Functional Analysis, Sobolev Spaces and Partial Differential Equations Solutions

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1.1 Properties of the duality map

Let E be an n.v.s. The duality map F is defined for every $x \in E$ by

$$F(x) = \{f \in E^* : \|f\| = \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}$$

1. Prove that
$$F(x) = \{f \in E^* : \|f\| \leq \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}.$$
and deduce that $F(x)$ is nonempty, closed, and convex.

Proof. To discard with the trivial case, observe that when $x = 0$, the statement follows from the fact that $\|f\| \geq 0$ for all $f \in E^*$. Hence, we may assume WLOG that $x \neq 0$. Fix nonzero $x \in E$ and define $S_x := \{f \in E^* : \|f\| \leq \|x\| \text{ and } \langle f, x\rangle = \|x\|^2\}$. Clearly $F(x) \subset S_x$, moreover for any $f \in S_x$, $\langle f, x\rangle = \|x\|^2$ implies that $\|x\| = \langle f, \frac{x}{\|x\|}\rangle \leq \|f\| \leq \|x\|$, so that $\|f\| = \|x\|$. Hence $f \in F(x)$, and it follows that $F(x) = S_x$.

The fact that $F(x)$ is nonempty follows from the Hahn-Banach theorem and is the content of Corollary 1.3. To see that $F(x)$ is closed, let $J_x \in E^{**} : f \mapsto \langle f, x\rangle$ be the embedding of x in E^{**} and observe that $F(x) = S_x = \overline{B_{E^*}(0, \|x\|)} \cap J_x^{-1}(\{\|x\|^2\})$, which is closed by the continuity of J_x. Finally, to see that $F(x)$ is convex, fix $f, g \in F(x)$ and $\lambda \in [0,1]$. Observe that $\|\lambda f + (1-\lambda)g\| \leq \lambda\|f\| + (1-\lambda)\|g\| \leq \|x\|$, and $\langle \lambda f + (1-\lambda)g, x\rangle = \lambda\langle f, x\rangle + (1-\lambda)\langle g, x\rangle = \lambda\|x\|^2 + (1-\lambda)\|x\|^2 = \|x\|^2$. Hence, $\lambda f + (1-\lambda)g \in S_x = F(x)$ for all $f, g \in F(x)$ and $\lambda \in [0,1]$, proving convexity. ☐

2. Prove that if E^* is strictly convex, then $F(x)$ contains a single point.

Proof. Fix $x \in E$, $f, g \in F(x)$ and suppose E^* is strictly convex. If $x = 0$, then $\|f\| = \|g\| = 0$ implies that $f = g = 0$, so we may assume WLOG that $x \neq 0$. By the convexity of $F(x)$, $\frac{f+g}{2} \in F(x)$ so that $\left\|\frac{f+g}{2}\right\| = \|x\|$. Define $f' := \frac{f}{\|x\|}$ and $g' := \frac{g}{\|x\|}$ and observe that $\|f'\| = \|g'\| = 1$ and $\left\|\frac{f'}{2} + \frac{g'}{2}\right\| = \frac{1}{\|x\|}\left\|\frac{f+g}{2}\right\| = 1$. Since E^* is strictly convex, this is only possible if $f' = g'$, and by rescaling, we see that $f = g$, which gives the desired result. ☐

3. Prove that
$$F(x) = \{f \in E^* : \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x\rangle \quad \forall y \in E\}.$$

Proof. Fix $x, y \in E$, $f \in F(x)$ and define $R_x := \{f \in E^* : \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2 \geq \langle f, y - x\rangle \quad \forall y \in E\}$. Observe that $\langle f, y - x\rangle = \langle f, y\rangle - \|x\|^2 \leq \|f\|\|y\| - \|x\|^2 = \|x\|(\|y\| - \|x\|)$. There are two cases to consider: when $\|y\| \geq \|x\|$, we have that $\|x\| \leq \frac{\|y\|+\|x\|}{2}$ and $\|y - x\| \geq 0$, so that $\langle f, y - x\rangle \leq \|x\|(\|y\| - \|x\|) \leq \frac{(\|y\|+\|x\|)}{2}(\|y\| - \|x\|)$, and the desired inequality follows. Otherwise, when $\|y\| \leq \|x\|$, then $\|x\| \geq \frac{\|y\|+\|x\|}{2}$ and $\|y\| - \|x\| < 0$, so that again $\langle f, y - x\rangle \leq \|x\|(\|y\| - \|x\|) \leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2$. This shows that the desired inequality holds for all $y \in E$, and therefore $f \in R_x$ giving the first inclusion $F(x) \subset R_x$.

Towards showing the other inclusion, suppose that $f \in R_x$. Then for any $\lambda > 1$, we have that $(\lambda - 1)\langle f, x\rangle \leq \frac{\lambda^2 - 1}{2}\|x\|^2$ and it follows that $\langle f, x\rangle \leq \frac{\lambda+1}{2}\|x\|^2$ for all $\lambda > 1$. Taking the limit as $n \to 1$ gives $\langle f, x\rangle \leq \|x\|^2$.

Similarly, we see that for all $\lambda \in (0,1)$, $\langle f, x\rangle \geq \frac{\lambda+1}{2}\|x\|^2$ and, in the limit, we see that $\langle f, x\rangle = \|x\|^2$. Using what we just showed, it follows that for all $y \in E$, $\langle f, y\rangle \leq \frac{1}{2}\|y\|^2 + \frac{1}{2}\|x\|^2$ and therefore for any $\varepsilon > 0$, we have

$$\|f\| = \sup_{y\in E, \|y\|=1} \langle f, y\rangle = \frac{1}{\varepsilon}\sup_{y\in E, \|y\|=\varepsilon} \langle f, y\rangle \leq \frac{\varepsilon}{2} + \frac{1}{2\varepsilon}\|x\|^2.$$

Assuming that $x \neq 0$, taking $\varepsilon = \|x\|$ gives $\|f\| \leq \|x\|$ so that $f \in F(x)$. In the case where $x = 0$, note that for every $y \in E$, we have that $\langle f, y\rangle = \frac{1}{\varepsilon}\langle f, \varepsilon y\rangle \leq \frac{\varepsilon}{2}\|y\|^2$ for all $\varepsilon > 0$. It follows that $f = 0 \in F(0)$, proving that $F(x) = R_x$ for all $x \in E$. □

4. Deduce that

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \forall x, y \in E,$$

and more precisely that

$$\langle f - g, x - y\rangle \geq 0 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Show that, in fact,

$$\langle f - g, x - y\rangle \geq (\|x\| - \|y\|)^2 \quad \forall x, y \in E, \quad \forall f \in F(x), \quad \forall g \in F(y).$$

Proof. Fix $x, y \in E$, $f \in F(x)$ and $g \in F(y)$. From (3), we have

$$\langle f - g, x - y\rangle = -\langle f, y - x\rangle - \langle g, x - y\rangle \geq \left(\frac{1}{2}\|x\|^2 - \frac{1}{2}\|y\|^2\right) + \left(\frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2\right) = 0,$$

proving the first inequality.

For the second inequality, we have

$$\begin{aligned} \langle f - g, x - y\rangle &= \langle f, x\rangle - \langle f, y\rangle - \langle g, x\rangle + \langle g, y\rangle \\ &= \|x\|^2 - \langle f, y\rangle - \langle g, x\rangle + \|y\|^2 \\ &\geq \|x\|^2 - \|f\|\|y\| - \|g\|\|x\| + \|y\|^2 \\ &= (\|x\| - \|y\|)^2. \end{aligned}$$

□

5. Assume again that E^* is strictly convex and let $x, y \in E$ be such that

$$\langle F(x) - F(y), x - y\rangle = 0.$$

Show that $Fx = Fy$.

Proof. From the last inequality in (4), we see that $(\|x\| - \|y\|)^2 \leq \langle F(x) - F(y), x - y\rangle = 0$, so that $\|x\| = \|y\|$. Moreover, since $0 = \|x\|^2 - \langle Fx, y\rangle + \|y\|^2 - \langle Fy, x\rangle$ and $\|x\|^2 - \langle Fx, y\rangle \geq \|x\|^2 - \|Fx\|\|y\| = 0$ and similarly $\|y\|^2 - \langle Fy, x\rangle \geq 0$, it follows that $\langle Fx, y\rangle = \|x\|^2 = \langle Fy, x\rangle$. Since $\left\langle \frac{1}{2}\frac{Fx}{\|x\|} + \frac{1}{2}\frac{Fy}{\|x\|}, \frac{x}{\|x\|}\right\rangle = 1$, it follows that $\left\|\frac{1}{2}\frac{Fx}{\|x\|} + \frac{1}{2}\frac{Fy}{\|x\|}\right\| \geq 1$. Finally, observing that $\left\|\frac{Fx}{\|x\|}\right\| = \left\|\frac{Fy}{\|x\|}\right\| = 1$, the fact that E^* is strictly convex implies that $Fx = Fy$. □

1.2

Let E be a vector space of dimension n and let $(e_i)_{1\leq i\leq n}$ be a basis of E. Given $x \in E$, write $x = \sum_{i=1}^n x_i e_i$ with $x_i \in \mathbb{R}$; given $f \in E^*$, set $f_i = \langle f, e_i\rangle$.

2. Consider on E the norm

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

   (a) Compute explicitly, in terms of the $f_i's$, the dual norm $\|f\|_{E^*}$ of $f \in E^*$.

Solution

Fix $f \in E^*$ and note that for any $x \in E$, $\langle f, x\rangle = \sum_i x_i f_i \le \sum_i |x_i||f_i| \le \|x\|_\infty(\sum_i |f_i|)$. Thus, $\|f\|_{E^*} \le \sum_i |f_i|$. Now let $y := (sgn(f_i))_{1\le i\le n}$, where we set $sgn(f_i) = 1$ if $f_i = 0$. Clearly $\langle f, y\rangle = \sum_i |f_i|$ and $\|y\|_\infty = 1$, hence $\|f\| = \sum_i |f_i|$.

(b) Determine explicitly the set $F(x)$ (duality map) for every $x \in E$.

Solution

Fix $x \in E$ and suppose that $f \in F(x)$. Then $\sum_i x_i f_i = \|x\|_\infty^2 = \max_i |x_i|^2$ and $\max_i |x_i| = \|f\| = \sum_i |f_i|$. Note that $\max_i |x_i|^2 = \sum_i x_i f_i \le \sum_i |x_i||f_i| \le \|f\|\|x\|_\infty = \max_i |x_i|^2$. It follows that for each i, $x_i f_i \ge 0$. Let $A := \{1 \le i \le n : |x_i| = \max_i |x_i|\}$. I claim that for all $j \notin A$, $f_j = 0$. Towards proving this claim, suppose for a contradiction that for some $j \notin A$, $|f_j| > 0$. Then $\max_i |x_i|^2 = \sum_i |x_i||f_i| = |x_j|(\max_k |x_k| - \sum_{i\ne j} |f_j|) + \sum_{i\ne j} |x_i||f_i| < \max_k |x_k|^2 - \max_k |x_k| \sum_{i\ne j} |f_i| + \max_k |x_k| \sum_{i\ne j} |f_i|$, a contradiction. Hence, $F(x) = \{f \in E^* : \sum_{i\in A} x_i f_i = \max_{1\le i\le n} |x_i| \text{ and } \forall j \notin A : f_j = 0 \text{ and } \forall j \in A : x_j f_j \ge 0\}$.

1.3

Let $E = \{u \in C([0,1];\mathbb{R}) : u(0) = 0\}$ with its usual norm

$$\|u\| = \max_{t\in[0,1]} |u(t)|.$$

Consider the linear functional

$$f : u \in E \mapsto f(u) = \int_0^1 u(t)dt.$$

1. Show that $f \in E^*$ and compute $\|f\|_{E^*}$.

Proof. The linearity of f follows from the linearity of the integral over $[0,1]$. Note that for any $u \in E$, $f(u) \le \int_0^1 |u(t)|dt \le \max_{t\in[0,1]} |u(t)| \int_0^1 dt = \|u\|$. Thus, $f \in E^*$ and $\|f\|_{E^*} \le 1$. To see that $\|f\| = 1$, for each $n \ge 1$, define $u_n \in E$ by $u_n(t) = \begin{cases} nx & 0 \le t \le \frac{1}{n} \\ 1, & \frac{1}{n} \le t \le 1. \end{cases}$ Clearly $\|u_n\| = 1$ for all n and $f(u_n) = (1 - \frac{1}{n}) + \frac{1}{2n} = 1 - \frac{1}{2n}$. It follows that $\|f\|_{E^*} \ge 1 - \frac{1}{2n}$ for all $n \ge 1$, so that $\|f\|_{E^*} = 1$. □

2. Can one find some $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|_{E^*}$?

Solution

No. Observe that for any $u \in E$ with $\|u\| = 1$, the fact that u is continuous and $u(0) = 0$ implies that there exists some $\varepsilon > 0$ such that $|u(t)| < \frac{1}{2}$ for all $t \in [0, \varepsilon)$. Thus,

$$f(u) \le \left|\int_0^1 u(t)dt\right| \le \int_0^1 |u(t)|dt < \frac{\varepsilon}{2} + \int_\varepsilon^1 |u(t)|dt \le \frac{\varepsilon}{2} + (1-\varepsilon) = 1 - \frac{\varepsilon}{2} < \|f\|_{E^*}.$$

1.6

Let E be an n.v.s. and let $H \subset E$ be a hyperplane. Let $V \subset E$ be an affine subspace containing H.

