U = S$, suppose that $\varphi_S : \mathcal{C} \to \mathcal{B}$ is another $*$-representation such that $\varphi_S \circ U = S$. Then for any basis element

$\chi_{p,q}$ of $\mathcal{C}$, $\varphi_S(\chi_{p,q}) = \varphi_S(\chi_{p,e}\chi_{q,e}^*) = \varphi_S \circ U(p)(\varphi_S \circ U(q))^* = S_p S_q^* = U(\chi_{p,q})$. Hence, φ_S must agree with π_S over the dense subset $\mathcal{A}$. Because φ_S and π_S are continuous and agree on a dense subset of $\mathcal{C}$, they must agree over all of $\mathcal{C}$, proving the uniqueness of π_S.

To show that $\mathcal{C}$ is the unique (up to isomorphism) C^*-algebra generated by a Nica-covariant representation satisfying the conditions of Theorem 3.1.8, suppose that $\mathcal{C}'$ is another C^*-algebra generated by a Nica-covariant representation $U' : P \to \mathcal{C}'$ that satisfies the conditions of Theorem 3.1.8. Then there must exist unique unital $*$-representations $\pi_{\mathcal{C}'} : \mathcal{C} \to \mathcal{C}'$ and $\pi_{\mathcal{C}} : \mathcal{C}' \to \mathcal{C}$ such that $\pi_{\mathcal{C}'} \circ U = U'$ and $\pi_{\mathcal{C}} \circ U' = U$. It follows that $\pi_{\mathcal{C}} \circ \pi_{\mathcal{C}'}$ is a unital $*$-representation such that $(\pi_{\mathcal{C}} \circ \pi_{\mathcal{C}'}) \circ U = \pi_{\mathcal{C}} \circ U' = U$. Because the identity $Id_{\mathcal{C}}$ on $\mathcal{C}$ is the unique unital $*$-representation such that $Id_{\mathcal{C}} \circ U = U$, it must be the case that $\pi_{\mathcal{C}} \circ \pi_{\mathcal{C}'} = Id_{\mathcal{C}}$. Moreover, $\pi_{\mathcal{C}'} \circ \pi_{\mathcal{C}}$ is a unital $*$-representation such that $(\pi_{\mathcal{C}'}\pi_{\mathcal{C}}) \circ U' = \pi_{\mathcal{C}'} \circ U = U'$. But the identity $Id_{\mathcal{C}'}$ on $\mathcal{C}'$ is the unique unital $*$-representation such that $Id_{\mathcal{C}'} \circ U' = U'$ and so it follows that $\pi_{\mathcal{C}'} \circ \pi_{\mathcal{C}} = Id_{\mathcal{C}'}$. That is, $\pi_{\mathcal{C}'} : \mathcal{C} \to \mathcal{C}'$ is a bounded unital $*$-isomorphism. Because continuous $*$-isomorphisms between C^*-algebras are automatically isometric, it follows that $\mathcal{C}$ is isometrically isomorphic to $\mathcal{C}'$, which proves the uniqueness of $\mathcal{C}$. □

Definition 3.1.9. *Let (G, P) be a weakly quasi-lattice ordered group. The **Universal algebra** $C^*(G, P)$ and **Universal Nica-covariant representation** $U : P \to C^*(G, P)$ are defined to be the unique (up to isomorphism) C^*-algebra and Nica-covariant representation that satisfy the conditions of Theorem 3.1.8.*

Corollary 3.1.10. *Every C^*-algebra generated by a Nica-covariant representation of a weakly quasi-lattice ordered group (G, P) is isomorphic to a quotient of the universal algebra $C^*(G, P)$.*

Proof. Fix a weak quasi-lattice (G, P) and let $S : P \to B(\mathcal{H})$ be a Nica-covariant representation of (G, P) on some Hilbert space $\mathcal{H}$. By Theorem 3.1.8, there exists a unique unital $*$-representation $\pi_S : C^*(G, P) \to B(\mathcal{H})$ such that $\pi_S \circ U = S$ where π_S sends a basis element $\chi_{p,q}$ to $\pi_S(\chi_{p,q}) = S_p S_q^*$. Note that to prove that the C^*-algebra $C^*(\{S_p : p \in P\})$ generated by S is a quotient of $C^*(G, P)$, it suffices to prove that $C^*(\{S_p : p \in P\}) \cong C^*(G, P)/\ker \pi_S$. By Theorem 2.2.12, the set $span(\{S_p S_q^* : p, q \in P\})$ is a dense unital $*$-subalgebra of the C^*-algebra $C^*(\{S_p : p \in P\})$ generated by S. Because the image of π_S is a C^*-algebra with dense $*$-subalgebra $span(\{S_p S_q^* : p, q \in P\})$, it follows that the image of π_S is equal to $\overline{span}\{S_p S_q^* : p, q \in P\} = C^*(\{S_p : p \in P\})$. The first isomorphism theorem for C^*-algebras implies that $C^*(G, P)/\ker \pi_S \cong C^*(\{S_p : p \in P\})$, proving that the C^*-algebra generated by S is a quotient of $C^*(G, P)$. □

3.2 Amenable Weakly Quasi-Lattice Ordered Groups

Because every Nica-covariant representation of a weakly quasi-lattice ordered group (G, P) generates a C^*-algebra that is isomorphic to a quotient of the universal algebra $C^*(G, P)$, studying the quotient spaces of $C^*(G, P)$ provides a systematic method for studying the Nica-covariant representations of (G, P). Of particular importance is the case when $C^*(G, P)$ is isomorphic to the Toeplitz algebra $\mathcal{T}(G, P)$. Due to the fact that computation is relatively simple in the algebra $\mathcal{T}(G, P)$, studying the quotient spaces of $\mathcal{T}(G, P)$ is, in general, an easier task than studying the quotient spaces of $C^*(G, P)$. Hence, studying the class of Nica-covariant representations of (G, P) is simplified in the case when $C^*(G, P)$ is isomorphic to $\mathcal{T}(G, P)$.

Definition 3.2.1. *Let $U : P \to C^*(G, P)$ be the universal representation of a weak quasi-lattice (G, P) and $T : P \to \mathcal{T}(G, P)$ its Toeplitz representation. By Theorem 3.1.8, there exists a unique unital $*$-representation $\pi_{\mathcal{T}} : C^*(G, P) \to \mathcal{T}(G, P)$ such that $\pi_{\mathcal{T}}(\chi_{p,q}) = T_p T_q^*$ for all basis elements $\chi_{p,q}$ of $C^*(G, P)$. For convenience we define $\pi_{\mathcal{T}}$ to be the* ***Toeplitz extension*** *of (G, P).*

Definition 3.2.2. *A weakly quasi-lattice ordered group (G, P) is called* ***amenable*** *if and only if the Toeplitz extension is injective. As the Toeplitz extension is surjective, amenability of (G, P) is equivalent to the condition that,*

$$C^*(G, P) \cong \mathcal{T}(G, P).$$

Because the universal algebra $C^*(G, P)$ is uniquely determined up to isomorphism by the condition that every Nica-covariant representation $S : P \to B(\mathcal{H})$ extends uniquely to a unital $*$-representation $\pi_S : C^*(G, P) \to B(\mathcal{H})$, it follows that the following is an equivalent formulation of amenability.

Proposition 3.2.3. *A weak quasi-lattice (G, P) is amenable if and only if for any Nica-covariant representation $S : P \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, there exists a unique unital $*$-representation $\pi_S : \mathcal{T}(G, P) \to B(\mathcal{H})$ such that $\pi_S \circ T = S$.*

Chapter 4

The Conditional Expectation E_U

In this chapter we consider distinguished Abelian sub-C^*-algebras $\mathcal{D}_U$ of $C^*(G, P)$ and $\mathcal{D}$ of $\mathcal{T}(G, P)$. We then show that there exists a conditional expectation $E : \mathcal{T}(G, P) \to \mathcal{D}$. After proving that $\mathcal{D}$ and $\mathcal{D}_U$ are isomorphic, we then obtain a conditional expectation $E_U : C^*(G, P) \to \mathcal{D}_U$ by composing with the Toeplitz representation. We then prove that E_U provides a convenient way of checking whether a given weak quasi-lattice is amenable. Finally, we provide an example of an amenable weak quasi-lattice.