1. Prove that either $V = H$ or $V = E$.

Proof. Let f be a linear functional on E and $\alpha \in \mathbb{R}$ such that $H = [f = \alpha]$. Since V is an affine subspace, there exists a linear subspace V' of E and $v_0 \in E$ such that $V = v_0 + V'$. Observe that WLOG, we may assume that $\langle f, v_0\rangle = \alpha$, so that $v_0 \in H$. Indeed, if $\langle f, v_0\rangle \ne \alpha$, then there must exist some $w \in V'$ such that $\langle f, v_0 + w\rangle = \alpha$, and we can simply take $V = (v_0 + w) + V'$. With this assumption in mind, observe that for any $w \in V'$, $w \in \ker f$ implies that $\langle f, v_0 + w\rangle = \alpha$, so that $v_0 + w \in H$, showing that $v_0 + \ker f \subset H$. Moreover, since $H \subset V$, if $w \in V'$ such that $v_0 + w \in H$, then $\langle f, v_0 + w\rangle = \langle f, v_0\rangle$, which implies that $w \in \ker f$. Thus, we have $H = v_0 + \ker f$. Suppose that $V \neq H$ so that $V' \neq \ker f$. Then there must exist some $w_0 \in V'$ such that $\langle f, w_0 \rangle \neq 0$. By homogeneity, it follows that for all $t \in \mathbb{R}$, there exists some $w_t \in V'$ such that $\langle f, w_t \rangle = t$. Clearly $E = \bigcup_{t\in\mathbb{R}}[f = t]$. Fix $t_0 \in \mathbb{R}$ and $y \in [f = t_0]$. Then taking $w_{-t_0} \in V'$. we have that $y + w_{-t_0} \in \ker f \subset V'$, so that $y = (y + w_{-t_0}) - w_{-t_0} \in V'$. It follows that $E = \bigcup_{t\in\mathbb{R}}[f = t] \subset V'$, which proves that $V = E$. □

2. Deduce that H is either closed or dense in E.

   Proof. Let $v_0 \in H$ and observe that since $H = v_0 + \ker f \subset v_0 + \overline{\ker f}$, the fact that $v_0 + \overline{\ker f}$ is an affine subspace containing H implies that either $H = v_0 + \overline{\ker f}$, so that H is closed, or $\overline{H} = \overline{v_0 + \ker f} = v_0 + \overline{\ker f} = E$, so that H is dense in E. □

1.8

Let E be an n.v.s. with norm $\| \ \|$. Let $C \subset E$ be an open convex set such that $0 \in C$. Let p denote the gauge of C.

1. Assuming C is symmetric (i.e., $-C = C$) and C is bounded, prove that p is a norm which is equivalent to $\| \ \|$.

   Proof. The gauge p is defined by $p(x) = \inf\{\alpha > 0 : x \in \alpha C\}$. From Lemma 1.2 (9) and (10), we see that there exists a constant M such that $0 \leq p(x) \leq M\|x\| \quad \forall x \in E$, and $C = \{x \in E : p(x) < 1\}$. The triangle inequality holds for p by definition. Towards proving homogeneity of p, fix $\lambda \leq 0$, $x \in E$ and observe that for any $\alpha > p(x)$, $x \in \alpha C$ by the definition of p. By the symmetry of C, it follows that $-x \in \alpha C$ so that $p(-x) \leq \alpha$. Thus, $p(-x) \leq p(x)$. By symmetry, it's clear that $p(-x) = p(x)$. It follows that $p(\lambda x) = p(-|\lambda| x) = |\lambda| p(-x) = |\lambda| p(x)$, proving homogeneity. To finish the proof that p defines a norm on E, note that it suffices to find some $m > 0$ such that $m\|x\| \leq p(x)$ for all $x \in E$. Since C is bounded, there exists some $c > 0$ such that $\|x\| \leq c$ for all $x \in C$. Pick $y \in E$, fix $\varepsilon > 0$ and note that $\frac{1}{p(y)+\varepsilon} y \in C$ so that $\frac{1}{p(y)+\varepsilon}\|y\| \leq c$. It follows that $\frac{1}{c}\|y\| \leq p(y) + \varepsilon$. Since this inequality holds for all $\varepsilon > 0$ and $y \in E$, we have $m = \frac{1}{c} > 0$ gives the desired constant. Note that since $m\|x\| \leq p(x) \leq M\|x\|$ for all $x \in E$, p and $\| \ \|$ are equivalent norms. □

2. Let $E = C([0, 1]; \mathbb{R})$ with its usual norm
   $$\|u\| = \max_{t\in[0,1]} |u(t)|.$$
   Let
   $$C = \Big\{u \in E : \int_0^1 |u(t)|^2 dt < 1\Big\}.$$
   Check that C is convex and symmetric and that $0 \in C$. Is C bounded in E? Compute the gauge p of C and show that p is a norm on E. Is p equivalent to $\| \ \|$?

   Solution

   Fix $u_1, u_2 \in C$ and $\lambda \in [0, 1]$. By the convexity of $x \mapsto x^2$, we have
   $$\int_0^1 |\lambda u_1(t) + (1-\lambda)u_2(t)|^2 dt \leq \lambda \int_0^1 |u_1(t)|^2 dt + (1-\lambda) \int_0^1 |u_2(t)|^2 dt < 1.$$
   Since $\lambda u_1 + (1-\lambda)u_2$ is obviously continuous, $\lambda u_1 + (1-\lambda)u_2 \in C$ which shows that C is convex. That C is symmetric simply follows from the fact that for all $u \in C$, $-u$ is continuous and $\int_0^1 |-u(t)|^2 dt = \int_0^1 |u(t)|^2 dt < 1$. Since $0 \in E$ and $\int_0^1 |0(t)|^2 dt = 0 < 1$, $0 \in C$. Observe that C is not bounded: for each $n \geq 1$, $u_n := \begin{cases} \sqrt{n(1-nt)}, & 0 \leq t \leq \frac{1}{n} \\ 0, & \frac{1}{n} \leq t \leq 1 \end{cases} \in E$ and $\int_0^1 |u_n(t)|^2 dt = \frac{1}{2} < 1$, so that $u_n \in C$. The fact that $\|u_n\| = \sqrt{n} \to \infty$ as $n \to \infty$ proves that C is unbounded in E.

   Towards computing the gauge p of C, note that for any $\alpha > 0$, $\alpha^{-1}u \in C$ if and only if $\|u\|^2_{L^2([0,1])} < \alpha^2$. Thus, after taking square roots, taking the inf over all such α gives $p(u) = \|u\|_{L^2([0,1])}$. That $\| \ \|_{L^2([0,1])}$ is a norm on E is immediate given that E can be realized as a subspace of $L^2([0,1])$. Clearly $\| \ \|$ and $\| \ \|_{L^2([0,1])}$ are not equivalent norms on E since $C \subset E$ is bounded with respect to the latter and unbounded with respect to the former.

1.14

Let $E = \ell^1$ and consider the two sets

$$X = \Big\{ x = (x_n)_{n\ge1} \in E : x_{2n} = 0 \quad \forall n \ge 1 \Big\}$$

and

$$Y = \Big\{ y = (y_n)_{n\ge1} \in E : y_{2n} = \frac{1}{2^n} y_{2n-1} \quad \forall n \ge 1 \Big\}.$$

1. Check that X and Y are closed linear spaces and that $\overline{X+Y} = E$.

   Solution

   Fix $x, x' \in X$ and $\lambda \in \mathbb{R}$. Observe that $(x+x')_{2n} = x_{2n} + x'_{2n} = 0$, $(\lambda x)_{2n} = \lambda x_{2n} = 0$ and $0_{2n} = 0$ for all $n \ge 1$, which shows that X is linear subspace of E. Now suppose that $(x^k)_{k\ge1} \subset X$ converges in ℓ^1 to some point $x \in E$. Then for any $n \ge 1$, since $|x_{2n}| = |x^k_{2n} - x_{2n}| \le \sum_m |x^k_m - x_m| = \|x^k - x\| \to 0$ as $k \to \infty$, it follows that $x \in X$, and so X is a closed linear space.

   Now fix $y, y' \in Y$ and observe that $(y+y')_{2n} = \frac{1}{2^n} y_{2n-1} + \frac{1}{2^n} y'_{2n-1} = \frac{1}{2^n}(y+y')_{2n-1}$, $(\lambda y)_{2n} = \frac{1}{2^n}\lambda y_{2n-1}$, and $0_{2n} = 0 = \frac{1}{2^n} 0_{2n-1}$ for all $n \ge 1$, which shows that Y is also a linear subspace of E. Suppose that $(y^k)_{k\ge1} \subset Y$ converges in ℓ^1 to some point $y \in E$. Then for all $n \ge 1$, $|y_{2n} - \frac{1}{2^n} y_{2n-1}| \le |y_{2n} - y^k_{2n}| + \frac{1}{2^n}|y^k_{2n-1} - y_{n-1}| \le (1 + \frac{1}{2^n})\|y - y^k\| \to 0$ as $k \to \infty$, proving that Y is a closed linear space.

   Towards proving that $\overline{X+Y} = E$, fix $(a_n)_{n\ge1} \in E$. For each $N \ge 1$, define the elements $x^N \in X$ and $y^N \in Y$ as follows: for $n > 2N$, define $x^N_n = y^N_n = 0$ and for $1 \le n \le N$, define $y_{2n-1} = 2^n a_{2n}$, $y_{2n} = \frac{1}{2^n} y_{2n-1}$, $x_{2n} = 0$ and $x_{2n-1} = a_{2n-1} - 2^n a_{2n}$. Note that $(x^N + y^N)_n = a_n$ for all $0 \le n \le 2N$, and so $\|a - (x^N + y^N)\| = \sum_{n\ge 2N+1} |a_n| \to 0$ as $N \to \infty$. Thus, $a \in \overline{X+Y}$, which proves that $\overline{X+Y} = E$.

2. Let $c \in E$ be defined by

   $$\begin{cases} c_{2n-1} = 0 & \forall n \ge 1, \\ c_{2n} = \frac{1}{2^n} & \forall n \ge 1. \end{cases}$$

   Check that $c \notin X + Y$.

   Solution

   Towards a contradiction, suppose that $c \in X+Y$. Then $c = x + y$ for some $x \in X$ and $y \in Y$. Since $x_{2n} = 0$ for all $n \ge 1$, it follows that $y_{2n} = \frac{1}{2^n}$ and therefore $y_{2n-1} = 2^n y_{2n} = 1$ for all $n \ge 1$. But then obviously $\|y\| = \infty$, contradicting the fact that y belongs to ℓ^1. Thus, $c \notin X + Y$

3. Set $Z = X - c$ and check that $Y \cap Z = \varnothing$. Does there exist a closed hyperplane in E that separates Y and Z? Compare with Theorem 1.7 and Exercise 1.9.

   Solution

   That $Y \cap Z = \varnothing$ follows immediately from part 2. since otherwise there would be some $y \in Y$ such that $y = x - c$ for some $x \in X$, contradicting that $c \notin X + Y$. To see that no closed hyperplane in E separates Y and Z, suppose for a contradiction that there were some nonzero $f \in E^*$ and α such that $\langle f, y\rangle \le \alpha \le \langle f, x - c\rangle$ for all $y \in Y$ and $x \in X$. Since X and Y are both linear subspaces, the only way this is possible is if $\ker f \supset X, Y$. But then $X + Y \subset \ker f$, so that $E = \overline{X+Y} \subset \overline{\ker f} = \ker f$, contradicting our assumption that f is nonzero. Note that this result does not conflict with the Hahn-Banach, second geometric form (Theorem 1.7) since neither X nor Y are compact (it's easy to see that neither are bounded in ℓ^1).

4. Same questions in $E = \ell^p$, $1 < p < \infty$, and in $E = c_0$.

Solution

Checking that X and Y are still closed linear subspaces when we set $E = \ell^p$ or $E = c_0$ is a matter of adding a pth power or using the sup norm in place of the ℓ^1 norm above, and is trivial. Similarly, my proof that $\overline{X+Y} = E$ works equally well with $E = \ell^p$ or $E = c_0$, just adding a power of p or observing that the trailing sequence converges to 0. My proofs for 3. and 4. work without any changes.

1.16

Let $E = \ell^1$, so that $E^* = \ell^\infty$. Consider $N = c_0$ as a closed subspace of E^*. Determine

$$N^\perp = \{x \in E : \langle f, x\rangle = 0 \quad \forall f \in N\}$$

and

$$N^{\perp\perp} = \{f \in E^* : \langle f, x\rangle = 0 \quad \forall x \in N^\perp\}.$$

Check that $N^{\perp\perp} \neq N$.

Solution

Note that $N^\perp = \{0\}$. To see why this holds, fix $x \in N^\perp$ and $n \in \mathbb{N}$ and observe that $(\delta_{i,n})_{i\geq 1}$ clearly belongs to N so that $0 = \langle (\delta_{i,n})_{i\geq 1}, x\rangle = x_n$. The claim then follows by noting that this identity holds for all $n \geq 1$. Thus, $N^{\perp\perp} = \{f \in E^* : \langle f, x\rangle = 0 \quad \forall x \in \{0\}\} = E^* = \ell^\infty$. Since $(1)_{i\geq 1} \in \ell^\infty \setminus c_0$, it follows that $N^{\perp\perp} \neq N$.

1.17

Let E be an n.v.s. and let $f \in E^*$ with $f \neq 0$. Let M be the hyperplane $[f = 0]$.

1. Determine $M^\perp$.

   Solution

   Clearly $\text{span}(f) \subset M^\perp$. Fix $x \in E \setminus M$ so that $\langle f, x\rangle \neq 0$. Observe that for any $y \in E$, $\left\langle f, y - \frac{\langle f,y\rangle)}{\langle f,x\rangle} x\right\rangle = 0$, so that $y - \frac{\langle f,y\rangle}{\langle f,x\rangle} x \in M$ for all $y \in E$. It follows that for all $g \in M^\perp$ and all $y \in E$, $\langle g, y\rangle = \frac{\langle g,x\rangle}{\langle f,x\rangle}\langle f, y\rangle$. Thus, $g \in \text{span}(f)$, proving that $M^\perp = \text{span}(f)$.

2. Prove that for every $x \in E$, $\text{dist}(x, M) = \inf_{y\in M} \|x - y\| = \frac{|\langle f,x\rangle|}{\|f\|}$.

   Proof. From Example 1.3 of section 1.4 and part 1. above, we have that for any $x \in E$,

   $$\text{dist}(x, M) = \max_{g\in M^\perp, \|g\|\leq 1} |\langle g, x\rangle| = \max_{\lambda\in\mathbb{R}} \frac{|\langle \lambda f, x\rangle|}{\|\lambda f\|} = \frac{|\langle f, x\rangle|}{\|f\|}.$$

   □

3. Assume now that $E = \{u \in C([0, 1]; \mathbb{R}) : u(0) = 0\}$ and that

   $$\langle f, u\rangle = \int_0^1 u(t)dt, \quad u \in E.$$

   Prove that $\text{dist}(u, M) = |\int_0^1 u(t)dt| \,\forall u \in E$. Show that $\inf_{v\in M} \|u - v\|$ is never achieved for any $u \in E \setminus M$.

   Solution

   I showed in problem 1.3 part 1. that $\|f\| = 1$, so that by part 2. above, for all $u \in E$, $\text{dist}(u, M) = \frac{|\langle f,u\rangle|}{\|f\|} = |\int_0^1 u(t)dt|$. In part 2. of problem 1.3, I showed that there exists no $u \in E$ such that $\|u\| = 1$ and $\langle f, u\rangle = 1 = \|u\|$. Since $|\langle f, u\rangle| \leq \|f\|\|u\| = \|u\|$ for all $u \in E$, it's clear from the previous sentence that for all nonzero $u \in E$, $|\langle f, u\rangle| < \|u\|$. Thus, for all $u \in E \setminus M$ and all $v \in M$, $\|u - v\| > |\langle f, u - v\rangle| = |\langle f, u\rangle| = \text{dist}(u, M)$, which proves that $\inf_{v\in M} \|u - v\|$ is never achieved for any $u \in E \setminus M$.

2.1 Continuity of convex functions.

Let E be a Banach space and let $\varphi : E \to (-\infty, +\infty]$ be a convex l.s.c. function. Assume $x_0 \in \text{Int}D(\varphi)$.

1. Prove that there exist two constants $R > 0$ and M such that

$$\varphi(x) \le M \quad \forall x \in E \text{ with } \|x - x_0\| \le R.$$

Proof. Since $x_0 \in \text{Int}D(\varphi)$, there exists a neighborhood V of x_0 such that $\varphi(y) < \infty$ for all $y \in V$. Hence, there exists some $\rho > 0$ such that $\overline{B(x_0, \rho)} \subset V$. Now for each $n \ge 1$, define $F_n := \{x \in E : \|x - x_0\| \le \rho \text{ and } \varphi(x) \le n\}$. Note that $\bigcup_{n=1}^{\infty} F_n = \overline{B(x_0, \rho)}$ and each F_n is closed by the lower semicontinuity of φ since $F_n = \overline{B(x_0, \rho)} \cap [\varphi \le n]$. By the Baire category theorem, the fact that $\overline{B(x_0, \rho)}$ is not meager implies that there must exist some $n_0 \ge 1$ such that $\text{Int}\, F_{n_0} \ne \varnothing$. It follows that there exists some $y_0 \in F_{n_0}$ and $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset F_{n_0}$. Observe that for any $x \in \overline{B(x_0, \frac{\varepsilon}{2})}$, $x = \frac{1}{2}(y + 2(x - x_0)) + \frac{1}{2}(x_0 + (x_0 - y))$. Applying the convexity of φ, we have that $\varphi(x) \le \frac{1}{2}\varphi(y + 2(x - x_0)) + \frac{1}{2}\varphi(x_0 + (x_0 - y))$. Observing that $y + 2(x - x_0) \in B(y, \varepsilon)$, it follows that $\varphi(x) \le \frac{n_0}{2} + \frac{1}{2}\varphi(x_0 + (x_0 - y))$ for all $x \in \overline{B(x_0, \frac{\varepsilon}{2})}$. Since $x_0 + (x_0 - y) \in \overline{B(x_0, \rho)} \subset D(\varphi)$, we can take $R = \frac{\varepsilon}{2}$ and $M = \frac{n_0}{2} + \frac{1}{2}\varphi(2x_0 + y)$. □

2. Prove that $\forall r < R, \exists L \ge 0$ such that

$$|\varphi(x_1) - \varphi(x_2)| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in E \text{ with } \|x_i - x_0\| \le r,\ i = 1, 2.$$

More precisely, one may choose $L = \frac{2[M - \varphi(x_0)]}{R - r}$.

Proof. Clearly we may assume WLOG that $x_0 = 0$. Fix $r \ge 0$ with $r < R$ and $x_1, x_2 \in \overline{B(0, r)}$. The inequality is trivial if $x_1 = x_2$, so WLOG assume that $x_1 \ne x_2$. Let $y = \frac{R}{\|x_1 - x_2\|}(x_1 - x_2)$. Then $x_1 = ty + (1 - t)x_2$ for some $t \in [0, 1]$, so that $\varphi(x_1) \le t\varphi(y) + (1-t)\varphi(x_2) \le tM + (1-t)\varphi(x_2)$. It follows that $\varphi(x_1) - \varphi(x_2) \le t(M - \varphi(x_2))$. Since $x_1 - x_2 = t(y - x_2)$, it follows that $\|x_1 - x_2\| \ge t(R - r)$, and so $\varphi(x_1) - \varphi(x_2) \le \frac{\|x_1 - x_2\|}{R - r}(M - \varphi(x_2))$. Applying the same reasoning except replacing x_1 with 0, we have that $\varphi(0) - \varphi(x_2) \le t(M - \varphi(x_2))$. Since $-x_2 = t(y - x_2)$, so that $\|x_2\| = t\|(-\frac{\|x_2\|}{R} - 1)x_2\|$. Solving for t, we get that $t = \frac{\|x_2\|}{R + \|x_2\|} \le \frac{1}{2}$. Hence, $\varphi(0) - \varphi(x_2) \le \frac{1}{2}(M - \varphi(x_2))$. Rearranging, we have $-\varphi(x_2) \le M - 2\varphi(0)$. Plugging this back into our prior inequality, we have $\varphi(x_1) - \varphi(x_2) \le \frac{2(M - \varphi(0))}{R - r}\|x_1 - x_2\|$. By symmetry, we must also have $\varphi(x_2) - \varphi(x_1) \le \frac{2(M - \varphi(0))}{R - r}\|x_2 - x_1\|$, and the desired inequality follows. □

2.3

Let E and F be two Banach spaces and let (T_n) be a sequence in $\mathcal{L}(E, F)$. Assume that for every $x \in E$, $T_n x$ converges as $n \to \infty$ to a limit denoted by Tx. Show that if $x_n \to x$ in E, then $T_n x_n \to Tx$ in F.

Proof. Suppose that $x_n \to x \in E$. Because $T_n y \to Ty$ for all $y \in E$, it follows that $\|T_n y\| \to \|Ty\|$ for all $y \in E$, so that $\sup_n \|T_n y\| < \infty$ for all $y \in E$. By the uniform boundedness principle, there exists $C \in \mathbb{R}$ such that $\sup_n \|T_n\| \le C$. Thus, for all $n \ge 1$

$$\|T_n x_n - Tx\| \le \|T_n(x_n - x)\| + \|T_n x - Tx\| \le C\|x_n - x\| + \|T_n x - Tx\| \to 0 \quad \text{as } n \to \infty.$$

□

2.4

Let E and F be two Banach spaces and let $a : E \times F \to \mathbb{R}$ be a bilinear form satisfying:

(i) for each fixed $x \in E$, the map $y \mapsto a(x, y)$ is continuous;

(ii) for each fixed $y \in F$, the map $x \mapsto a(x, y)$ is continuous.

Prove that there exists a constant $C \ge 0$ such that

$$|a(x, y)| \le C\|x\|\|y\| \quad \forall x \in E, \quad \forall y \in F.$$

Proof. For every $x \in E$, define $T_x : y \in F \mapsto a(x, y)$. By assumption each $T_x \in F^*$. Define the map $T : E \to F^*$; $x \mapsto T_x$. Note that the proof will be complete if I can show that T is a bounded linear operator since then for any $x \in E$ and $y \in F$,

$$|a(x, y)| = |\langle T_x, y\rangle| \leq \|T_x\|\|y\| \leq \|T\|\|x\|\|y\|.$$

That T is linear follows from the fact that a is bilinear: $\forall x_1, x_2 \in E \quad \forall \lambda_1, \lambda_2 \in \mathbb{R} \quad \forall y \in F : \quad \langle T(\lambda_1 x_1 + \lambda_2 x_2), y\rangle = a(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 a(x_1, y) + \lambda_2 a(x_2, y) = \langle \lambda_1 T(x_1) + \lambda_2 T(x_2), y\rangle$. To prove that T is bounded, let $T(B) := \{T(x) : \|x\| \leq 1\} \subset F^*$. Fix $y \in F$ and observe that by the assumed continuity of the linear map $x \mapsto a(x, y)$, there exists $C_y \in \mathbb{R}$ such that $|a(x, y)| \leq C_y\|x\|$ for all $x \in E$. Hence, for all $T(x) \in T(B)$

$$|\langle T(x), y\rangle| = |a(x, y)| \leq C_y\|x\| \leq C_y.$$

It follows that for each $y \in F$, the set $\langle T(B), y\rangle$ is bounded in $\mathbb{R}$ and so by corollary 2.5, $T(B)$ is bounded in F^*. That is, there exists $C \in \mathbb{R}$ such that $\|T(x)\| \leq C$ for all $x \in E$ with $\|x\| \leq 1$. This proves that T is bounded and has operator norm $\|T\| \leq C$. The desired inequality follows. □

2.5

Let E be a Banach space and let ε_n be a sequence of positive numbers such that $\lim \varepsilon_n = 0$. Further, let (f_n) be a sequence in E^* satisfying the property

$$\begin{cases} \exists r > 0, \quad \forall x \in E \quad \text{with } \|x\| < r, \exists C(x) \in \mathbb{R} \quad \text{such that} \\ \langle f_n, x\rangle \leq \varepsilon_n\|f_n\| + C(x) \quad \forall n. \end{cases}$$

Prove that (f_n) is bounded.

Proof. For each $n \geq 1$, define $\frac{1}{1+\varepsilon_n\|f_n\|} f_n$. Fix $x \in E$. I claim that the set $\langle (g_n)_{n\geq 1}, x\rangle$ is bounded in $\mathbb{R}$. If $x = 0$, this statement is obvious so suppose WLOG that $x \neq 0$. Then by assumption, for all $n \geq 1$,

$$\begin{aligned}\langle g_n, \frac{r}{2\|x\|}x\rangle &= \frac{1}{1+\varepsilon_n\|f_n\|}\langle f_n, \frac{r}{2\|x\|}x\rangle \\ &\leq \frac{\varepsilon_n\|f_n\| + C(\frac{r}{2\|x\|}x)}{1+\varepsilon_n\|f_n\|} \\ &\leq 1 + C\Big(\frac{r}{2\|x\|}x\Big).\end{aligned}$$

It follows that for all $x \in E$ and $n \geq 1$, $\langle g_n, x\rangle \leq \frac{2\|x\|}{r}\Bigg(1 + C\Big(\frac{r}{2\|x\|}x\Big)\Bigg)$. Hence, for all $x \in E$, the set $\langle (g_n)_{n\geq 1}, x\rangle$ is bounded and so by corollary 2.5, the set $(g_n)_{n\geq 1}$ is a bounded subset of E^*. That is, there exists some $C \in \mathbb{R}$ $\sup_n \|g_n\| = \sup_n \frac{1}{1+\varepsilon_n\|f_n\|}\|f_n\| \leq C$. Thus, for any n such that $\|f_n\| > 0$, $1 - \varepsilon_n \leq \frac{C}{\|f_n\|}$. Since $\varepsilon_n \to 0$, there exists N such that $\varepsilon_n \leq \frac{1}{2}$ for all $n \geq N$, so that for all $n \geq N$ such that $\|f_n\| > 0$, $\|f_n\| \leq 2C$, proving that (f_n) is bounded in E^*. □

2.7

Let $\alpha = (\alpha_n)$ be a given sequence of real numbers and let $1 \leq p \leq \infty$. Assume that $\sum|\alpha_n||x_n| < \infty$ for every element $x = (x_n)$ in ℓ^p. Prove that $\alpha \in \ell^{p'}$.

Proof. For $p = \infty$, set $(x_n)_{n\geq} = (1)_{n\geq 1}$ and observe that $\sum|\alpha_n| = \sum|\alpha_n||x_n| < \infty$, so that $\alpha \in \ell^1$. For $p = 1$, suppose for a contradiction that $\alpha \notin \ell^\infty$. Then for each $k, N \geq 1$, there must exist some $n_k \geq N$ such that $|\alpha_{n_k}| \geq 2^k$. Thus, we can construct an increasing sequence (n_k) such that $|\alpha_{n_k}| \geq 2^k$ for all $k \geq 1$. For each k, define $(x_n)_{n\geq 1} = \sum_k (\partial_{n,n_k}\frac{1}{2^k})_{n\geq 1}$ and note that $(x_n) \in \ell^1$ but $\sum_n |\alpha_n||x_n| \geq \sum_k |\alpha_{n_k}|\frac{1}{2^k} = \infty$. By contradiction, $\alpha \in \ell^\infty$.

Having dealt with the cases $p = 1$ and $p = \infty$, we may assume WLOG that $1 < p < \infty$. For each $n \geq 1$, define the map $T_n : \ell^p \to \mathbb{R}$; $(x_j) \mapsto \sum_{j=1}^n \alpha_j x_j$. Clearly each T_n is a continuous linear functional and, by assumption, for all $x \in \ell^p$, $\langle T_n, x\rangle$ converges as $n \to \infty$ to some point which we shall denote Tx. Then by corollary 2.3, $T \in \ell^{p*}$ and so there exists $C \in \mathbb{R}$ such that $|\sum_n \alpha_n x_n| = |\langle T, x\rangle| \le C\|x\|_p$ for all $x \in \ell^p$. Now for each n, define $\alpha_{k,n} := \begin{cases} \alpha_k, & k \le n \\ 0, & k > n. \end{cases}$ Clearly $(a_{k,n})_{k\ge1} \in \ell^p$ for all n. For each n, define $\beta_n : \mathbb{N} \to \mathbb{R};\ k \mapsto \frac{\operatorname{sgn}(\alpha_{k,n})|\alpha_{k,n}|^{p'-1}}{\|(\alpha_{m,n})_{m\ge1}\|_{p'}^{p'/p}}$. Note that $\|\beta_n\|_p^p = \sum_{k=1}^n \frac{|\alpha_k|^{pp'-p}}{\|(\alpha_{m,n})_{m\ge1}\|_{p'}^{p'}} = 1$ and $\langle T, \beta_n\rangle = \sum_{k=1}^n \frac{|\alpha_k|^{p'}}{\|(\alpha_{m,n})_{m\ge1}\|_{p'}^{p'/p}} = \left(\sum_{k=1}^n |\alpha_k|^{p'}\right)^{\frac{1}{p'}}$. Thus, for all $n \ge 1$, $\left(\sum_{k=1}^n |\alpha_k|^{p'}\right)^{\frac{1}{p'}} = \langle T, \beta_n\rangle \le C\|\beta_n\|_p = C$, proving that $\alpha \in \ell^{p'}$. □

2.8

Let E be a Banach space and let $T : E \to E^*$ be a linear operator satisfying

$$\langle Tx, x\rangle \ge 0 \quad \forall x \in E.$$

Prove that T is a bounded operator.