Definition 4.0.1. *Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. A linear map $\theta : \mathcal{A} \to \mathcal{B}$ is defined to be a **$\mathcal{B}$-linear** mapping if $\theta|_{\mathcal{B}} = Id_{\mathcal{B}}$ and for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$:*

$$\theta(ba) = b\theta(a), \text{ and}$$
$$\theta(ab) = \theta(a)b.$$

Definition 4.0.2. *Let $\mathcal{A}$ be a C^*-algebra. A linear map $\varphi : \mathcal{A} \to \mathcal{A}$ is defined to be completely positive if the maps $\varphi^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ preserve positivity for any natural number n, where $\varphi^{(n)}((A_{ij})_{ij}) = (\varphi(A_{ij}))_{ij}$.*

Definition 4.0.3. *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras. A $\mathcal{B}$-linear mapping $E : \mathcal{A} \to \mathcal{B}$ that is a completely positive contraction is defined to be a **conditional expectation** of $\mathcal{A}$ onto $\mathcal{B}$.*

As one might expect, conditional expectation operators generalize the notion of a conditional expectation on a probability space. When $\mathcal{B} \subset \mathcal{C}$ are both spaces of random variables on some probability space, definition 4.0.1 specialises to the standard definition of a conditional expectation in classical probability.

The following theorem, proven by Tomiyama in [9], provides a convenient way to check whether a linear mapping is a conditional expectation.

Theorem 4.0.4. *Let $\mathcal{B}$ be a C^*-subalgebra of the C^*-algebra $\mathcal{A}$, and let $E : \mathcal{A} \to \mathcal{B}$ be an idempotent linear mapping such that $\|E\| = 1$. Then E is a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.*

In Corollary II.6.10.3 of [3], Blackadar proved that Theorem 4.0.4 implies the following.

Corollary 4.0.5. *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras and suppose that $E : \mathcal{A} \to \mathcal{B}$ is an idempotent postive $\mathcal{B}$-linear mapping. Then E is a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.*

4.1 The Toeplitz Conditional Expectation

We start by defining the diagonal subalgebra $\mathcal{D} \subset \mathcal{T}(G, P)$ of the Toeplitz algebra and the Toeplitz conditional expectation $E : \mathcal{T}(G, P) \to \mathcal{D}$. The Toeplitz conditional expectation will be used to prove an equivalent formulation of amenability for a weak quasi-lattice in terms of the universal conditional expectation.

Definition 4.1.1. *Let (G, P) be a weak quasi-lattice and $T : P \to B(\ell^2(P))$ its Toeplitz representation. Define,*

$$\mathcal{D} := \overline{span}\{T_pT_p^* : p \in P\} \subset \mathcal{T}(G, P).$$

*$\mathcal{D}$ is called the **diagonal subalgebra**.*

Proposition 4.1.2. *The diagonal subalgebra $\mathcal{D}$ is a unital Abelian C^*-subalgebra of $\mathcal{T}(G, P)$.*

Proof. By Lemma 2.2.11, to prove that $\mathcal{D}$ is a unital C^*-subalgebra of $\mathcal{T}(G, P)$, it suffices to prove that $span\{T_pT_p^* : p \in P\}$ is a unital $*$-subalgebra of $\mathcal{T}(G, P)$. By linearity, the set $span\{T_pT_p^* : p \in P\}$ is a unital $*$-subalgebra of $\mathcal{T}(G, P)$ if and only if the set $B := \{T_pT_p^* : p \in P\} \cup \{0\}$ contains the identity element of $\mathcal{T}(G, P)$ and is closed under multiplication and involution. Because for all $p \in P$, the inequality $p \geq e$ holds, it follows that for all basis elements δ_p of $\ell^2(P)$,

$$T_eT_e^*(\delta_p) = \delta_{ee^{-1}p} = \delta_p,$$

proving that the identity $I = T_eT_e^*$ is contained in B. Moreover, B is closed under involution since $0^* = 0$ and $(T_pT_p^*)^* = T_pT_p^*$ for all elements $T_pT_p^*$ of B. Towards proving that B is closed under multiplication, fix non-zero elements $T_pT_p^*, T_qT_q^* \in B$ and note that,

$$T_pT_p^*T_qT_q^* = \begin{cases} T_{p\vee q}T_{p\vee q}^* & p \vee q < \infty \\ 0 & p \vee q = \infty \end{cases} \in B. \tag{4.1}$$

Thus, $span\{T_pT_p^* : p \in P\}$ is a unital $*$-subalgebra of $\mathcal{T}(G, P)$, which proves that the diagonal subalgebra $\mathcal{D}$ is a C^*-subalgebra of the Toeplitz algebra. Moreover, because for any $p, q \in P$ we have that $p \vee q = q \vee p$, Equation 4.1 implies that multiplication is commutative in $\mathcal{D}$, proving that $\mathcal{D}$ is an Abelian C^*-subalgebra of $\mathcal{T}(G, P)$. □

Proposition 4.1.3. *For each $p \in P$, write $Q_p : \ell^2(P) \to \ell^2(P)$ for the orthogonal projection with range $span\{\delta_p\}$. For each $S \in B(\ell^2(P))$ there is an element $\sum_{p\in P} Q_pSQ_p$ of $B(\ell^2(P))$. The function $\Delta : B(\ell^2(P)) \to B(\ell^2(P))$ defined by,*

$$\Delta(S) := \sum_{p\in P} Q_pSQ_p, \qquad S \in B(\ell^2(P)),$$

is a well-defined bounded linear map. Moreover, for all $x, y \in P$,

$$\Delta(T_xT_y^*) = \begin{cases} T_xT_y^* & \text{if } x = y \\ 0 & \text{if } x \neq y, \end{cases}$$

where $T : P \to \mathcal{T}(G, P)$ is the Toeplitz representation of P.

The following proof of Proposition 4.1.3 fills in details omitted by Nica in his proof found in [8].

Proof. Towards proving that Δ is well-defined, fix some $S \in B(\ell^2(P))$ and suppose that $\{F_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of P such that for each $p \in P$, there exists some integer N such that $p \in F_n$ for all $n \geq N$. For each $n \in \mathbb{N}$, define $S_{F_n} := \sum_{p\in F_n} Q_pSQ_p$. To prove that for each $S \in B(\ell^2(P))$ there exists an element $\sum_{p\in P} Q_pSQ_p$ in $B(\ell^2(P))$, we need to prove that $\lim_{n,m\to\infty} \|S_{F_n} - S_{F_m}\|_{op}$ converges to zero. Clearly if $S = 0$ then $\Delta(S) = \sum_{p\in P} Q_p0Q_p = 0$, so we may suppose without loss of generality that S is a non-zero element. Fix $\varepsilon > 0$ and, using the supremum definition of the operator norm, pick some $h \in \ell^2(P)$ with norm $\|h\| \le 1$ such that $\|S_{F_m} - S_{F_n}\| - \|(S_{F_m} - S_{F_n})h\| < \varepsilon/2$. Then $\|S_{F_m} - S_{F_n}\| < \varepsilon/2 + \|(S_{F_m} - S_{F_n})h\|$. By Parseval's identity the sum $\sum_{p\in P} |\langle h\,,\,\delta_p\rangle|^2$ converges to $\|h\| \le 1$ so that there must exist some finite set $F \subset P$ such that,

$$\sum_{p\in P-F} |\langle h\,,\,\delta_p\rangle|^2 < \frac{\varepsilon^2}{4\|S\|^2}.$$

Moreover, for any $p \in P$,

$$\begin{aligned}
|\langle S(\delta_p)\,,\,\delta_p\rangle|^2 &\le \sum_{q\in P} |\langle S(\delta_p)\,,\,\delta_q\rangle|^2 \\
&= \|S(\delta_p)\|^2 \le \|S\|^2,
\end{aligned}$$

so that

$$\sum_{p\in P-F} |\langle h\,,\,\delta_p\rangle\langle S(\delta_p)\,,\,\delta_p\rangle|^2 \le \|S\|^2 \sum_{p\in P-F} |\langle h\,,\,\delta_p\rangle|^2 < \frac{\varepsilon^2}{4}.$$