Proof. Since T is a linear operator between two Banach spaces E and E^*, by the closed graph theorem, to prove that T is a bounded operator, it suffices to prove that T is closed. To this end, suppose that $(x_n, Tx_n) \subset E\times E^*$ converges to a point (x, f) in $E\times E^*$. We have that for all $y \in E$, $\langle Tx_n - Ty, x_n - y\rangle \ge 0$. Since each $T(x_n - y) \in E^*$, $T(x_n - y) \to f - Ty$ and $x_n - y \to x - y$ as $n \to \infty$, we can apply problem 2.3 to get that $\langle f - Ty, x - y\rangle = \lim_{n\to\infty}\langle T(x_n - y), x_n - y\rangle \ge 0$, which holds for all $y \in E$. Thus, fixing $u \in E$ and taking $y = x - \frac{1}{n}u$, we have that for all $n \ge 1$ $\langle f - Tx + \frac{1}{n}Tu, \frac{1}{n}u\rangle \ge 0$, which implies that $\langle f - Tx + \frac{1}{n}Tu, u\rangle \ge 0$ and taking the limit as $n \to \infty$, we get that $\langle f - Tx, u\rangle \ge 0$ for all $u \in E$. Doing the same trick but replacing $-\frac{1}{n}u$ with $\frac{1}{n}u$, we see that also $\langle f - Tx, u\rangle \le 0$ for all $u \in E$, proving that $f = Tx$. By the closed graph theorem, T is a bounded linear operator. □

2.20

Let E and F be two Banach spaces. Let $T \in \mathcal{L}(E, F)$ and let $A : D(A) \subset E \to F$ be an unbounded operator that is densely defined and closed. Consider the operator $B : D(B) \subset E \to F$ defined by

$$D(B) = D(A), \quad B = A + T.$$

1. Prove that B is closed.

   Proof. Suppose that $(x_n, Bx_n) \subset D(B) \times F$ converges to some point $(x, f) \in E \times F$. Then $x_n \to x$ in E and since T is continuous, $Tx_n \to Tx$ in F. Note that since $(x_n, Ax_n) = (x_n, Bx_n - Tx_n) \to (x, f - Tx)$ as $n \to \infty$, the fact that A is closed and $(x_n) \subset D(B) = D(A)$ implies that $x \in D(A)$ and $Ax = f - Tx$. Thus, $x \in D(B)$ and $f = Bx$, proving that B is closed. □

2. Prove that $D(B^*) = D(A^*)$ and $B^* = A^* + T^*$.

   Proof. Fix $v \in D(A^*)$. By definition, there exists $C \in \mathbb{R}$ such that $|\langle v, Au\rangle| \le C\|u\|$ for all $u \in D(A) = D(B)$. Thus, for all $u \in D(B)$, $|\langle v, Bu\rangle| = |\langle v, Au + Tu\rangle| \le (C + \|T\|)\|u\|$, and it follows that $v \in D(B^*)$ so that $D(A^*) \subset D(B^*)$. Further, if $v \in D(B^*)$, then there exists some $C \in \mathbb{R}$ such that for all $u \in D(B) = D(A)$, $|\langle v, Bu\rangle| \le C\|u\|$. It follows that for all $u \in D(A)$, $|\langle v, Au\rangle| \le |\langle v, Bu\rangle| + |\langle v, Tu\rangle| \le (C + \|T\|)\|u\|$, so that $v \in D(A^*)$, which proves that $D(A^*) = D(B^*)$. Note that for any $v \in D(B^*) = D(A^*)$ and $u \in D(B) = D(A)$, $\langle B^*v, u\rangle = \langle v, Bu\rangle = \langle v, Au + Tu\rangle = \langle A^*v + T^*v, u\rangle$. By the continuity of B^* and $A^* + T^*$ and the fact that $D(B)$ is dense in E, it follows that for all $v \in D(B^*) = D(A^*)$, $B^*v = A^*v + T^*v$, proving that $B^* = A^* + T^*$. □

2.21

Let E be an infinite dimensional Banach space. Fix an element $a \in E$, $a \ne 0$, and a discontinuous linear functional $f : E \to \mathbb{R}$. Consider the operator $A : E \to E$ defined by

$$D(A) = E, \quad Ax = x - f(x)a.$$

1. Determine $N(A)$ and $R(A)$.

Solution

Clearly $N(A) \subset \text{span}(a)$. In fact, if $\lambda a \in N(A)$ and $\lambda \neq 0$, then $\lambda a = \lambda f(a)a \iff a = f(a)a \iff f(a) = 1$. Hence, either $f(a) \neq 1$ and $N(A) = \{0\}$ or $f(a) = 1$ and $N(A) = \text{span}(a)$. Towards finding $R(A)$, note that if $x \in N(f)$ then $Ax = x - f(x)a = x$, which shows that $N(f) \subset R(A)$. In fact, if $f(a) = 1$, then $u \in R(A)$ implies that for some $x \in E$, $f(u) = f(Ax) = f(x) - f(x)f(a) = 0$, which shows that $R(A) = N(f)$ when $f(a) = 1$. If $f(a) \neq 1$, then for any $u \in E$, set $x = u + \frac{f(u)}{1-f(a)}a$ and note that $Ax = u + \frac{f(u)}{1-f(a)}a - f(u)a - \frac{f(u)}{1-f(a)}f(a)a = u$, showing that $R(A) = E$ when $f(a) \neq 1$.

2. Is A closed?

Solution

No. Since A is a linear operator from the Banach space E to itself, if A were closed then it would be continuous by the closed graph theorem. In particular, it would be continuous at 0. However, since f is discontinuous, it is necessarily discontinuous at 0 and so there exists a sequence $x_n \subset E$ that converges to 0 such that $f(x_n)$ does not converge to 0. But then Ax_n cannot converge to 0, and so A cannot be closed as it is not continuous.

3. Determine A^*

Solution

Suppose that $v \in D(A^*)$, then there exists C such that for all $x \in E$,

$$\begin{aligned} |f(x)||\langle v, a\rangle| - ||\langle v, x\rangle| &\leq ||\langle v, x\rangle| - |f(x)||\langle v, a\rangle|| \\ &\leq |\langle v, x - f(x)a\rangle| \\ &\leq C\|x\|. \end{aligned}$$

Observe that this forces $\langle v, a\rangle = 0$, since otherwise we would have that for all $x \in E$, $|f(x)| \leq \frac{(C+\|v\|)}{|\langle v,a\rangle|}\|x\|$, contradicting the assumption that f is discontinuous. Thus, $D(A^*) \subset N(a \in E^{**})$. Clearly if $v \in N(a \in E^{**})$, then for all $x \in E$, $|\langle v, Ax\rangle| = |\langle v, x\rangle| \leq \|v\|\|x\|$, which shows that $D(A^*) = N(a \in E^{**})$. Thus, it follows that for all $v \in D(A^*)$ and for all $x \in E$, $\langle A^*v, x\rangle = \langle v, x - f(x)a\rangle = \langle v, x\rangle$, showing that $A^* = Id_{D(A^*)}$.

4. Determine $N(A^*)$ and $R(A^*)$.

Solution

From part 3. above, it follows that $N(A^*) = \{0\}$ and $R(A^*) = D(A^*) = N(a \in E^{**})$.

5. Compare $N(A)$ with $R(A^*)^\perp$ as well as $N(A^*)$ with $R(A)^\perp$.

Solution

$R(A^*)^\perp = \{x \in E : \langle v, x\rangle \quad \forall v \in D(A^*)\} = \text{span}(a)$ (equality follows from an obvious application of Hahn-Banach, second geometric form). Comparing this to $N(A)$, we see that $N(A) = \{0\} \subsetneq R(A^*)^\perp$ if $f(a) \neq 1$ and $N(A) = \text{span}(a) = R(A^*)^\perp$ if $f(a) = 1$. Further, $R(A)^\perp = \{0\} = N(A^*)$ since by problem 1.6, the fact that $N(f)$ is not closed implies that $N(f)$ is dense in E. (That $N(f)$ is not closed follows from the closed graph theorem and the fact that f is discontinuous.)

6. Compare with the results of Exercise 2.18 (skipping since 2.18 was not included in the assignment).

2.22

The purpose of this exercise is to construct an unbounded operator $A : D(A) \subset E \to E$ that is densely defined, closed, and such that $\overline{D(A^*)} \neq E^*$. Let $E = \ell^1$, so that $E^* = \ell^\infty$. Consider the operator $A : D(A) \subset E \to E$ defined by

$$D(A) = \left\{u = (u_n) \in \ell^1 : (nu_n) \in \ell^1\right\} \text{ and } Au = (nu_n).$$

1. Check that A is densely defined and closed.

Solution

Towards first proving that A is densely defined, fix $x = (x_n) \in \ell^1$ and $\varepsilon > 0$. Pick N such that $\sum_{n=N+1}^{\infty} |x_n| < \varepsilon$ and define $u = (u_n) = \left(\begin{cases} x_n, & n \le N \\ 0, & n > N \end{cases} \right)_{n\ge1}$. Clearly $u \in \ell^1$ and $(nu_n) \in \ell^1$ since the sum is finite. The fact that $\|x - u\|_1 = \sum_{n>N} |x_n| < \varepsilon$ proves that $D(A)$ is dense in ℓ^1.

To see that A is closed, suppose that $x_n = (x_{k,n}) \subset D(A)$ is a sequence that converges to some point $x = (x_k) \in \ell^1$, and $Ax_n = (kx_{k,n})$ converges to $f = (f_k)$ in ℓ^1. Then $|kx_{k,n} - f_k| \le \|Ax_n - f\|_1 \to 0$ as $n \to \infty$, so that $f_k = \lim_{n\to\infty} kx_{n,k}$ for each $k \ge 1$. But also $|kx_k - kx_{k,n}| \le k\|x - x_n\|_1 \to 0$ as $n \to \infty$, so that $f_k = \lim_{n\to\infty} kx_{n,k} = kx_k$. Thus, $x \in D(A)$ and $f = Ax$, proving that A is closed.

2. Determine $D(A^*)$, A^*, and $\overline{D(A^*)}$.

Solution

Note that for all $v = (v_n) \in D(A^*)$, there exists some C such that for every $x = (x_n) \in \ell^1$, $|\langle v, Ax\rangle| = |\sum_n nv_n x_n| \le C\|x\|_1 < \infty$. By exercise 2.7, $(nv_n) \in \ell^\infty$ and so $D(A^*) \subset \{v \in \ell^\infty : (nv_n) \in \ell^\infty\}$. Clearly if $v \in \ell^\infty$ and $(nv_n) \in \ell^\infty$, then for all $x \in \ell^1$, $|\langle v, Ax\rangle| = |\sum_n nv_n x_n| \le \|(nv_n)\|_\infty \|x\|_1$, which shows that $D(A^*) = \{v \in \ell^\infty : (nv_n) \in \ell^\infty\}$. For any $v \in D(A^*)$, $(A^*v)_j = \langle A^*v, (\partial_{n,j})\rangle = \langle v, A(\partial_{n,j})\rangle = jv_j$, which shows that $A^*v = (nv_n)$ on $D(A^*)$. Clearly for every $v \in D(A^*)$, there must exist some $C \in \mathbb{R}$ such that $|nv_n| \le C$ for all n, which implies that $|v_n| \le C/n \to 0$ as $n \to \infty$. Hence, $\overline{D(A^*)} \subset c_0$. Using the same method as above to show that $D(A)$ is dense in ℓ^1, it is clear that $D(A^*)$ is dense in c_0, so that $\overline{D(A^*)} = c_0 \subsetneq \ell^\infty$.

3.1

Let E be a Banach space and let $A \subset E$ be a subset that is compact in the weak topology $\sigma(E, E^*)$. Prove that A is bounded.

Proof. By Corollary 2.4, to prove that A is bounded it suffices to prove that for every $f \in E^*$, the set $f(A)$ is bounded in $\mathbb{R}$. To this end, fix $f \in E^*$ and for each $x \in A$, define $U_x = \{y \in E : |\langle f, y - x\rangle| < 1\}$. Clearly each U_x is weakly open and the collection $\{U_x\}_{x\in A}$ covers A. Since A is weakly compact, there exist $x_1, \dots, x_n \in A$ such that $A \subset \bigcup_{i=1}^n U_{x_i}$. Thus, for any $y \in A$, there is some x_i such that $|\langle f, y\rangle| < 1 + |\langle f, x_i\rangle| \le 1 + \max_{1\le k\le n} |\langle f, x_k\rangle| < \infty$, proving that $f(A)$ is bounded for each $f \in E^*$. It follows that A is bounded. □

3.2

Let E be a Banach space and let (x_n) be a sequence such that $x_n \rightharpoonup x$ in the weak topology $\sigma(E, E^*)$. Set

$$\sigma_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n).$$

Prove that $\sigma_n \rightharpoonup x$ in the weak topology $\sigma(E, E^*)$.

Proof. Fix $f \in E^*$. Since $x_n \rightharpoonup x$, $\langle f, x_n\rangle \to \langle f, x\rangle$. Fix $\varepsilon > 0$ and pick N_1 such that $|\langle f, x - x_n\rangle| < \frac{\varepsilon}{2}$ for all $n \ge N_1$. Pick N_2 large enough such that $\frac{1}{N_2}\sum_{i=1}^{N_1} |\langle f, x - x_i\rangle| < \frac{\varepsilon}{2}$. Then for all $n \ge \max(N_1, N_2)$,

$$\begin{aligned} |\langle f, x - \sigma_n\rangle| &\le \sum_{i=1}^{n} \frac{1}{n} |\langle f, x - x_i\rangle| \\ &\le \frac{1}{N_2}\sum_{i=1}^{N_1} |\langle f, x - x_i\rangle| + \frac{1}{n}\sum_{i=N_1+1}^{n} |\langle f, x - x_i\rangle| \\ &< \varepsilon. \end{aligned}$$

Thus, $\langle f, \sigma_n\rangle \to \langle f, x\rangle$. Since $\langle f, \sigma_n\rangle \to \langle f, x\rangle$ for every $f \in E^*$, it follows that $\sigma_n \rightharpoonup x$ in the weak topology $\sigma(E, E^*)$. □

Lemma 1

Let X be a first countable topological vector space and suppose that $C \subset X$ is convex. Then the closure of C is convex.