There must exist some $N \in \mathbb{N}$ such that $F \subset F_n$ for all $n \ge N$. Then for all $m, n \ge N$,

$$\begin{aligned}
\|\sum_{p\in F_m-F_n} (Q_pSQ_p)h\|^2 &= \|\sum_{p\in F_m-F_n} (Q_pS)\langle h\,,\,\delta_p\rangle\delta_p\|^2 \\
&= \|\sum_{p\in F_m-F_n} Q_p\langle h\,,\,\delta_p\rangle S(\delta_p)\|^2 \\
&= \|\sum_{p\in F_m-F_n} \langle h\,,\,\delta_p\rangle\langle S(\delta_p)\,,\,\delta_p\rangle\delta_p\|^2 \\
&= \sum_{p\in F_m-F_n} |\langle h\,,\,\delta_p\rangle\langle S(\delta_p)\,,\,\delta_p\rangle|^2 \\
&\le \sum_{p\in P-F_n} |\langle h\,,\,\delta_p\rangle\langle S(\delta_p)\,,\,\delta_p\rangle|^2 \\
&< \frac{\varepsilon^2}{4}.
\end{aligned}$$

It follows that for all $n, m \ge N$,

$$\begin{aligned}
\|S_{F_m} - S_{F_n}\| &< \frac{\varepsilon}{2} + \|(S_{F_m} - S_{F_n})h\| \\
&= \frac{\varepsilon}{2} + \|\sum_{p\in F_m-F_n} (Q_pSQ_p)h\| \\
&< \varepsilon.
\end{aligned}$$

Because the sequence $\{S_{F_n}\}_{n\in\mathbb{N}} \subset B(\ell^2(P))$ is Cauchy, it must converge uniquely to a bounded linear operator $\sum_{p\in P} Q_pSQ_p \in B(\ell^2(P))$. This gives a well-defined map $\Delta : B(\ell^2(P)) \to B(\ell^2(P))$ defined as in the statement of Proposition 4.1.3.

Towards proving that Δ is a bounded linear operator, fix elements S_1, S_2 in $B(\ell^2(P))$ and scalars $c_1, c_2 \in \mathbb{C}$. Then by the algebra of limits,

$$\begin{aligned}
\Delta(c_1S_1 + c_2S_2) &= \sum_{p\in P} Q_p(c_1S_1 + c_2S_2)Q_p \\
&= c_1\sum_{p\in P} Q_pS_1Q_p + c_2\sum_{p\in P} Q_pS_2Q_p \\
&= c_1\Delta(S_1) + c_2\Delta(S_2),
\end{aligned}$$

proving that Δ is linear. Now fix a sequence $\{S_n\}_{n\in\mathbb{N}} \subset B(\ell^2(P))$ that converges to some element $S \in B(\ell^2(P))$. Fix $\varepsilon > 0$. Because composition and addition are continuous, for any finite set $F \subset P$, the sequence $\{\sum_{p\in F} Q_p S_n Q_p\}_{n\in\mathbb{N}}$ converges to $\sum_{p\in F} Q_p S Q_p$. Construct an increasing sequence of finite sets $\{F_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $\| \sum_{p\in F_n} Q_p S_n Q_p - \sum_{p\in P} Q_p S_n Q_p\| < \varepsilon/3$, and for all $p \in P$, there exists some $n \in \mathbb{N}$ such that $p \in F_n$. Then pick $N \in \mathbb{N}$ large enough such that for all $n \geq N$, $\| \sum_{p\in P} Q_p S Q_p - \sum_{p\in F_n} Q_p S Q_p\| < \varepsilon/3$, and $\| \sum_{p\in F_N} Q_p S Q_p - \sum_{p\in F_N} Q_p S_n Q_p\| < \varepsilon/3$. We have that for all $n \geq N$,

$$\begin{aligned}
\|\Delta(S) - \Delta(S_n)\| &= \| \sum_{p\in P} Q_p S Q_p - \sum_{p\in P} Q_p S_n Q_p\| \\
&\leq \| \sum_{p\in P} Q_p S Q_p - \sum_{p\in F_N} Q_p S Q_p\| + \| \sum_{p\in F_N} Q_p S Q_p - \sum_{p\in F_N} Q_p S_n Q_p\| \\
&+ \| \sum_{p\in F_N} Q_p S_n Q_p - \sum_{p\in P} Q_p S_n Q_p\| \\
&< \varepsilon.
\end{aligned}$$

Hence, Δ is a continuous linear operator and is therefore a bounded linear operator.

It remains to show that for any elements $x, y \in P$, the following identity holds:

$$\Delta(T_x T_y^*) = \begin{cases} T_x T_y^* & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

To this end, fix elements $x, y \in P$ and note that for any basis element δ_q of $\ell^2(P)$,

$$\begin{aligned}
\Delta(T_x T_y^*)(\delta_q) &= \sum_{p\in P} \langle \delta_q \, , \, \delta_p \rangle \langle T_x T_y^*(\delta_p) \, , \, \delta_p \rangle \delta_p \\
&= \langle T_x T_y^*(\delta_q) \, , \, \delta_q \rangle \delta_q \\
&= \begin{cases} \delta_q & \text{if } y \leq p \text{ and } x = y \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} T_x T_y^*(\delta_q) & \text{if } x = y \\ 0(\delta_q) & \text{if } x \neq y \end{cases}
\end{aligned}$$

Hence, for the case where $x = y$, the above equation along with the linearity of $\Delta(T_x T_y^*)$ imply that $\Delta(T_x T_y^*)$ must agree with $T_x T_y^*$ on the dense subspace $span\{\delta_p : p \in P\}$. By the continuity of these two linear operators, they agree over all of $\ell^2(P)$. For the case where $x \neq y$, because $\Delta(T_x T_y^*)$ is a continuous linear operator that sends all basis elements of $\ell^2(P)$ to zero, it must be the case that $\Delta(T_x T_y^*)$ maps all elements of $\ell^2(P)$ to zero. $\square$

Proposition 4.1.4. *There exists a unique bounded linear map $E : \mathcal{T}(G, P) \to \mathcal{D}$ such that for any $p, q \in P$:*

$$E(T_p T_q^*) = \begin{cases} T_p T_q^* & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

*Moreover, E is a conditional expectation which we define to be the **Toeplitz conditional expectation**.*

Proof. Let Δ be as defined in Proposition 4.1.3 and let $E := \Delta|_{\mathcal{T}(G, P)}$. To see that the range of E is indeed contained in $\mathcal{D}$, note that $\mathcal{T}(G, P) = \overline{span}\{T_p T_q^* : p, q \in P\}$. Since $\Delta(T_p T_q^*) =$

$\begin{cases} T_pT_p^* & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$, it is clear that range$E = \mathcal{D}$.

By Corollary 4.0.5, to prove that E is a conditional expectation, it suffices to prove that E is an idempotent positive $\mathcal{D}$-linear mapping. Towards proving that E is a $\mathcal{D}$-linear map, fix elements basis elements $T_xT_x^* \in \mathcal{D}$, $T_yT_z^* \in \mathcal{T}(G, P)$ and $\delta_q \in \ell^2(P)$, then

$$\begin{aligned}
E(T_xT_x^*T_yT_z^*)(\delta_q) &= \sum_{p\in P} \langle \delta_q \,,\, \delta_p \rangle \langle T_xT_x^*T_yT_z^*(\delta_p) \,,\, \delta_p \rangle \delta_p. \\
&= \langle T_xT_x^*T_yT_z^*(\delta_q) \,,\, \delta_q \rangle \delta_q \\
&= \begin{cases} \delta_q & \text{if } q \geq y \geq x \text{ and } y = z \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} T_yT_z^*(\delta_q) & \text{if } q \geq x \text{ and } y = z \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} E(T_yT_z^*)(\delta_q) & \text{if } q \geq x \\ 0 & \text{otherwise} \end{cases} \\
&= T_xT_x^*E(T_yT_z^*)(\delta_q)
\end{aligned}$$

Because $E(T_xT_x^*T_yT_z^*)$ and $T_xT_x^*E(T_yT_z^*)$ are both bounded linear operators, the above working implies that these operators must agree over all elements in the span of the basis for $\ell^2(P)$. Because this spanning set is dense in $\ell^2(P)$ and $E(T_xT_x^*T_yT_z^*)$ and $T_xT_x^*E(T_yT_z^*)$ are both bounded, and therefore continuous, linear operators, it must be the case that these operators agree over all of $\ell^2(P)$. Moreover, because $E = \Delta|_{\mathcal{T}(G,P)}$ is a bounded linear map, by Proposition 4.1.3, it follows that $E(T_xT_x^* \sum_{(p,q)\in F} c_{p,q}T_pT_q^*) = T_xT_x^*E(\sum_{(p,q)\in F} c_{p,q}T_pT_q^*)$ for any linear combination $\sum_{(p,q)\in F} c_{p,q}T_pT_q^*$ of basis elements of $\mathcal{T}(G, P)$. Because the span of basis elements of $\mathcal{T}(G, P)$ is dense in $\mathcal{T}(G, P)$ and E and $T_xT_x^*E$ are both bounded, and therefore continuous, maps that agree over this dense set, it follows that these maps agree over all of $\mathcal{T}(G, P)$. Hence, we have that for all $S \in \mathcal{T}(G, P)$, $E(T_xT_x^*S) = T_xT_x^*E(S)$.