Proof. Suppose that $a, b \in \overline{C}$ and $\lambda \in [0,1]$. Since the topology on X is first countable, $\overline{C}$ is equal to the set of all limits of sequences in C. Thus, there exist sequences $(a_n), (b_n) \subset C$ such that $a_n \to a$ and $b_n \to b$. Since $\lambda a_n + (1-\lambda)b_n \in C$ for all n and $\lambda a_n + (1-\lambda)b_n \to \lambda a + (1-\lambda)b$ by the continuity of the addition and scalar multiplication operations on X, it follows that $\lambda a + (1-\lambda)b \in \overline{C}$, proving the convexity of $\overline{C}$. □

3.3

Let E be a Banach space. Let $A \subset E$ be a convex subset. Prove that the closure of A in the strong topology and that in the weak topology $\sigma(E, E^*)$ are the same.

Proof. Define $\overline{A}$ to be the strong closure of A and $\bar{A}^\sigma$ the weak closure of A. Since the strong and weak topologies on an n.v.s. are obviously first countable, it follows by Lemma 1 above that $\overline{A}$ and $\bar{A}^\sigma$ are both convex subsets. Thus, by Theorem 3.7, $\overline{A}$ is a weakly closed subset including A, proving that $\bar{A}^\sigma \subset \overline{A}$. Since all weakly closed subsets are strongly closed, $\bar{A}^\sigma$ is a strongly closed subset including A, proving that $\overline{A} = \bar{A}^\sigma$. □

3.5

Let E be a Banach space and let $K \subset E$ be a subset of E that is compact in the strong topology. Let (x_n) be a sequence in K such that $x_n \rightharpoonup x$ weakly $\sigma(E, E^*)$. Prove that $x_n \to x$ strongly.

Proof. Suppose for a contradiction that x_n does not converge strongly to x. Then there must exist some $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k} - x\| > \varepsilon$. In particular, no subsequence of (x_{n_k}) converges to x. Since the strong topology on E is obviously metrizable, K being strongly compact is equivalent to K being sequentially compact with respect to the norm on E. Thus, the sequence $(x_{n_k}) \subset K$ has a convergent subsequence $(x_{n_{k_m}})$ which must converge to a point $y \in K$. But then by Proposition 3.5, $x_{n_{k_m}} \rightharpoonup y$ in $\sigma(E, E^*)$. Since $x_{n_{k_m}} \rightharpoonup x$ and $\sigma(E, E^*)$ is Hausdorff, it follows that $x_{n_{m_k}} \to y = x$, a contradiction. Thus, by contradiction $x_n \to x$ strongly. □

3.7

Let E be a Banach space and let $A \subset E$ be a subset that is closed in the weak topology $\sigma(E, E^*)$. Let $B \subset E$ be a subset that is compact in the weak topology $\sigma(E, E^*)$.

1. Prove that $A + B$ is closed in $\sigma(E, E^*)$.

   Proof. Let $O := E \setminus (A + B)$ and fix a point $x \in O$. For each $b \in B$, since $A + b$ is weakly closed and $x \notin A + b$, there exists a weakly open neighborhood U_b of 0 such that $(x + U_b) \cap (A + b) = \varnothing$. Moreover, since $\sigma(E, E^*)$ is a locally convex topology, we can assume WLOG that each U_b is convex. Finally, it is clear from the local bases of 0 in the weak topology that we can make the further assumption that each U_b is symmetric. Observe that the collection $\{\frac{1}{2}U_b + b\}_{b \in B}$ is a weak open cover of B and so by weak compactness, there exist $\frac{1}{2}U_{b_1} + b_1, \dots, \frac{1}{2}U_{b_n} + b_n$ that cover B. Thus, $A + B \subset A + \bigcup_{i=1}^n (\frac{1}{2}U_{b_i} + b_i)$. I claim that $(x + \bigcap_{i=1}^n \frac{1}{2}U_{b_i}) \cap (A + B) = \varnothing$. Towards proving this claim, suppose for a contradiction that there exists $u \in \bigcap_{i=1}^n \frac{1}{2}U_{b_i}$ such that $x + u \in A + B$. Then there exists some $a \in A$, $1 \le k \le n$ and $u' \in \frac{1}{2}U_{b_k}$ such that $x + u = a + u' + b_k$. But then by the symmetry and convexity of U_{b_k}, $u - u' \in U_{b_k}$, so that $x + u - u' \in (x + U_{b_k}) \cap (A + b_k)$, which is impossible. Thus, by contradiction, $(x + \bigcap_{i=1}^n U_{b_i}) \cap (A + B) = \varnothing$. Since $x + \bigcap_{i=1}^n U_{b_i} \subset O$ is a weak open neighborhood of x, it follows that O is weakly open, proving that $E \setminus O = A + B$ is weakly closed. □

2. Assume, in addition, that A and B are convex, nonempty, and disjoint. Prove that there exists a closed hyperplane strictly separating A and B.

Proof. Note that since B is weakly compact and convex, so is $-B$. By part 1. above, $A - B$ is weakly closed and therefore strongly closed. Since the sum of two convex sets is convex, $A - B$ is a nonempty, convex, strongly closed subset of E that does not include $\{0\}$ (since $A \cap B = \emptyset$). By the second geometric form of the Hahn-Banach theorem, there exists $f \in E^*$ and $\alpha \in \mathbb{R}$ such that $f(a-b) < \alpha < f(0) = 0$ for all $a \in A$ and $b \in B$. It follows that $f(a) < \alpha + f(b) < f(b)$ for all $a \in A$ and $b \in B$. Thus, $\sup_{a\in A} f(a) \leq \alpha + \inf_{b\in B} f(b) < \inf_{b\in B} f(b)$. Pick $\alpha' \in (\sup_{a\in A} f(a), \inf_{b\in B} f(b))$ and $\varepsilon > 0$ such that $(\alpha' - \varepsilon, \alpha' + \varepsilon) \subset (\sup_{a\in A} f(a), \inf_{b\in B} f(b))$ and observe that for all $a \in A$ and $b \in B$, $f(a) < \alpha' - \varepsilon < \alpha' + \varepsilon < f(b)$. Thus, A and B are strictly separated by the hyperplane $[f = \alpha']$. ☐

3.8

Let E be an infinite-dimensional Banach space. Our purpose is to show that E equipped with the weak topology is not metrizable. Suppose, by contradiction, that there is a metric $d(x,y)$ on E that induces on E the same topology as $\sigma(E, E^*)$.

1. For every integer $k \geq 1$ let V_k denote a neighborhood of 0 in the topology $\sigma(E, E^*)$, such that

$$V_k \subset \Big\{x : d(x,0) < \frac{1}{k}\Big\}.$$

   Prove that there exists a sequence (f_n) in E^* such that every $g \in E^*$ is a (finite) linear combination of the $f_n's$.

   Proof. We may assume WLOG that for each k, there exist $\varepsilon_k > 0$ and bounded linear functionals $f_{k,1}, \ldots, f_{k,n_k} \in E^*$ such that $V_k = \{x \in E : |\langle f_{k,i}, x\rangle| < \varepsilon_k \quad \forall i : 1 \leq i \leq n_k\}$. Let (f_n) be an ordering of these functionals $f_{k,i}$ for all $k \geq 1$ and $1 \leq i \leq n_k$. Fix $g \in E^*$. I claim that there exists $m_1, \ldots, m_j$ and $\lambda_1, \ldots, \lambda_j \in \mathbb{R}$ such that $g = \sum_{i=1}^{j} \lambda_i f_{m_i}$. Observe that by Lemma 3.2, to prove this claim, it suffices to prove that there exists $m_1, \ldots, m_j$ such that $\bigcap_{i=1}^{j} \ker f_{m_i} \subset \ker g$. Suppose for a contradiction that for any finite subset $F \subset \mathbb{N}$, $\bigcap_{i\in F} \ker f_i$ is not a subset of $\ker g$. Then for every $k \geq 1$, there exists $x_k \in \bigcap_{i=1}^{n_k} \ker f_{k,i}$ such that $x \notin \ker g$. Then $\lambda x_k \in \bigcap_{i=1}^{n_k} \ker f_{k,i} \setminus \ker g$ for all $\lambda \neq 0$ and so by potentially rescaling each x_k, we may assume WLOG that $\langle g, x_k\rangle = \frac{1}{2}$. But since each x_k clearly belongs to V_k, it follows that $d(x_k, 0) < \frac{1}{k} \to 0$ as $k \to \infty$ and so $x_k \rightharpoonup 0$ weakly, forcing $\frac{1}{2} = \langle g, x_k\rangle \to \langle g, 0\rangle = 0$, which is clearly absurd. Thus, by contradiction, g must be equal to a finite linear combination of the functionals in the sequence (f_n). ☐

2. Deduce that E^* is finite-dimensional.

   Proof. From part 1. we have a sequence $(f_n) \subset E^*$ such that every $g \in E^*$ is equal to a finite linear combination of the f_n's. Now for each $n \geq 1$, define $F_k = \text{span}(f_1, \ldots, f_k)$ and observe that each F_k is a finite-dimensional subspace so strongly closed in E^*. Since $\bigcup_{n\geq 1} F_n = E^*$ and E^* is a complete metric space with respect to the operator norm, it follows by the Baire category theorem that there exists some N such that $\text{Int}(F_N) \neq \emptyset$. That is, there exists some $g \in F_N$ and an open neighborhood V of 0 such that $g + V \subset F_N$. It follows that $V = (g + V) - g \subset F_N$. Using the bases for the topology induced by the operator norm on E^*, there exists some $\varepsilon > 0$ such that $B_{E^*}(0, \varepsilon) \subset V \subset F_N$. Observe that if $\{b_i\}_{i\in I}$ is a basis for E^*, and b_i is any vector belonging to this basis, then $\frac{\varepsilon}{2\|b_i\|} b_i \in V \subset F_N$ and so $b_i \in F_N$, proving that $\{b_i\}_{i\in I} \subset F_N$. Since F_N is finite-dimensional and $\{b_i\}_{i\in I} \subset F_N$ is a linearly independent collection of vectors in F_N, it follows that $|I| < \infty$. That is, E^* is finite-dimensional. ☐

3. Conclude.

   Solution

   Towards proving that E^* can never be finite-dimensional when E is infinite-dimensional, fix a linearly independent collection $f_1, \ldots, f_n \in E^*$. Define the map $\varphi : E \to \mathbb{R}^n;\ x \mapsto (f_1(x), \ldots, f_n(x))$. Since φ is linear, continuous and its image is finite-dimensional well its domain is infinite-dimensional, φ cannot be injective and so there must exist nonzero $x \in \bigcap_{i=1}^{n} \ker f_i$. By Corollary 1.6, there exists $f \in E^*$ such that $\langle f, x\rangle = \|x\|^2 \neq 0$ and so $\bigcap_{i=1}^{n} \ker f_i$ is not a subset of $\ker f$, which implies that f cannot be a linear combination of the f_i's. Thus, E^* cannot be finite-dimensional, contradicting our conclusion from part 2. above. By contradiction, the weak topology $\sigma(E, E^*)$ on E cannot be metrizable when E is an infinite-dimensional Banach space.

4. Prove by a similar method that E^* equipped with the weak* topology $\sigma(E^*, E)$ is not metrizable.

Proof. Suppose for a contradiction that there exists a metric $d(f,g)$ in E^* that induces the same topology as $\sigma(E^*, E)$. For every integer $k \geq 1$ let V_k denote a neighborhood of 0 in the topology $\sigma(E^*, E)$, such that

$$V_k \subset \Big\{ f : d(f,0) < \frac{1}{k} \Big\}.$$

Then we may assume WLOG that for each k, there exists $\varepsilon_k > 0$ and $x_{k,1}, \ldots, x_{k,n_k} \in E$ such that $V_k = \{f \in E^* : |\langle f, x_{k,i}\rangle| < \varepsilon_k \quad \forall i : 1 \leq i \leq n_k\}$. Let (x_n) be an ordering of the $x_{k,i}$'s and fix $x \in E$. I claim that x must be equal to a finite linear combination of the x_n's. Again using Lemma 3.2, to prove this claim, it suffices to find $m_1, \ldots, m_n \in \mathbb{N}$ such that $\bigcap_{i=1}^n \ker J(x_i) \subset \ker J(x)$, where J is the embedding of E into E^{**} (since then there will exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\langle f, x - \sum_{i=1}^n \lambda_i x_{m_i}\rangle = 0$ for all $f \in E^*$, so that $\|x - \sum_{i=1}^n \lambda_i x_{m_i}\| = 0$). Suppose for a contradiction that x were not a finite linear combination of some of the x_n's. Then for all $k \geq 1$, there would exist $f_k \in \bigcap_{i=1}^{n_k} \ker J(x_{k,i}) \setminus \ker J(x)$, and we may assume WLOG that $\langle f_k, x\rangle = \frac{1}{2}$. Since each $f_k \in V_k$, $d(f_k, 0) < \frac{1}{k} \to 0$ as $n \to \infty$, implying that $f_k \overset{*}{\rightharpoonup} 0$ in $\sigma(E^*, E)$. However, this implies that $\frac{1}{2} = \langle f_k, x\rangle = \langle J(x), f_k\rangle \to 0$, a contradiction. Thus, by contradiction, x is a finite linear combination of the x_n's. For each n, define $F_n = \operatorname{span}(x_1, \ldots, x_n) \subset E$. Each F_n is a finite-dimensional subspace of E so strongly closed in E, and from our conclusion above, it follows that $E = \bigcup_{n \geq 1} F_n$. Since E is a complete metric space with respect to the metric induced by its norm, by the Baire category theorem, there exists some $N \geq 1$ such that $\operatorname{Int}(F_N) \neq \varnothing$. And the same reasoning as in part 2. above shows that E would then by finite-dimensional, contradicting our assumption that E is infinite-dimensional. By contradiction, it follows that the weak* topology on E^* cannot be metrizable whenever E is infinite-dimensional. ☐

3.10

Let E and F be two Banach spaces. Let $T \in \mathcal{L}(E, F)$, so that $T^* \in \mathcal{L}(F^*, E^*)$. Prove that T^* is continuous from F^* equipped with $\sigma(F^*, F)$ into E^* equipped with $\sigma(E^*, E)$.

Proof. Note that, by definition, $\sigma(E^*, E)$ is the weakest topology that makes all maps $J_x : f \in E^* \mapsto \langle f, x\rangle$ for each $x \in E$ continuous. Thus, by Proposition 3.2, to prove that $T^* : (F^*, \sigma(F^*, F)) \to (E^*, \sigma(E^*, E))$ is continuous, it suffices to check that for each $x \in E$, the map $J_x \circ T^*$ is continuous. But for any $v \in F^*$, $J_x \circ T^*(v) = \langle T^* v, x\rangle_{E^*,E} = \langle v, Tx\rangle_{F^*,F}$, and since $v \in F \mapsto \langle v, Tx\rangle_{F^*,F}$ is a continuous map from $(F^*, \sigma(F^*, F))$ into $\mathbb{R}$ by the definition of $\sigma(F^*, F)$, it follows that T^* is continuous between the weak* topologies. ☐

3.13

Let E be a Banach space. Let (x_n) be a sequence in E and let $x \in E$. Set

$$K_n = \overline{\operatorname{conv}\bigg(\bigcup_{i=n}^{\infty} \{x_i\} \bigg)}.$$

1. Prove that if $x_n \rightharpoonup x$ weakly $\sigma(E, E^*)$, then
$$\bigcap_{i=1}^{\infty} K_n = \{x\}.$$

Proof. Note that if the sequence $(x_n)_{n \geq 1}$ converges weakly to x in $\sigma(E, E^*)$ then clearly all subsequence of (x_n) also converge weakly to x and, in particular, all sequences $(x_k)_{k \geq n}$ for any n. Thus, by Mazur's lemma, for each $n \geq 1$, there exists a sequence $(y_k) \subset \operatorname{conv}\Big(\bigcup_{i=n}^{\infty} \{x_i\} \Big)$ such that $y_k \to x$ strongly. It follows that for all $n \geq 1$, $x \in K_n$ and so $\{x\} \subset \bigcap_{n=1}^{\infty} K_n$. Now fix $y \in \bigcap_{n=1}^{\infty} K_n$. Towards proving that $y = x$, fix $\varepsilon > 0$, nonzero $f \in E^*$ and pick N such that $|\langle f, x_n - x\rangle| < \frac{\varepsilon}{2}$ for all $n \geq N$. Since $y \in K_N$, there exists $n_1, \ldots, n_m \geq N$ and

$\lambda_1, \ldots, \lambda_m \in [0,1]$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\|y - \sum_{i=1}^m \lambda_i x_{m_i}\| < \frac{\varepsilon}{2\|f\|}$. Thus,

$$
\begin{aligned}
|\langle f, x-y\rangle| &\le |\langle f, x - \sum_{i=1}^m \lambda_i x_{n_i}\rangle| + |\langle f, \sum_{i=1}^m \lambda_i x_{n_i} - y\rangle| \\
&= |\langle f, \sum_{i=1}^m \lambda_i (x - x_{n_i})\rangle| + |\langle f, \sum_{i=1}^m \lambda_i x_{n_i} - y\rangle| \\
&\le \sum_{i=1}^m \lambda_i \frac{\varepsilon}{2} + \|f\| \|\sum_{i=1}^m \lambda_i x_{n_i} - y\| < \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ and $f \in E^*$ were arbitrary, it follows that $\langle f, x-y\rangle = 0$ for all $f \in E^*$. Thus, $\|x-y\| = 0$, and so $x = y$. The statement to prove follows. □

2. Assume that E is reflexive. Prove that if (x_n) is bounded and if $\bigcap_{n=1}^\infty K_n = \{x\}$, then $x_n \rightharpoonup x$ weakly $\sigma(E, E^*)$.

   Proof. Towards a contradiction, suppose that x_n does not converge weakly to x. Then, there would necessary exist a subsequence (x_{n_k}) of (x_n), $f \in E^*$ and $\varepsilon > 0$ such that $|\langle f, x_{n_k} - x\rangle| > \varepsilon$ for all $k \ge 1$. Since E is reflexive and (x_{n_k}) is bounded, by Theorem 3.18 there exists a subsequence $(x_{n_{k_m}})$ of (x_{n_k}) that converges weakly to a point $y \in E$. Observe that for any $n \ge 1$, there exists N such that for all $m \ge N$, $n_{k_m} \ge n$, so that $x_{n_{k_m}} \in K_n$ for all $m \ge N$. It follows that y is a weak limit point of each K_n and since each K_n is convex and strongly closed, each K_n is also weakly closed, so $y \in \bigcap_{i=1}^\infty K_n = \{x\}$. It follows that $x_{n_{k_m}} \rightharpoonup x$ weakly, and so $\langle g, x_{n_{k_m}}\rangle \to \langle g, x\rangle$ as $m \to \infty$ for all $g \in E^*$. But then there must exist some m such that $|\langle f, x_{n_{k_m}} - x\rangle| < \varepsilon$, a contradiction. Thus, by contradiction, $x_n \rightharpoonup x$ weakly. □

3. Assume that E is finite-dimensional and $\bigcap_{i=1}^\infty K_n = \{x\}$. Prove that $x_n \to x$.

   Proof. I claim that (x_n) must be a bounded sequence. Towards proving this claim, suppose for a contradiction that (x_n) is unbounded. Fix a basis $v_1, \ldots, v_n$ for E and let $\langle \cdot, \cdot\rangle_E$ be the canonical inner product on E that makes the basis $v_1, \ldots, v_n$ orthonormal. Let $\|\cdot\|_E$ be the norm induced by this inner product. Since all norms defined on a finite-dimensional vector space are equivalent, the sequence (x_n) is bounded if and only if it is bounded with respect to $\|\cdot\|_E$. For $m \ge 1$, let $(x_{m,1}, \ldots, x_{m,n})$ be the components of x_m with respect to the fixed basis $v_1, \ldots, v_n$. Then since (x_n) must be unbounded with respect to $\|\cdot\|_E$, it follows that there must exist $1 \le i \le n$ such that $(x_{m,i})$ is an unbounded sequence in $\mathbb{R}$. For all $m \ge 1$, writing K_m with respect to the basis $v_1, \ldots, v_n$ gives

$$
K_m = \overline{\Big\{ \big(\sum_{k\in F} \lambda_k x_{k,1}, \ldots, \sum_{k \in F} \lambda_k x_{k,n}\big) : F \subset \{m, m+1, \ldots\} \text{ and } |F| < \infty \text{ and } \lambda_1, \ldots, \lambda_k \in [0,1] : \sum_{k\in F} \lambda_k = 1 \Big\}}.
$$

   Thus, since $\{(x_1, \ldots, x_n)\} = \bigcap_{n=1}^\infty K_n$, we must have that $\{x_i\} = \overline{\operatorname{conv}(\bigcup_{j=m}^\infty \{x_{j,i}\})} \supset [\liminf_{j\to\infty} x_{j,i}, \limsup_{j\to\infty} x_{j,i}]$. Clearly we must therefore have that $\limsup_{j\to\infty} x_{j,i} = \liminf_{j\to\infty} x_{j,i} = x_i$, but this contradicts the sequence $(x_{m,i})_{m\ge1}$ being unbounded. Thus, by contradiction, (x_n) is a bounded sequence. Since E is finite-dimensional, E is reflexive. By part 2., $x_n \rightharpoonup x$ weakly. But since the weak topology on any finite-dimensional Banach space is the same as the strong topology, it follows that $x_n \to x$ strongly. □

4. In ℓ^p, $1 < p < \infty$, construct a sequence (x_n) such that $\bigcap_{n=1}^\infty K_n = \{x\}$, and (x_n) is not bounded.

   Solution

   Define the function $o : \mathbb{N} \to \mathbb{N}$ by $o(n) = \begin{cases} n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$, and define the sequence $(o(n)\partial_{n,m})_{n\ge1}$. Observe that for all $n \ge 1$, $\|(o(n)\partial_{n,m})\|_2 = o(n)$ and so $(o(n)\partial_{n,m})_{n\ge1} \subset \ell^2$ is unbounded. Now suppose that $y \in \bigcap_{n=1}^\infty K_n$. Then for all $n \ge 1$, since $y \in K_{n+1}$, it follows that the nth component of y must be equal to 0 and so $y = 0$. Clearly $0 \in \bigcap_{n=1}^\infty K_n$, proving that $\bigcap_{n=1}^\infty K_n = \{0\}$.

3.16

Let E be a Banach space.

1. Let (f_n) be a sequence in E^* such that for every $x \in E$, $\langle f_n, x\rangle$ converges to a limit. Prove that there exists some $f \in E^*$ such that $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$.

   Proof. For each $x \in E$, denote the limit $\lim_{n\to\infty}\langle f_n, x\rangle$ by $\langle f, x\rangle$. By Corollary 2.3, $f \in \mathcal{L}(E, \mathbb{R}) = E^*$. Since $\langle f - f_n, x\rangle \to 0$ for all $x \in E$, it follows that $f_n \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$. □

2. Assume here that E is reflexive. Let (x_n) be a sequence in E such that for every $f \in E^*$, $\langle f, x_n\rangle$ converges to a limit. Prove that there exists some $x \in E$ such that $x_n \rightharpoonup x$ in $\sigma(E, E^*)$.

   Proof. Let $J : E \to E^{**}$ be the embedding of E in E^{**}. Observe that $(J(x_n))$ is a sequence in $E^{**} = (E^*)^*$ such that for all $f \in E^*$, $\langle J(x_n), f\rangle = \langle f, x_n\rangle$ converges to a limit. Applying part 1., it follows that there exists some $\xi \in E^{**}$ such that $J(x_n) \overset{*}{\rightharpoonup} \xi$ in $\sigma(E^{**}, E^*)$. Since E is reflexive, there exists some $x \in E$ such that $\xi = J(x)$. Moreover, for every $f \in E^*$, $\langle f, x - x_n\rangle = \langle J(x) - J(x_n), f\rangle \to 0$ as $n \to \infty$. Thus, $x_n \rightharpoonup x$ weakly in E. □

3. Construct an example in a nonreflexive space E where the conclusion of 2 fails.

   Solution

   Take $E = c_0$ (which is not reflexive since $c_0^{**} = (\ell^1)^* = \ell^\infty$) and for each $n \geq 1$, define the sequence $1_n : \mathbb{N} \to \{0,1\}; k \mapsto \begin{cases} 1, & k \leq n \\ 0, & k > n \end{cases}$. Fix some $f \in c_0^* = \ell^1$. Observe that $\langle f, 1_n\rangle = \sum_{i=1}^n f_i$, which converges as $n \to \infty$ since the series $\sum_{i=1}^\infty f_i$ is absolutely convergent. However, for any $x \in c_0$, since $x_n \to 0$, there exists some N such that $|x_N| < \frac{1}{2}$, and so $|\langle \partial_{N,k}, 1_n - x\rangle| = |1 - x_N| > \frac{1}{2}$ for all $n \geq N$, proving that 1_n does not converge weakly to x for any $x \in c_0$.

3.17

1. Let (x^n) be a sequence in ℓ^p with $1 \leq p \leq \infty$. Assuming $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$ prove that:

   (a) (x^n) is bounded in ℓ^p,

   Proof. This is just Proposition 3.5 (iii) for $p < \infty$ and Proposition 3.13 (iii) for $p = \infty$. □

   (b) $x_i^n \xrightarrow[n\to\infty]{} x_i$ for every i, where $x^n = (x_1^n, x_2^n, \ldots)$ and $x = (x_1, x_2, \ldots)$.

   Proof. First suppose that $1 \leq p < \infty$. For each $n \geq 1$, the projection map $\pi_n : \ell^p \to \mathbb{R}; x \mapsto x_n$ is obviously bounded and linear for all $1 \leq p < \infty$ and so $\pi_n \in \ell^{p'}$. Since $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$, it follows that for every $i \geq 1$, $x_i^n = \langle \pi_i, x^n\rangle \to \langle \pi_i, x\rangle = x_i$ as $n \to \infty$. Now when $p = \infty$, we have that $x^n \overset{*}{\rightharpoonup} x$ in the weak* topology $\sigma(\ell^\infty, \ell^1)$. Thus, for any $y \in \ell^1$, $\langle x^n, y\rangle \to \langle x, y\rangle$. Fix $i \geq 1$ and note that $\pi_i = (\partial_{i,n})_{n\geq 1} \in \ell^1$ so that $x_i^n = \langle x^n, \pi_i\rangle \to \langle x, \pi_i\rangle = x_i$ as $n \to \infty$ for all $i \geq 1$. □

2. Conversely, suppose (x^n) is a sequence in ℓ^p with $1 < p \leq \infty$. Assume that (a) and (b) hold (for some limit denoted by x_i). Prove that $x \in \ell^p$ and that $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$.

   Proof. First consider $1 < p < \infty$. Suppose for a contradiction that x^n does not weakly converge to x. Then there must exist some subsequence (x^{n_k}) as well as some $y \in \ell^{p'}$ and $\varepsilon > 0$ such that $|\langle y, x^{n_k} - x\rangle| > \varepsilon$ for all $k \geq 1$. Since ℓ^p is reflexive and (x^n) is bounded, by Theorem 3.18, there exists a subsequence $(x^{n_{k_m}})$ that converges in the weak topology $\sigma(\ell^p, \ell^{p'})$. Let $a \in \ell^p$ be the weak limit of $(x^{n_{k_m}})$ and observe that by (b) above, $x_i^{n_{k_m}} \to a_i$ as $m \to \infty$ for all $i \geq 1$. Since $x_i^n \to x_i$ for all $i \geq 1$, it follows that $a = x$. But then $\langle y, x^{n_{k_m}} - x\rangle \to 0$ as $m \to \infty$, contradicting that $|\langle y, x^{n_k} - x\rangle| > \varepsilon$ for all $k \geq 1$. Thus, by contradiction, $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$.