Now fix $S \in \mathcal{T}(G, P)$. Since $T_xT_x^*$ was an arbitrary basis element of $\mathcal{D}$, the above equality holds for any basis element $T_xT_x^* \in \mathcal{D}$. Then applying the linearity of E we have that $E(dS) = dE(S)$ for any element d in the span of the basis of $\mathcal{D}$. Because this set is dense in $\mathcal{D}$, the continuity of the maps $E(- \circ S)$ and $- \circ E(S)$ implies that this maps agree over all elements $d \in \mathcal{D}$.

Observe also that for any basis elements $T_xT_x^* \in \mathcal{D}$, $T_yT_z^* \in \mathcal{T}(G, P)$ and $\delta_q \in \ell^2(P)$,

$$\begin{aligned}
E(T_yT_z^*T_xT_x^*)(\delta_q) &= \sum_{p\in P} \langle \delta_q \,,\, \delta_p \rangle \langle T_yT_z^*T_xT_x^*(\delta_p) \,,\, \delta_p \rangle \delta_p \\
&= \begin{cases} \langle T_yT_z^*(\delta_q) \,,\, \delta_q \rangle \delta_q & \text{if } q \geq x \\ 0 & \text{otherwise} \end{cases} \\
&= \langle T_xT_x^*(\delta_q) \,,\, \delta_q \rangle \langle T_yT_z^*(\delta_q) \,,\, \delta_q \rangle \delta_q \\
&= \sum_{p\in P} \langle T_xT_x^*(\delta_q) \,,\, \delta_p \rangle \langle T_yT_z^*(\delta_p) \,,\, \delta_p \rangle \delta_p = E(T_yT_z^*) \circ T_xT_x^*(\delta_q).
\end{aligned}$$

Analogous reasoning as above then proves that for all $d \in \mathcal{D}$ and all $S \in \mathcal{T}(G, P)$, $E(Sd) = E(S) \circ d$. It follows that E is a $\mathcal{D}$-linear mapping.

Towards proving that E is idempotent, note that by the continuity and linearity of E and E^2, and the density of $span\{T_xT_y^* : x,y \in P\}$ in $\mathcal{T}(G, P)$ (Corollary 2.2.13), it suffices to prove that $E^2(T_xT_y^*) = E(T_xT_y^*)$ for all $x,y \in P$. To this end, fix some basis elements $x,y \in P$. Then by applying Proposition 4.1.3 three times, we have that,

$$\begin{aligned} E^2(T_xT_y^*) &= \begin{cases} E(T_xT_y^*) & \text{if } x = y \\ E(0) & \text{if } x \neq y \end{cases} \\ &= \begin{cases} T_xT_y^* & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \\ &= E(T_xT_y^*), \end{aligned}$$

which proves that E is indeed idempotent.

To prove that E is a conditional expectation, it only remains to prove that E is positive. To this end, fix some positive element $S \in \mathcal{T}(G, P)$ and note that for any $p \in P$ and $h \in \ell^2(P)$,

$$\begin{aligned} \langle Q_pSQ_p(h)\,,\,h\rangle &= \Big\langle \langle h\,,\,\delta_p\rangle\langle S(\delta_p)\,,\,\delta_p\rangle\delta_p\,,\,h\Big\rangle \\ &= |\langle h\,,\,\delta_p\rangle|^2\langle S(\delta_p)\,,\,\delta_p\rangle \geq 0. \end{aligned}$$

It follows that for all $p \in P$, the operator Q_pSQ_p is a positive operator. Note that because sums and limits of positive operators are again positive, the operator $E(S) = \sum_{p\in P} Q_pSQ_p$, being the limit of a sequence of finite sums of positive operators, is a positive operator. It follows that E maps positive elements of $\mathcal{T}(G, P)$ to positive elements of $\mathcal{D}$, proving that E is a positive and completing the proof that E is a conditional expectation.

Finally, to see that E is the unique bounded linear map such that for any elements $p,q \in P$,

$$E(T_pT_q^*) = \begin{cases} T_pT_q^* & \text{if } p = q \\ 0 & \text{if } p \neq q, \end{cases}$$

note that, by the continuity of bounded linear maps and the density of $span\{T_pT_q^* : p,q \in P\}$ in $\mathcal{T}(G, P)$, any other bounded linear map that agrees with E over $span\{T_pT_q^* : p,q \in P\}$ must agree with E over all of $\mathcal{T}(G, P)$. □

Definition 4.1.5. *Let $\mathcal{B} \subset \mathcal{A}$ be C^*-algebras. A conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is defined to be **faithful** if for all positive elements $a \in \mathcal{A}$, $E(a) = 0$ implies that $a = 0$.*

Proposition 4.1.6. *The Toeplitz conditional expectation is faithful.*

Proof. Fix some basis element $\delta_s \in \ell^2(P)$. Note that for any spanning element $T_pT_q^*$ of $\mathcal{T}(G, P)$,

$$\begin{aligned} \langle E(T_pT_q^*)(\delta_s)\,,\,\delta_s\rangle &= \begin{cases} \langle T_pT_q^*(\delta_s)\,,\,\delta_s\rangle & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases} \\ &= \begin{cases} 1 & \text{if } p = q \text{ and } s \geq q \\ 0 & \text{otherwise.} \end{cases} \\ &= \langle T_pT_q^*(\delta_s)\,,\,\delta_s\rangle. \end{aligned}$$

Then by the linearity and continuity of both E and the first component of the inner product, it must be the case that for all elements $a \in \mathcal{T}(G, P)$, $\langle E(a)(\delta_s)\,,\,\delta_s\rangle = \langle a(\delta_s)\,,\,\delta_s\rangle$. Now suppose that $a \in \mathcal{T}(G, P)$ is a positive element such that $E(a) = 0$. Then by the functional calculus on $\mathcal{T}(G, P)$, there must exist some positive element $b \in \mathcal{T}(G, P)$ such that $b^2 = a$. Because b is a positive element, there exists some $c \in \mathcal{T}(G, P)$ such that $b = cc^*$. Then for all basis elements $\delta_s \in \ell^2(P)$,

$$\begin{aligned}
\|b(\delta_s)\|^2 &= \langle cc^*(\delta_s)\,,\, cc^*(\delta_s)\rangle \\
&= \langle cc^*cc^*(\delta_s)\,,\, \delta_s\rangle \\
&= \langle a(\delta_s)\,,\, \delta_s\rangle \\
&= \langle E(a)(\delta_s)\,,\, \delta_s\rangle = 0.
\end{aligned}$$

It follows that $b(\delta_s) = 0$ for all basis elements δ_s and, therefore, that $b = 0$. Hence, $a = b^2 = 0$. $\square$

4.2 The Universal Conditional Expectation

We are now in position to be able to define the universal conditional expectation E_U from the universal algebra $C^*(G, P)$ onto the universal diagonal subalgebra $\mathcal{D}_U$ defined below.

Definition 4.2.1. *Let (G, P) be a weakly quasi-lattice ordered group and $C^*(G, P)$ its universal algebra. Define the **universal diagonal subalgebra**,*

$$\mathcal{D}_U := \overline{span}\{\chi_{p,p} : p \in P\}.$$

Proposition 4.2.2. *The universal diagonal subalgebra $\mathcal{D}_U$ is a unital Abelian C^*-subalgebra of $C^*(G, P)$.*

The proof of Proposition 4.2.2 is practically identical to the proof of Proposition 4.1.2 and is therefore omitted.

The following lemmas will be used to prove the existence of the universal conditional expectation E_U.