Now suppose that $p = \infty$ and fix $y \in \ell^1$ as well as $\varepsilon > 0$. Pick N_1 such that $\sum_{n=N+1}^{\infty} |y_i| < \frac{\varepsilon}{2\max(\sup \|x^n\|_\infty + \|x\|_\infty, 1)}$ (which we can do since $(\|x^n\|_\infty)$ is bounded). Choose N_2 large enough such that for any $n \geq N_2$ and $1 \leq i \leq N_1$ where $y_i \neq 0$, $|x_i^n - x_i| < \frac{\varepsilon}{2N_1|y_i|}$. Then for all $n \geq N_2$,

$$
\begin{aligned}
|\langle x^n - x, y\rangle_{\ell^\infty,\ell^1}| &\leq \sum_{i=1}^{\infty} |y_i||x_i^n - x_i| \\
&\leq \sum_{i=1}^{N_1} \frac{\varepsilon}{2N_1} + (\sup_n \|x^n\|_\infty + \|x\|_\infty) \sum_{i=N_1+1}^{\infty} |y_i| \\
&< \varepsilon.
\end{aligned}
$$

It follows that $|\langle x^n - x, y\rangle_{\ell^\infty,\ell^1}| \to 0$ as $n \to \infty$ for all $y \in \ell^1$, and so $x^n \overset{*}{\rightharpoonup} x$ in $\sigma(\ell^\infty, \ell^1)$. ☐

3.18

For every integer $n \geq 1$ let

$$e^n = (\partial_{n,m})_{m \geq 1}.$$

1. Prove that $e^n \xrightarrow[n\to\infty]{} 0$ in ℓ^p weakly $\sigma(\ell^p, \ell^{p'})$ with $1 < p \leq \infty$.

   Proof. For $1 < p < \infty$, that $e^n \rightharpoonup 0$ weakly simply expresses the fact that for any $y \in \ell^{p'}$, $\langle y, e^n\rangle = y_n \to 0$ as $n \to \infty$, which follows from the fact that $\sum_{i=1}^{\infty} |y_i|^{p'} < \infty$. The case $p = \infty$ is essentially the same: for any $y \in \ell^1$, $\langle e^n, y\rangle = y_n \to 0$ as $n \to \infty$ since $\sum_{i=1}^{\infty} |y_i| < \infty$. Thus, $e^n \overset{*}{\rightharpoonup} 0$ in the weak* topology $\sigma(\ell^\infty, \ell^1)$. ☐

2. Prove that there is no subsequence (e^{n_k}) that converges in ℓ^1 with respect to $\sigma(\ell^1, \ell^\infty)$.

   Proof. Fix a subsequence (e^{n_k}) and $x \in \ell^1$. Pick N such that $\sum_{i=N}^{\infty} |x_i| < \frac{1}{2}$. Define $1_{\geq N}(k) = \begin{cases} 0, & k < N \\ 1, & k \geq N \end{cases} \in \ell^\infty$ and observe that for all k such that $n_k \geq N$, $|\langle 1_{\geq N}, e^{n_k} - x\rangle| = |1 - \sum_{i=N}^{\infty} x_i| > \frac{1}{2}$, proving that (e^{n_k}) cannot converge weakly to x for any $x \in \ell^1$. ☐

3. Construct an example of a Banach space E and a sequence (f_n) in E^* such that $\|f_n\| = 1 \quad \forall n$ and such that (f_n) has no subsequence that converges in $\sigma(E^*, E)$. Is there a contradiction with the compactness of B_{E^*} in the topology $\sigma(E^*, E)$?

   Proof. Pick $E = \ell^\infty$. Let $J : \ell^1 \to (\ell^\infty)^*$ be the canonical embedding of ℓ^1 inside $(\ell^1)^{**}$. For each $n \geq 1$, set $f_n = J(e^n) \in J(\ell^1) \subset (\ell^\infty)^*$. Since J is an isometry, $\|f_n\|_{E^*} = \|e^n\|_1 = 1$ for all n. Towards a contradiction, suppose that the sequence (f_n) has a subsequence (f_{n_k}) that converges in $\sigma((\ell^\infty)^*, \ell^\infty)$. Then for any $x \in \ell^\infty$, $\langle f_{n_k}, x\rangle_{(\ell^\infty)^*, \ell^\infty}$ must converge in $\mathbb{R}$ as $k \to \infty$. Define a sequence $x \in \ell^\infty$ by $x_n = \begin{cases} 0, & n \notin \{n_k : k \geq 1\} \\ 1, & n = n_k \text{ and } k \text{ is odd} \\ -1, & n = n_k \text{ and } k \text{ is even} \end{cases}$.

   Observe that $\langle f_{n_k}, x\rangle = \langle x, e^{n_k}\rangle = \begin{cases} 1, & k \text{ is odd} \\ -1, & k \text{ is even.} \end{cases}$, which clearly does not converge in $\mathbb{R}$ as $k \to \infty$, a contradiction. Thus, (f_n) is a sequence with the desired properties. Note that this conclusion does not contradict the compactness of $B_{(\ell^\infty)^*}$ in the weak* topology $\sigma((\ell^\infty)^*, \ell^\infty)$ since compactness is only equivalent to sequential compactness for metric spaces and $B_{(\ell^\infty)^*}$ is not metrizable in the weak* topology as ℓ^∞ is not separable. ☐

3.19

Let $E = \ell^p$ and $F = \ell^q$ with $1 < p < \infty$ and $1 < q < \infty$. Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|a(t)| \leq C|t|^{p/q} \quad \forall t \in \mathbb{R}.$$

Given

$$x = (x_1, x_2, \ldots, x_i, \ldots) \in \ell^p,$$

set

$$Ax = (a(x_1), a(x_2), \ldots, a(x_i), \ldots).$$

1. Prove that $Ax \in \ell^q$ and that the map $x \mapsto Ax$ is continuous from ℓ^p (strong) into ℓ^q (strong).

   Proof. Fix $x \in \ell^p$ and observe that $\sum_{n\geq 1} |a(x_n)|^q \leq \sum_{n\geq 1} C^q |x_n|^p = C^q \|x\|_p^p < \infty$, proving that $Ax \in \ell^q$. Towards proving that A is continuous between the strong topologies on ℓ^p and ℓ^q, fix $\varepsilon > 0$ and a sequence $(x^k) \subset \ell^p$ that converges to some point $x \in \ell^p$. Then since $(x_n^k)_{n\geq N} \to (x_n)_{n\geq N}$ strongly in ℓ^p for any $N \geq 1$, it follows that $\|(x_n^k)_{n\geq N}\|_p^p \to \|(x_n)_{n\geq N}\|_p^p$ for any $N \geq 1$. Observe also that for each $n \geq 1$, $x_n^k \to x_n$ as $k \to \infty$. Pick N_1 such that $\sum_{n\geq N_1} |x_n|^p < \frac{\varepsilon^q}{6C^q}$, pick N_2 such that $|\|(x_n^k)_{n\geq N_1}\|_p^p - \|(x_n)_{n\geq N_1}\|_p^p| < \frac{\varepsilon^q}{3C^q}$ for all $k \geq N_2$, and by the continuity of a, pick N_3 such that $|a(x_n^k) - a(x_n)| < \frac{1}{3}\frac{\varepsilon^q}{2^n}$ for every $n \leq N_1$ and $k \geq N_3$. Then for every $k \geq \max(N_1, N_2, N_3)$

   $$\begin{aligned}\|Ax^k - Ax\|_q^q &= \sum_{n\geq 1} |a(x_n^k) - a(x_n)|^q \\ &< \frac{1}{3}\sum_{n=1}^{N_1-1} \frac{\varepsilon^q}{2^n} + C^q \sum_{n\geq N_1} (|x_n^k|^p + |x_n|^p) \\ &< \frac{\varepsilon^q}{3} + C^q\Big(\frac{\varepsilon^q}{3C^q} + 2\sum_{n\geq N_1} |x_n|^p\Big) \\ &< \varepsilon^q.\end{aligned}$$

   Thus, $Ax^k \to Ax$ strongly in ℓ^q, proving that A is a continuous map between the strong topologies on ℓ^p and ℓ^q. ☐

2. Prove that if (x^n) is a sequence in ℓ^p such that $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$ then $Ax^n \rightharpoonup Ax$ in $\sigma(\ell^q, \ell^{q'})$.

   Proof. Suppose that (x^n) is a sequence in ℓ^p such that $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$. Then by Exercise 3.17, (x^n) is bounded in ℓ^p and $x_i^n \to x_i$ for every $i \geq 1$. Using the inequality from part 1. above, we have that $\|Ax^n\|_q^q \leq C^q \|x^n\|_p^p \leq C^q (\sup_{k\geq 1} \|x^k\|_p)^p$, so that (Ax^n) is a bounded sequence in ℓ^q. Moreover, by the continuity of a, $(Ax^n)_i = a(x_i^n) \to a(x_i) = (Ax)_i$ as $n \to \infty$ for all $i \geq 1$. Thus, again by Exercise 3.17, $Ax^n \rightharpoonup Ax$ in $\sigma(\ell^q, \ell^{q'})$. ☐

3. Deduce that A is continuous from B_E equipped with $\sigma(E, E^*)$ into F equipped with $\sigma(F, F^*)$.

   Proof. Since $1 < p' < \infty$ and $1 < q' < \infty$, $\ell^{p'} = E^*$ and $\ell^{q'} = F^*$ are both separable and so, by Theorem 3.29, B_E and $C^q B_F \supset A(B_E)$ are metrizable in the weak topologies $\sigma(E, E^*)$ and $\sigma(F, F^*)$, respectively. Since metric spaces are first countable and sequentially continuous functions between first countable spaces are continuous, we conclude by part 2. that A is continuous from B_E equipped with $\sigma(E, E^*)$ into $C^q B_F \subset F$ equipped with (the subspace topology induced by) $\sigma(F, F^*)$. Finally, since the inclusion map $i : (C^q B_F, \sigma(F, F^*)) \to (F, \sigma(F, F^*))$ is obviously continuous, it follows that A is continuous from B_E equipped with $\sigma(E, E^*)$ into F equipped with $\sigma(F, F^*)$. ☐

3.21

Let E be a separable Banach space and let (f_n) be a bounded sequence in E^*. Prove directly–without using the metrizability of E^*–that there exists a subsequence (f_{n_k}) that converges in $\sigma(E^*, E)$.

Proof. Let $\{x_n\}$ be a countable dense subset of B_E. Since (f_n) is a bounded sequence in E^*, the sequence $(\langle f_n, x_1\rangle)$ is bounded in $\mathbb{R}$ and therefore, by Bolzano-Weierstrass, there exists a subsequence $(f_n^1) \subset (f_n)$ such that $\langle f_n^1, x_1\rangle$ converges. Now suppose that we have defined subsequences $(f_n^k) \subset (f_n^{k-1}) \subset \cdots \subset (f_n^1) \subset (f_n)$ such that for all $1 \leq i \leq k$, $\langle f_n^i, x_i\rangle$ converges as $n \to \infty$. Then observe that since $\langle f_n^k, x_{k+1}\rangle$ is a bounded sequence in $\mathbb{R}$, there exists a subsequence $(f_n^{k+1}) \subset (f_n^k)$ such that $\langle f_n^{k+1}, x_{k+1}\rangle$ converges as $n \to \infty$. Thus, we inductively have a sequence of nested subsequences $(f_n^k)_{k\geq 1}$ such that for all $k \geq 1$, $\langle f_n^k, x_k\rangle$ converges. For each $k \geq 1$, define $f_{n_k} = f_k^k$. Observe that by the construction of the nested subsequences, for any $m \geq 1$, the sequence $\langle f_{n_k}, x_m \rangle$ converges as $k \to \infty$. Moreover, for any $x \in E$ and $\varepsilon > 0$, there exists m such that $\|x - \|x\|x_m\| < \frac{\varepsilon}{2(\sup \|f_n\|+1)}$, as well as N such that for all $k, j \geq N$, $|\langle f_{n_k} - f_{n_j}, x_m \rangle| < \frac{\varepsilon}{2(\|x\|+1)}$. Thus, for any $k, j \geq N$,

$$\begin{aligned}
|\langle f_{n_k} - f_{n_j}, x \rangle| &\leq |\langle f_{n_k}, x - \|x\|x_m \rangle| + |\langle f_{n_k} - f_{n_j}, \|x\|x_m \rangle| + |\langle f_{n_j}, x - \|x\|x_m \rangle| \\
&\leq 2 \sup_n \|f_n\| \|x - \|x\|x_m\| + \|x\| |\langle f_{n_k} - f_{n_j}, x_m \rangle| \\
&< \varepsilon.
\end{aligned}$$

It follows that for every $x \in E$, the sequence $(\langle f_{n_k}, x \rangle)$ is Cauchy and therefore converges to some point in $\mathbb{R}$. By Corollary 2.3, $f = \lim_{k \to \infty} f_{n_k} \in E^*$ and since $\langle f - f_{n_k}, x \rangle \to 0$ as $k \to \infty$ for all $x \in E$, it follows that $f_{n_k} \overset{*}{\rightharpoonup} f$ in $\sigma(E^*, E)$. □

Lemma 1

Let V be an n.v.s. and suppose that M is a closed proper subspace of V. Then for any $\varepsilon > 0$, there exists some unit vector $x \in E$ such that $\text{dist}(x, M) \geq 1 - \varepsilon$.

Proof. Fix $\varepsilon > 0$ and some $y \notin M$. Let $\lambda = \text{dist}(y, M)$. Since M is closed, $\lambda > 0$. Pick some $\delta > 0$ such that $\frac{\delta}{\lambda+\delta} \leq \varepsilon$. Note that by the definition of dist, there must exist some $m \in M$ such that $\|y - m\| \leq \lambda + \delta$, and since $y \notin M$, $\|y - m\| > 0$. Set $x = \frac{y-m}{\|y-m\|}$. Then for any $m' \in M$,

$$\begin{aligned}
\|x - m'\| &= \left\| \frac{y - m}{\|y - m\|} - m' \right\| \\
&= \left\| \frac{y - (m - \|y - m\| m')}{\|y - m\|} \right\| \\
&\geq \frac{\lambda}{\lambda + \delta} \geq 1 - \varepsilon.
\end{aligned}$$

□

3.22

Let E be an infinite-dimensional Banach space satisfying *one* of the following assumptions:

(a) E^* is separable,

(b) E is reflexive.

Prove that there exists a sequence (x_n) in E such that

$$\|x_n\| = 1 \quad \forall n \quad \text{and} \quad x_n \rightharpoonup 0 \text{ weakly } \sigma(E, E^*).$$

Proof. (a) If E^* is separable, then B_E is metrizable in the weak topology $\sigma(E, E^*)$ by Theorem 3.29. Now since E is infinite-dimensional, we saw in Example 1 of Chapter 3 that the weak closure of the unit sphere $S = \{x \in E : \|x\| = 1\}$ is $\bar{S}^{\sigma(E, E^*)} = B_E$. It follows that S is a dense subset of the metric space B_E (with respect to $\sigma(E, E^*)$), and so every point $x \in B_E$ is equal to the weak limit of some sequence in S. In particular, there must exist some sequence $(x_n) \subset S$ such that $x_n \rightharpoonup 0$ weakly.