Lemma 4.2.3. *Fix some Hilbert space $\mathcal{H}$ with orthonormal basis $\{\delta_a : a \in A\}$ and let $\{S_i\}_{i=1}^n \subset B(\mathcal{H})$ be a finite family of projections such that $S_iS_j = 0$ whenever $i \neq j$. Then,*

$$\|\sum_{i=1}^{n} c_iS_i\| = \sup_{1\le j\le n} |c_j|, \qquad \{c_i\}_{i=1}^n \subset \mathbb{C}.$$

Proof. Fix some element $h \in \mathcal{H}$ and note that because for any $1 \le i < j \le n$, the identity

$S_iS_jh = 0$ holds, it follows that for any $1 \leq i < j \leq n$, $S_jh \in Im(S_i)^\perp$. Hence,

$$\begin{aligned}
\|\sum_{i=1}^{n} c_i S_i h\|^2 &= \langle \sum_{i=1}^{n} c_i S_i h \,,\, \sum_{j=1}^{n} c_j S_j h \rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i \overline{c_j} \langle S_i h \,,\, S_j h \rangle \\
&= \sum_{i=1}^{n} |c_i|^2 \|S_i h\|^2 \\
&= \sum_{i=1}^{n} |c_i|^2 \sum_{\substack{a \in A \\ \delta_a \in Im(S_i)}} |\langle h \,,\, \delta_a \rangle|^2 \\
&\leq \sum_{i=1}^{n} \sup_{1 \leq j \leq n} |c_j|^2 \sum_{\substack{a \in A \\ \delta_a \in Im(S_i)}} |\langle h \,,\, \delta_a \rangle|^2 \\
&\leq \sup_{1 \leq j \leq n} |c_j|^2 \sum_{a \in A} |\langle h \,,\, \delta_a \rangle|^2 \\
&= \sup_{1 \leq j \leq n} |c_j|^2 \|h\|^2,
\end{aligned}$$

where the second inequality follows since it must be the case that $Im(S_i) \perp Im(S_j)$ for all $1 \leq i < j \leq n$. From the infimum formulation of the operator norm, it follows that $\| \sum_{i=1}^{n} c_i S_i \| \leq \sup_{1 \leq j \leq n} |c_j|$. On the other hand, by fixing $m \in \{1, \ldots, n\}$ such that $|c_m| = \sup_{1 \leq j \leq n} |c_j|$ and picking an orthonormal basis element $\delta_{a_m} \in Im(S_m)$, we have,

$$\| \sum_{i=1}^{n} c_i S_i \delta_{a_m} \| = \|c_m \delta_{a_m}\| = \sup_{1 \leq j \leq n} |c_j|.$$

By the supremum formulation of the operator norm, it follows that $\| \sum_{i=1}^{n} c_i S_i \| \geq \sup_{i \leq j \leq n} |c_j|$, completing the proof. □

Lemma 4.2.4. *Let (G, P) be a weakly quasi-lattice ordered group and $\pi_{\mathcal{T}} : C^*(G, P) \to \mathcal{T}(G, P)$ be the Toeplitz representation of (G, P). Then $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$ is an isomorphism between the C^*-algebras $\mathcal{D}_U$ and $\mathcal{D}$.*

The following proof of Lemma 4.2.4 fills in some details omitted by Nica in his proof found in [4].

Proof. Towards proving that $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$ is a surjection onto $\mathcal{D}$, note that for all basis elements $\chi_{p,p}$ of the universal diagonal subalgebra $\mathcal{D}_U$, $\pi_{\mathcal{T}}(\chi_{p,p}) = T_pT_p^*$, which is an element of $\mathcal{D}$. By the linearity of $\pi_{\mathcal{T}}$, it follows that $\pi_{\mathcal{T}}(span\{\chi_{p,p} : p \in P\})$ is a subset of $\mathcal{D}$. Hence,

$$\begin{aligned}
\pi_{\mathcal{T}}(\mathcal{D}_U) &= \pi_{\mathcal{T}}(\overline{span}\{\chi_{p,p} : p \in P\}) \\
&= \overline{\pi_{\mathcal{T}}(span\{\chi_{p,p} : p \in P\})} \\
&= \overline{span}\{\pi_{\mathcal{T}}(\chi_{p,p}) : p \in P\} \\
&= \overline{span}\{T_pT_p^* : p \in P\} = \mathcal{D},
\end{aligned}$$

proving that $\pi_{\mathcal{T}}|_{\mathcal{D}_U} : \mathcal{D}_U \to \mathcal{D}$ is a surjective unital $*$-homomorphism onto $\mathcal{D}$. Towards defining an inverse of $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$, we define a map,

$$\hat{\rho} : span\{T_pT_p^* : p \in P\} \to \mathcal{D}_U$$

$$\sum_{p\in F} c_pT_pT_p^* \mapsto \sum_{p\in F} c_p\chi_{p,p}, \qquad F\subset P,\ |F|<\infty,\ \{c_p\}_{p\in F}\subset \mathbb{C}.$$

Clearly $\hat{\rho}$ is linear, unital and preserves that product and involution operations on $\mathcal{D}$. Because $span\{T_pT_p^* : p\in P\}$ is dense is $\mathcal{D}$, to prove that $\hat{\rho}$ extends uniquely to a bounded unital $*$-homomorphism $\rho:\mathcal{D}\to\mathcal{D}_U$, it suffices to show that $\hat{\rho}$ is bounded. To this end, fix some element $\sum_{p\in F} c_pT_pT_p^* \in span\{T_pT_p^* : p\in P\}$, where F is a finite subset of P and $\{c_p\}_{p\in F}\subset\mathbb{C}$. Note that because for any $p\in F$ and basis element $\delta_q\in\ell^2(P)$,

$$T_pT_p^*(\delta_q)=\begin{cases}\delta_q & \text{if } q\geq p\\ 0 & \text{otherwise,}\end{cases}$$

it follows that $T_pT_p^*$ is diagonal with respect to the canonical basis of $\ell^2(P)$. Because linear combinations of diagonal operators are diagonal, the element $\sum_{p\in F} c_pT_pT_p^*$ is diagonal relatively to the canonical basis. The norm of $\sum_{p\in F} c_pT_pT_p^*$, being diagonal with respect to the canonical basis, is therefore equal to

$$\begin{aligned}\sup_{q\in P}|\langle(\sum_{p\in F} c_pT_pT_p^*)\delta_q\,,\,\delta_q\rangle| &= \sup_{q\in P}|\langle\sum_{\substack{p\in F\\ q\geq p}} c_p\delta_q\,,\,\delta_q\rangle|\\ &= \sup_{q\in P}|\sum_{\substack{p\in F\\ q\geq p}} c_p|.\end{aligned}$$

Towards computing the norm $\|\sum_{p\in F} c_p\chi_{p,p}\|$ define,

$$\chi_{\vee A,\vee A} := \begin{cases}\chi_{\vee_{t\in A}t,\vee_{t\in A}t} & \text{if } \vee_{t\in A}\, t<\infty\\ 0 & \text{otherwise,}\end{cases}$$

and for each $p\in F$ write,

$$\begin{aligned}\chi_{p,p} &= \chi_{p,p}\prod_{\substack{q\in F\\ q\neq p}}(\chi_{q,q}+(\chi_{e,e}-\chi_{q,q}))\\ &= \sum_{\{A\in\mathcal{P}(F):p\in A\}}\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}).\end{aligned}$$

Then the element $\sum_{p\in F} c_p\chi_{p,p}$ becomes,

$$\begin{aligned}\sum_{p\in F} c_p\chi_{p,p} &= \sum_{p\in F} c_p\chi_{p,p}\prod_{\substack{q\in F\\ q\neq p}}(\chi_{q,q}+(\chi_{e,e}-\chi_{q,q}))\\ &= \sum_{p\in F} c_p\sum_{\{A\in\mathcal{P}(F):p\in A\}}\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s})\\ &= \sum_{A\in\mathcal{P}(F)-\varnothing}(\sum_{p\in A} c_p)\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}).\end{aligned}$$

Fix some $A\in\mathcal{P}(F)-\varnothing$ and $s\in F-A$ and note that the elements $\chi_{\vee A,\vee A}$, $\chi_{s,s}$ and, thus, $\chi_{e,e}-\chi_{s,s}=Id_{\mathcal{D}_U}-\chi_{s,s}$ are all projections. Because multiplication in $\mathcal{D}_U$ is commutative, it follows that the element $\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s})$ is a projection. Moreover, for any $B\in\mathcal{P}(F)-\varnothing$ such that $B\neq A$, there must exist some $q\in(A\cap(F-B))\cup(B\cap(F-A))$. Because $\chi_{q,q}(\chi_{e,e}-\chi_{q,q})=\chi_{q,q}-\chi_{q,q}=0$, commutativity of multiplication in $\mathcal{D}_U$ implies that,

$$\begin{aligned}(\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}))(\chi_{\vee B,\vee B}\prod_{t\in B}(\chi_{e,e}-\chi_{t,t}))\\ &=\prod_{p\in A}\chi_{p,p}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s})\prod_{r\in B}\chi_{r,r}\prod_{t\in B}(\chi_{e,e}-\chi_{t,t})\\ &=0.\end{aligned}$$

Fix some $*$-homomorphism $\pi : C^*(G, P) \to B(\mathcal{H})$ on some Hilbert space $\mathcal{H}$ and, for each $A \in \mathcal{P}(F) - \varnothing$, define $\pi_A := \pi(\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}))$. Then each π_A is a projection such that for any $B \in \mathcal{P}(F) - \varnothing$ with $A \neq B$, we have that $\pi_A\pi_B = 0$. By Lemma 4.2.3,

$$\begin{aligned}\|\pi(\sum_{A\in\mathcal{P}(F)-\varnothing}(\sum_{p\in A}c_p)\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}))\| &= \|\sum_{\substack{A\in\mathcal{P}(F)-\varnothing\\ \pi_A\neq 0}}(\sum_{p\in A}c_p)\pi_A\|\\ &=\sup_{\substack{A\in\mathcal{P}(F)-\varnothing\\ \pi_A\neq 0}}|\sum_{p\in A}c_p|.\end{aligned}$$

Because the $*$-homomorphism $\pi : C^*(G, P) \to B(\mathcal{H})$ was arbitrary, it must therefore be the case that,

$$\begin{aligned}\|\sum_{p\in F}c_p\chi_{p,p}\| &= \|\sum_{A\in\mathcal{P}(F)-\varnothing}(\sum_{p\in A}c_p)\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s})\|\\ &=\sup_{A\in\mathcal{P}(F)-\varnothing}\{|\sum_{p\in A}c_p| : \chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s})\neq 0\}.\end{aligned}$$

Thus, comparing the norm above to the expression for the norm of $\sum_{p\in P}c_pT_pT_p^*$, to show that $\hat{\rho}$ is bounded, it suffices to take any subset $A \in \mathcal{P}(F) - \varnothing$ and find an element $q \in P$ such that $A = \{p \in F : q \geq p\}$. To this end, fix $A \in \mathcal{P}(F) - \varnothing$ and note that if $\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e} - \chi_{s,s}) \neq 0$, then $\vee_{p\in A}p < \infty$. That is, every element of A must share a common upper bound $\vee_{p\in A}p$. Towards a contradiction, suppose that t is an element of $F - A$ such that $t \leq \vee_{p\in A}p$. Then,

$$\begin{aligned}\chi_{\vee A,\vee A}\prod_{s\in F-A}(\chi_{e,e}-\chi_{s,s}) &= \chi_{\vee A,\vee A}(\chi_{e,e}-\chi_{t,t})\prod_{s\in F-A-\{t\}}(\chi_{e,e}-\chi_{s,s})\\ &=(\chi_{\vee A,\vee A}-\chi_{t\vee(\vee_{p\in A}p),t\vee(\vee_{p\in A}p)})\prod_{s\in F-A-\{t\}}(\chi_{e,e}-\chi_{s,s})\\ &=(\chi_{\vee A,\vee A}-\chi_{\vee A,\vee A})\prod_{s\in F-A-\{t\}}(\chi_{e,e}-\chi_{s,s})\\ &=0,\end{aligned}$$

a contradiction. It follows that $A = \{r \in F : (\vee_{p\in A}p) \geq r\}$. Therefore $\hat{\rho}$ is a bounded unital $*$-homomorphism, which proves that there exists a unique bounded unital $*$-homomorphism $\rho : \mathcal{D} \to \mathcal{D}_U$ such that $\rho(T_pT_p^*) = \chi_{p,p}$ for all basis elements $T_pT_p^*$ of $\mathcal{D}$. Because $\rho \circ \pi_{\mathcal{T}}|_{\mathcal{D}_U}(\chi_{p,p}) = Id_{\mathcal{D}_U}(\chi_{p,p})$ for all basis elements $\chi_{p,p}$ of $\mathcal{D}_U$, by linearity $\rho \circ \pi_{\mathcal{T}}|_{\mathcal{D}_U}$ must agree with $Id_{\mathcal{D}_U}$ over $span\{\chi_{p,p} : p \in P\}$. By the continuity of these operators and the density of $span\{\chi_{p,p} : p \in P\}$ in $\mathcal{D}_U$, it must be the case that $\rho \circ \pi_{\mathcal{T}}|_{\mathcal{D}_U} = Id_{\mathcal{D}_U}$. Essentially the same argument shows that $\pi_{\mathcal{T}}|_{\mathcal{D}_U} \circ \rho = Id_{\mathcal{D}}$, which proves that $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$ is an isomorphism between the C^*-algebras $\mathcal{D}_U$ and $\mathcal{D}$. □

Proposition 4.2.5. *There exists a unique bounded linear map $E_U : C^*(G, P) \to \mathcal{D}_U$ such that for any $p, q \in P$:*

$$E_U(\chi_{p,q}) = \begin{cases}\chi_{p,q} & \text{if } p = q\\ 0 & \text{if } p \neq q.\end{cases}$$

Moreover, E_U is a conditional expectation which we call the ***universal conditional expectation****.*

Proof. By Lemma 4.2.4, $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$ is an isomorphism between the C^*-algebras $\mathcal{D}_U$ and $\mathcal{D}$. Let $\rho : \mathcal{D} \to \mathcal{D}_U$ be the bounded unital $*$-homomorphism that is the inverse of $\pi_{\mathcal{T}}|_{\mathcal{D}_U}$ and define the map

$$E_U := \rho \circ E \circ \pi_{\mathcal{T}} : C^*(G, P) \to \mathcal{D}_U,$$

where $E : \mathcal{T}(G, P) \to \mathcal{D}$ is the Toeplitz conditional expectation. Then E_U, being the composition of bounded linear maps is itself a bounded linear map. Moreover, for any elements $p, q \in P$,

$$\begin{aligned} E_U(\chi_{p,q}) &= \rho \circ E \circ \pi_{\mathcal{T}}(\chi_{p,q}) \\ &= \rho \circ E(T_p T_q^*) \\ &= \begin{cases} \rho(T_p T_q^*) & \text{if } p = q \\ \rho(0) & \text{if } p \neq q \end{cases} \\ &= \begin{cases} \chi_{p,q} & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases} \end{aligned}$$

The uniqueness of E_U follows from noting that $\{\chi_{p,q} : p, q \in P\}$ is a basis for $C^*(G, P)$ and that any two bounded linear maps that agree over all basis elements must agree over their entire domain.

It remains to prove that E_U is a conditional expectation. Towards proving that E_U is $\mathcal{D}_U$-linear, fix elements $a \in C^*(G, P)$ and $b \in \mathcal{D}_U$ and note that because $\pi_{\mathcal{T}}(b) \in \mathcal{D}$ and E is $\mathcal{D}$-linear, it follows that,

$$\begin{aligned} E_U(ab) &= \rho \circ E(\pi_{\mathcal{T}}(a)\pi_{\mathcal{T}}(b)) \\ &= \rho(E(\pi_{\mathcal{T}}(a))\pi_{\mathcal{T}}(b)) \\ &= E_U(a)b, \end{aligned}$$

and,

$$\begin{aligned} E_U(ba) &= \rho \circ E(\pi_{\mathcal{T}}(b)\pi_{\mathcal{T}}(a)) \\ &= \rho(\pi_{\mathcal{T}}(b)E(\pi_{\mathcal{T}}(a))) \\ &= bE_U(a). \end{aligned}$$

Hence, E_U is indeed $\mathcal{D}_U$-linear.

Towards proving that E_U is completely positive, note that because ρ and $\pi_{\mathcal{T}}$ are $*$-homomorphisms and E is a conditional expectation, the maps ρ, $\pi_{\mathcal{T}}$ and E are all completely positive. Hence, $E_U = \rho \circ E \circ \pi_{\mathcal{T}}$ must also be completely positive.

Finally, to see that E_U is a contraction, note that because all bounded $*$-homomorphisms between C^*-algebras are contractions, ρ and $\pi_{\mathcal{T}}$ are both be contractions. Moreover, E, being a conditional expectation, must be a contraction by definition. Thus, $\|E_U\| = \|\rho \circ E \circ \pi_{\mathcal{T}}\| \leq \|\rho\|\|E\|\|\pi_{\mathcal{T}}\| \leq 1$, which proves that E_U is a contraction and therefore a conditional expectation. □

Theorem 4.2.6. *Let (G, P) be a weak quasi-lattice. Then (G, P) is amenable if and only if the universal conditional expectation $E_U : C^*(G, P) \to \mathcal{D}_U$ is faithful.*

Proof. Suppose first that (G, P) is amenable. Then by definition the Toeplitz representation $\pi_{\mathcal{T}}$ is injective. Let $a \in C^*(G, P)$ be a positive element such that $E_U(a) = 0$. Then since $E_U = \rho \circ E \circ \pi_{\mathcal{T}}$ and ρ is injective, it follows that $E \circ \pi_{\mathcal{T}}(a) = 0$. Because $\pi_{\mathcal{T}}$ is positive,

$\pi_{\mathcal{T}}(a)$ is positive and so E being faithful implies that $\pi_{\mathcal{T}}(a) = 0$. Since we have assumed that $\pi_{\mathcal{T}}$ is injective, it must be the case that $a = 0$.

For the other direction, suppose that E_U is faithful and let $a \in C^*(G, P)$ be a positive element such that $\pi_{\mathcal{T}}(a) = 0$. Then,

$$E_U(a) = \rho \circ E \circ \pi_{\mathcal{T}}(a) = \rho \circ E(0) = 0.$$

Hence, it must be the case that $a = 0$, which proves that for all positive elements $a \in C^*(G, P)$, if $\pi_{\mathcal{T}}(a) = 0$, then $a = 0$. Let $b \in C^*(G, P)$ be any element such that $\pi_{\mathcal{T}}(b) = 0$. Then $\pi_{\mathcal{T}}(bb^*) = 0$ and because bb^* is a positive element, it follows that $bb^* = 0$. Hence, either $b = 0$ or $b^* = 0$. In either case, $b = 0$ and so $\pi_{\mathcal{T}}$ is injective, proving that (G, P) is amenable. □

4.3 An Example

Let F_n be the free group on n letters and FM_n the free monoid on n letters. In this section, we show that (F_n, FM_n) is an amenable weakly quasi-lattice ordered group.

Proposition 4.3.1. *Let F_n and FM_n be the free group and free monoid on n letters, respectively.Then the pair (F_n, FM_n) is a weak quasi-lattice.*

Proof. First note that the pair (F_n, FM_n) is a partially ordered group since the inverse of any non-identity element in FM_n is not also an element of FM_n. That is, $FM_n \cap FM_n^{-1} = \{e\}$. Now suppose that elements $p, q \in FM_n$ share a common upper bound, say $t \in FM_n$. Note that the set of all elements $s \in FM_n$ such that $s \leq t$ can be written as $\{e < a_{j_1} < a_{j_1}a_{j_2} < \cdots < a_{j_1} \cdots a_{j_m} = t\}$ where $a_1, \ldots, a_n$ are the generators of FM_n. Because the set of all elements in FM_n that are bounded above by t is totally ordered, the elements p and q must be comparable. Hence, either $p \leq q$ and $p \vee q = q$, or $q \leq p$ and $p \vee q = p$. This proves that any two elements in FM_n that share a common upper bound must have a least common upper bound and, thus, that the partially ordered group (F_n, FM_n) is a weak quasi-lattice. □

Note that in the above proof of Proposition 4.3.1, we proved the following:

Corollary 4.3.2. *For any elements $p, q \in FM_n$,*

1. $p \vee q < \infty \iff p \leq q$ *or* $q \leq p \iff p = qp'$ *or* $q = pq'$ *for some* $p', q' \in FM_n$
2. *if* $p \vee q < \infty$ *then* $p \vee q = p$ *or* $p \vee q = q$.

We will use Corollary 4.3.2 to simplify our proof of Theorem 4.3.6 below.

Before continuing to our proof that the free group on n letters is amenable as a weak quasi-lattice, we take a brief detour to introduce the notion of an amenable group. In 1924 Banach and Tarski [2] proved a result that was so counter-intuitive it is now known as the Banach-Tarski Paradox. In plain English the result states that given a ball in three-dimensional space, there is a way to decompose the ball into finitely many disjoint pieces that can be rearranged to form two balls of the same size as the original. This counter-intuitive result inspired mathematicians to search for all equivalent sets of necessary and sufficient conditions that ensure a topological group will not have this pathological behavior. Such groups are called amenable. The following is one of several equivalent definitions of amenability for groups (see, for example, Theorem 1.15 of [5]).

Definition 4.3.3. *Let G be a discrete group. Then G is said to be* ***amenable as a discrete group*** *if there exists a finitely additive probability measure μ on $\mathcal{P}(G)$ such that $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subseteq G$.*

As a side note, the definition of an amenable weak quasi-lattice was born out of the definition of an amenable discrete group. In particular, it is well known that a discrete group G is amenable if and only if the reduced and full C^*-algebras of G are isomorphic. Hence, we call (G, P) amenable if $C^*(G, P) = \mathcal{T}(G, P)$.

Our proof of the following lemma follows the outline provided by Garrido in Proposition 2.3 of [5]. However, our proof below fills in a substantial amount of detail omitted by Garrido.

Lemma 4.3.4. *$\mathbb{Z}$ is amenable.*

Proof. Fix an $\varepsilon > 0$. We begin by showing that there exists a finitely additive probability measure μ_ε on $\mathcal{P}(\mathbb{Z})$ such that,

$$|\mu_\varepsilon(A) - \mu_\varepsilon(1 + A)| < \varepsilon \quad \text{for all } A \subseteq \mathbb{Z}.$$

Pick some integer N such that $2/N < \varepsilon$ and for each $A \in \mathcal{P}(\mathbb{Z})$ define,

$$\mu_\varepsilon(A) := \frac{|\{1, \ldots, N\} \cap A|}{N}.$$

Then for any two disjoint sets $A, B \subseteq \mathbb{Z}$, the sets $\{1, \ldots, N\} \cap A$ and $\{1, \ldots, N\} \cap B$ are disjoint and finite. Hence,

$$\begin{aligned}
\mu_\varepsilon(A \cup B) &= \frac{|\{1, \ldots, N\} \cap (A \cup B)|}{N} \\
&= \frac{|(\{1, \ldots, N\} \cap A) \cup (\{1, \ldots, N\} \cap B)|}{N} \\
&= \frac{|\{1, \ldots, N\} \cap A|}{N} + \frac{|\{1, \ldots, N\} \cap B|}{N} \\
&= \mu_\varepsilon(A) + \mu_\varepsilon(B).
\end{aligned}$$

It follows that μ_ε is finitely additive. Moreover, $\mu_\varepsilon(\emptyset) = \frac{|\emptyset|}{N} = 0$ and $\mu_\varepsilon(\mathbb{Z}) = \frac{|\{1,\ldots,N\}|}{N} = 1$, proving that μ_ε is a finitely additive probability measure on $\mathcal{P}(\mathbb{Z})$.
Now to see that for all $A \subseteq \mathbb{Z}$, $|\mu_\varepsilon(A) - \mu_\varepsilon(1 + A)| < \varepsilon$, note that if $n \in \{1, \ldots, N-1\} \cap A$, then $1 + n \in \{1 \ldots, N\} \cap (1 + A)$ and if $n \in \{2, \ldots, N\} \cap (1 + A)$ then $n - 1 \in \{1, \ldots, N\} \cap A$. This shows that $|\{1, \ldots, N\} \cap A| \leq |\{1, \ldots, N\} \cap (1 + A)| + 1 \leq |\{1, \ldots, N\} \cap A| + 2$. It follows that,

$$\begin{aligned}
|\mu_\varepsilon(A) - \mu_\varepsilon(1 + A)| &= \left| \frac{|\{1, \ldots, N\} \cap A| - |\{1, \ldots, N\} \cap (1 + A)|}{N} \right| \\
&\leq \frac{2}{N} < \varepsilon.
\end{aligned}$$

Thus we have proven that for any $\varepsilon > 0$, the set S_ε is non-empty, where we define S_ε to be the set consisting of all finitely additive probability measures ν on $\mathcal{P}(\mathbb{Z})$ with the property that for all $A \subseteq \mathbb{Z}$, $|\nu(A) - \nu(1 + A)| < \varepsilon$. With the intention of eventually proving that the set $\cap_{\varepsilon > 0} S_\varepsilon$ is non-empty, we claim that each S_ε is closed. To this end, suppose that $\{\nu_i\} \subset S_\varepsilon$ is a sequence that converges to some measure ν on $\mathcal{P}(\mathbb{Z})$. Then it is easily checked using epsilon arguments that ν must also be a finitely additive probability measure. Fix $A \subseteq \mathbb{Z}$ and $\delta > 0$, and pick $M \in \mathbb{Z}$ large enough such that $|\nu_n(B) - \nu(B)| < \delta/2$ for all $B \subseteq \mathbb{Z}$ and for all $n \geq M$. Then,

$$\begin{aligned}|\nu(A) - \nu(1+A)| &\leq |\nu(A) - \nu_M(A)| + |\nu_M(A) - \nu_M(1+A)| + |\nu_M(1+A) - \nu(1+A)| \\ &< \delta + |\nu_M(A) - \nu_M(1+A)|.\end{aligned}$$

Since this result holds for all $\delta > 0$, it follows that,

$$|\nu(A) - \nu(1+A) \leq |\nu_M(A) - \nu_M(1+A)| < \varepsilon.$$

Thus, $\nu \in S_\varepsilon$ and we have proven that for all $\varepsilon > 0$, the set S_ε is closed.
Because for any $n \in \mathbb{Z}$ and any $\varepsilon_1, \ldots, \varepsilon_n > 0$, we have that,

$$\bigcap_{i=1}^{n} S_{\varepsilon_i} = S_{\min(\varepsilon_i)_{\{1 \leq i \leq n\}}} \in \{S_\varepsilon\}_{\varepsilon > 0},$$

it follows that the set $\{S_\varepsilon\}_{\varepsilon>0}$ satisfies the finite intersection property. By Tychonoff's theorem, the set $[0,1]^{\mathcal{P}(\mathbb{Z})}$ is compact and it follows that $\bigcap_{\varepsilon>0} S_\varepsilon$ is not equal to the empty set. That is, there exists some finitely additive probability measure μ on $\mathcal{P}(\mathbb{Z})$ such that for any $A \subseteq \mathbb{Z}$ and for all $\varepsilon > 0$, $|\mu(A) - \mu(1+A)| < \varepsilon$. It follows that for all $A \subseteq \mathbb{Z}$, $\cdots = \mu(-1+A) = \mu(A) = \mu(1+A) = \cdots = \mu(n+A) = \cdots$ which proves the amenability of $\mathbb{Z}$ as a discrete group. □

The following proposition, proven by Laca and Raeburn in Proposition 4.2 of [6], will be used to prove the amenability of (F_n, FM_n) as a weakly quasi-lattice ordered group.

Proposition 4.3.5. *Let (G, P) and (H, K) be two weakly quasi-lattice ordered groups. Suppose there exists an order-preserving homomorphism $\mu : G \to H$ such that for all $x, y \in P$ with $x \vee y < \infty$ we have,*

1. $\mu(x \vee y) = \mu(x) \vee \mu(y)$
2. $\mu(x) = \mu(y) \implies x = y$.

If H is an amenable group, then (G, P) is an amenable weak quasi-lattice.

We call such a map μ defined in Proposition 4.3.5 a controlled map.

Theorem 4.3.6. *Let F_n and FM_n be the free group and free monoid on n letters, respectively. Then the weak quasi-lattice (F_n, FM_n) is amenable.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the set of generators of F_n and define a map $\mu : F_n \to \mathbb{Z}$ that sends the empty word to 0 and any other reduced word $x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}$ to the sum $k_1 + \cdots + k_m$. Towards proving that μ is a homomorphism, fix two reduced words $x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}$ and $y_{b_1}^{l_1} \cdots y_{b_v}^{l_v}$. Now let j be the greatest integer such that $y_{b_j}^{l_j} \neq (x_{a_{m-j+1}}^{k_{m-j+1}})^{-1}$. Then,

$$x_{a_1}^{k_1} \cdots x_{a_m}^{k_m} y_{b_1}^{l_1} \cdots y_{b_v}^{l_v} = \begin{cases} x_{a_1}^{k_1} \cdots x_{a_{m-j+1}}^{k_{m-j+1}+l_j} y_{b_{j+1}}^{l_{j+1}} \cdots y_{b_v}^{l_v} & \text{if } a_{m-j+1} = b_j \\ x_{a_1}^{k_1} \cdots x_{a_{m-j+1}}^{k_{m-j+1}} y_{b_j}^{l_j} \cdots y_{b_v}^{l_v} & \text{if } a_{m-j+1} \neq b_j. \end{cases}$$

Moreover, for all natural numbers $i < j$, it must be the case that $b_i = a_{m-i+1}$ and $l_i = -k_{m-i+1}$. Hence, in both of the above cases,

$$\begin{aligned}\mu(x_{a_1}^{k_1} \cdots x_{a_m}^{k_m} y_{b_1}^{l_1} \cdots y_{b_v}^{l_v}) &= k_1 + \cdots + k_{m-j+1} + l_j + \cdots + l_v \\ &= k_1 + \cdots + k_{m-j+1} + (k_{m-j+2} + l_{j-1}) + \cdots + (k_m + l_1) + l_j + \cdots + l_v \\ &= \mu(x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}) + \mu(y_{b_1}^{l_1} \cdots y_{b_v}^{l_v}).\end{aligned}$$

It follows that μ is a homomorphism.

We now turn to the task of proving that μ is order-preserving. Since μ is a homomorphism it suffices to prove that $\mu(FM_n) \subseteq \mathbb{N}$. To this end, fix some reduced word $x_{a_1}^{k_1} \cdots x_{a_m}^{k_m} \in FM_n$. Then by definition $k_i \geq 0$ for all $1 \leq i \leq m$ and it follows that,

$$0 \leq K_1 + \cdots + k_m = \mu(x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}).$$

Thus, $\mu(x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}) \in \mathbb{N}$ and so μ is an order-preserving homomorphism.

Finally we prove that identities 1. and 2. of Proposition 4.3.5 hold. Suppose for elements $x, y \in FM_n$ we have that $x \vee y < \infty$. Note that by Corollary 4.3.2, either there exists some $x' \in FM_n$ such that $x = yx'$, or there exists some $y' \in FM_n$ such that $y = xy'$. Without loss of generality, we may assume that $x = yx'$ for some $x' \in FM_n$. By Corollary 4.3.2, either $x \vee y = x$ or $x \vee y = y$. Because $y^{-1}x = x' \in FM_n$ implies that $y \leq x$, it must be the case that $x \vee y = x$. Moreover, because μ is order-preserving, $y \leq x$ implies that $\mu(y) \leq \mu(x)$. Hence, by the standard properties of the order on $\mathbb{Z}$, $\mu(x) \vee \mu(y) = \mu(x)$ so that

$$\mu(x \vee y) = \mu(x) = \mu(x) \vee \mu(y),$$

proving that μ satisfies identity 1. of Proposition 4.3.5.

Now towards proving that μ also satisfies identity 2. of Proposition 4.3.5, suppose that $\mu(x) = \mu(y)$. Then because $x = yx'$ for some $x' \in FM_n$, we have that

$$\mu(x) = \mu(yx') = \mu(y) + \mu(x') = \mu(x) + \mu(x').$$

This proves that $\mu(x') = 0$. Writing x' as its unique reduced word $x' = x_{a_1}^{k_1} \cdots x_{a_m}^{k_m}$, note that because $x' \in FM_n$, it must be the case that $k_i \geq 0$ for all $i \in \{1, \ldots, m\}$. But then for all $i \in \{1, \ldots, m\}$,

$$0 \leq k_i \leq \sum_{j=1}^{m} k_j = \mu(x') = 0,$$

which implies that $k_i = 0$. It follows that $x' = e$ so that $x = yx' = y$, proving that if $x \vee y < \infty$ and $\mu(x) = \mu(y)$, then $x = y$.

We have shown that $\mu : F_n \to \mathbb{Z}$ is an order-preserving homomorphism that satisfies identities 1. and 2. of Proposition 4.3.5. Because, by Lemma 4.3.4, $\mathbb{Z}$ is amenable as a group, by Proposition 4.3.5, it follows that the weak quasi-lattice (F_n, FM_n) is amenable. □

This result may be surprising to some readers who are familiar with group amenability, as it is well known that the free group on two elements F_2 is not amenable as a group.

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