(b) Suppose that E is reflexive and infinite-dimensional. Using Lemma 1 above, I shall construct a sequence $\{x_n\}$ such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq \frac{1}{2}$ for any $n \neq m$. Begin by picking any $x_1 \in E$ such that $\|x_1\| = 1$. Now suppose that we have picked $x_1, \ldots, x_k \in E$ with the desired properties. Since $\text{span}(x_1, \ldots, x_k)$ is a closed proper subspace of E, by Lemma 1, there exists some unit vector $x_{k+1} \in E$ such that $\|x_{k+1} - x_i\| \geq \text{dist}(x_{k+1}, \text{span}(x_1, \ldots, x_k)) \geq \frac{1}{2}$. Thus, we can continue inductively to get the desired sequence $(x_n) \subset E$. Since (x_n) is a bounded sequence and E is reflexive, by Theorem 3.18, there exists a weakly convergent subsequence (x_{n_k}). Let $x \in E$ be the weak limit of this subsequence. By potentially removing at most one point in this subsequence, we may assume WLOG that $\|x_{n_k} - x\| \geq \frac{1}{4}$ for all k (if there were some k_0 such that $\|x - x_{n_{k_0}}\| < \frac{1}{4}$, then for all $k \neq k_0$,

$\|x - x_{n_k}\| \geq |\|x - x_{n_{k_0}}\| - \|x_{n_{k_0}} - x_{n_k}\|| \geq \frac{1}{4}$, then just remove $x_{n_{k_0}}$ from the sequence). Define the sequence (y_k) by $y_k = \frac{x_{n_k} - x}{\|x_{n_k} - x\|}$. Observe that for all k, $\|y_k\| = 1$ and for any $f \in E^*$,

$$|\langle f, y_k \rangle| = \left| \left\langle f, \frac{x_{n_k} - x}{\|x_{n_k} - x\|} \right\rangle \right|$$
$$\leq 4|\langle f, x_{n_k} - x \rangle| \to 0,$$

as $k \to \infty$. Thus, $y_k \rightharpoonup 0$ in $\sigma(E, E^*)$.

□

3.25

Let K be a compact metric space that is not finite. Prove that $C(K)$ is not reflexive.

Proof. Since K is a compact metric space with infinitely many points, only finitely many of the points in K can be isolated points (or else K would not be compact) and so there must exist some $a \in K$ that is a limit point of K. That is, there exists some sequence $(a_n) \subset K \setminus \{a\}$ that converges to a. Define a function $f : C(K) \to \mathbb{R}$; $u \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} u(a_n)$. To see that f is well-defined, fix $u \in C(K)$ and observe that $\sum_{n=1}^{\infty} |\frac{1}{2^n} u(a_n)| \leq \sup_{x \in K} |u(x)| \sum_{n=1}^{\infty} \frac{1}{2^n} = \|u\|_{C(K)} < \infty$, and so $\sum_{n=1}^{\infty} \frac{1}{2^n} u(a_n)$ converges absolutely for all $u \in C(K)$. Observe that since the sum converges absolutely over $C(K)$, it follows that for any $u_1, u_2 \in C(K)$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, $f(\lambda_1 u_1 + \lambda_2 u_2) = \sum_{n=1}^{\infty} (\lambda_1 u_1 + \lambda_2 u_2)(a_n) = \lambda_1 \sum_{n=1}^{\infty} u_1(a_n) + \lambda_2 \sum_{n=1}^{\infty} u_2(a_n) = \lambda_1 f(u_1) + \lambda_2 f(u_2)$. Thus, f is a linear functional such that for any $u \in C(K)$, $|f(u)| \leq \|u\|_{C(K)}$. It follows that $f \in C(K)^*$ and $\|f\| \leq 1$.

Define $M = \{u \in C(K) : u(a) = 0\}$. Clearly M is a linear subspace of $C(K)$ and since for any $(u_n) \subset M$ such that $u_n \to u$ in $C(K)$, then $u_n \to u$ uniformly and so $0 = u_n(a) \to u(a)$, which shows that M is a closed linear subspace of $C(K)$. By Proposition 3.20, to prove that $C(K)$ is not reflexive, it suffices to prove that M is not reflexive. To this end, set $g = f|_M$. Clearly $g \in M^*$ and $\|g\| \leq \|f\| = 1$. To see that $\|g\| = 1$, observe that for any $n \geq 1$, the map $u_n = n\, d(x, a) \wedge 1 \in M$ and $|g(u_n)| \geq \sum_{k=1}^{n} \frac{1}{2^k} \to 1$ as $n \to \infty$. Fix $u \in M$ with $\|u\| = 1$ and observe that since $a_n \to a$ and u is continuous, there exists some N such that for all $n \geq N$, $|u(a_n)| = |u(a_n) - u(a)| < \frac{1}{2}$. Thus, $|g(u)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} |u(a_n)| \leq \sum_{n=1}^{N} \frac{1}{2^n} + \frac{1}{2} \sum_{n=N+1}^{\infty} \frac{1}{2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. It follows that, for every $u \in M$ with $\|u\| = 1$, $|g(u)| < 1$. Let $J : M \to M^{**}$ be the canonical embedding of M into its double dual. Since $g \in M^*$, by the Hahn-Banach theorem, there exists some $\xi \in M^{**}$ such that $\langle \xi, g \rangle = \|g\|^2 = 1$ and $\|\xi\| = \|g\| = 1$. Observe that $\xi \notin J(M)$ since for any $u \in M$ with $\|u\| = \|J(u)\| = \|\xi\| = 1$, $\langle J(u), g \rangle = g(u) < 1$. Thus, M is not reflexive which proves that $C(K)$ cannot be reflexive. □

3.26

Let F be a separable Banach space and let (a_n) be a dense subset of B_F. Consider the linear operator $T : \ell^1 \to F$ defined by

$$Tx = \sum_{i=1}^{\infty} x_i a_i \quad \text{with } x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^1.$$

1. Prove that T is bounded and surjective.

   Proof. First observe that T is well defined since for any $x \in \ell^1$, the sequence $\sum_{i=1}^{n} x_i a_i$ is Cauchy (since $\|\sum_{i=n}^{m} x_i a_i\| \leq \sum_{i=n}^{m} |x_i| \to 0$ as $n, m \to \infty$) and therefore converges to a unique limit in F. Fix $x \in \ell^1$ with unit norm and note that $\|Tx\|_F = \lim_{n\to\infty} \|\sum_{i=1}^{n} x_i a_i\| \leq \lim_{n\to\infty} \sum_{i=1}^{n} |x_i| \|a_i\|_F \leq \lim_{n\to\infty} \sum_{i=1}^{n} |x_i| = 1$. Thus, $T \in \mathcal{L}(\ell^1, F)$ with $\|T\| \leq 1$.

   Clearly to prove that T is surjective, it suffices to prove that $B_F \subset T(\ell^1)$. To this end, fix $a \in B_F$. Since (a_n) is dense in B_F, there exists some n_1 such that $\|a - a_{n_1}\| < \frac{1}{2}$. Now suppose we have found $n_1, \ldots, n_k$ such that $n_i \neq n_j$ for $i \neq j$ and $\|a - a_{n_1} - \frac{1}{2} a_{n_2} - \cdots - \frac{1}{2^{k-1}} a_{n_k}\| < \frac{1}{2^k}$. Since F is a metric space with no isolated points (with respect to the norm on F), a dense set excluding finitely many points is still dense. Thus, the sequence $(\frac{1}{2^k} a_n)_{n \notin \{n_1, \ldots, n_k\}}$ is dense in $\frac{1}{2^k} B_F$, and so there exists some $n_{k+1} \notin \{n_1, \ldots, n_k\}$ such that $\|(a - a_{n_1} - \cdots - \frac{1}{2^{k-1}} a_{n_k}) - \frac{1}{2^k} a_{n_{k+1}}\| < \frac{1}{2^{k+1}}$. Continuing this process inductively, we get an injection $\xi : k \in \mathbb{N} \mapsto n_k$

such that $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} a_{n_k} = a$. Take the inverse $\xi^{-1} : \xi(\mathbb{N}) \to \mathbb{N}$ of this sequence and define the sequence x by $x_n = \begin{cases} \frac{1}{2^{\xi^{-1}(n)-1}}, & n \in \xi(\mathbb{N}) \\ 0, & n \notin \xi(\mathbb{N}). \end{cases}$ Note that $x \in \ell^1$ since $\sum_{n=1}^{\infty} |x_n| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ (since $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is absolutely convergent and so unconditionally convergent). Moreover, by a change of variables, we have that

$$\begin{aligned} Tx &= \sum_{n \in \xi(\mathbb{N})} \frac{1}{2^{\xi^{-1}(n)-1}} a_n \\ &= \sum_{k \in \mathbb{N}} \frac{1}{2^{k-1}} a_{n_k} = a. \end{aligned}$$

Thus, $B_F \subset T(\ell^1)$, which proves that T is surjective. □

In what follows, we assume, in addition, that F is infinite-dimensional and that F^* is separable.

2. Prove that T has no right inverse.

   Proof. Towards a contradiction, suppose that there exists some $S \in \mathcal{L}(F, \ell^1)$ such that $Id_F = TS$. Since F is infinite-dimensional and F^* is separable, by Exercise 3.22 there must exist some sequence $(b_n) \subset F$ such that $\|b_n\| = 1$ for all n and $b_n \rightharpoonup 0$ weakly $\sigma(F, F^*)$. By Theorem 3.10, S is continuous from $\sigma(F, F^*)$ on F to $\sigma(\ell^1, \ell^\infty)$ on ℓ^1. Thus, $Sb_n \rightharpoonup 0$ in $\sigma(\ell^1, \ell^\infty)$ and by Schur's theorem, it follows that $Sb_n \to 0$ strongly so that $\|Sb_n\| \to 0$ as $n \to \infty$. However, for all n, $1 = \|b_n\| = \|TSb_n\| \leq \|Sb_n\|$, a contradiction. Thus, by contradiction, T has no right inverse. □

3. Deduce that $N(T)$ has no complement in ℓ^1.

   Proof. By Theorem 2.12, $N(T)$ does not admit a complement in ℓ^1. □

4. Determine T^*.

   Solution

   For any $f \in F^*$, define $Af = \left(\langle f, a_i \rangle\right)_{i \geq 1}$. Observe that $Af \in \ell^\infty$ for all $f \in F^*$ since $\sup_n |\langle f, a_n \rangle| \leq \|f\|$. For any $f \in F^*$ and $x \in \ell^1$ we have $\langle T^* f, x \rangle = \langle f, Tx \rangle = \left\langle f, \sum_{i=1}^{\infty} x_i a_i \right\rangle = \sum_{i=1}^{\infty} x_i \langle f, a_i \rangle = \langle Af, x \rangle$. It follows that $T^* f = Af = \left(\langle f, a_i \rangle\right)_{i \geq 1}$.

3.27

Let E be a separable Banach space with norm $\| \ \|$. The dual norm on E^* is also denoted by $\| \ \|$. The purpose of this exercise is to construct an equivalent norm on E that is strictly convex and whose dual norm is also strictly convex.

Let $(a_n) \subset B_E$ be a dense subset of B_E with respect to the strong topology. Let $(b_n) \subset B_{E^*}$ be a countable subset of B_{E^*} that is dense in B_{E^*} for the weak* topology $\sigma(E^*, E)$. Why does such a set exist?

Solution

By Theorem 3.23, the weak* topology on B_{E^*} is metrizable and by Banach-Alaoglu, B_{E^*} is weak* compact. Since every compact metric space is separable, B_{E^*} is separable with respect to $\sigma(E^*, E)$. (Let K be a compact metric space and for each $n \geq 1$, consider the covering of K by $\frac{1}{n}$-balls indexed over $x \in K$. Apply compactness to conclude that the $\frac{1}{n}$-balls indexed over some finite set $F_n \subset K$ cover K. Then $\bigcup_{n \in \mathbb{N}} F_n$ is a countable dense subset of K.)

Given $f \in E^*$, set

$$\|f\|_1 = \left\{ \|f\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle f, a_n \rangle|^2 \right\}^{\frac{1}{2}}.$$

1. Prove that $\| \ \|_1$ is a norm equivalent to $\| \ \|$.

Proof. The homogeneity and positive-definiteness of $\| \ \|_1$ are obvious. Note that my proof of the strict convexity of $\| \ \|_1$ in part 2. below does not rely on $\| \ \|_1$ being a norm and so to prove that the triangle inequality is satisfied, I may use the fact that $\| \ \|_1$ satisfies the strict convexity property. Fix $f, g \in E^*$ and observe that if $f = g$ or one of f or g are zero, then the triangle inequality follows trivially. Thus, we may assume WLOG that $f, g \neq 0$ and $f \neq g$. Then since $\| \ \|_1$ satisfies strict convexity (my proof given below just uses the strict convexity of $x \mapsto x^2$), it follows that

$$\frac{1}{(\|f\| + \|g\|)}\|f+g\| = \left\| \frac{\|f\|}{\|f\|+\|g\|}\frac{f}{\|f\|} + \left(1 - \frac{\|f\|}{\|f\|+\|g\|}\right)\frac{g}{\|g\|}\right\|_1$$
$$< 1.$$

Thus $\| \ \|_1$ is a norm on E^*. Clearly for any $f \in E^*$, $\|f\|^2 \leq \|f\|_1^2$ and so $\|f\| \leq \|f\|_1$. Moreover, $\|f\|_1^2 \leq \|f\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n}\|f\|^2\|a_n\| \leq 2\|f\|^2$. It follows that, $\|f\| \leq \|f\|_1 \leq \sqrt{2}\|f\|$, proving that $\| \ \|$ and $\| \ \|_1$ are equivalent norms on E^*. □

2. Prove that $\| \ \|_1$ is strictly convex.

Proof. Fix $t \in (0,1)$ and $f, g \in E^*$ such that $\|f\|_1 = \|g\|_1 = 1$ and $f \neq g$. Since (a_n) is dense in B_E and $f \neq g$, there must exist some n_0 such that $\langle f, a_{n_0}\rangle \neq \langle g, a_{n_0}\rangle$. Thus, by the strict convexity of $x \mapsto x^2$, it follows that $|\langle tf + (1-t)g, a_{n_0}\rangle|^2 < t|\langle f, a_{n_0}\rangle|^2 + (1-t)|\langle g, a_{n_0}\rangle|^2$. Then again using the convexity of the $x \mapsto x^2$, it follows that

$$\begin{aligned}\|tf + (1-t)g\|_1^2 &< t\|f\|^2 + (1-t)\|g\|^2 + t\sum_{n=1}^{\infty}\frac{1}{2^n}|\langle f, a_n\rangle|^2 + (1-t)\sum_{n=1}^{\infty}\frac{1}{2^n}|\langle g, a_n\rangle|^2 \\ &= t\|f\|_1^2 + (1-t)\|g\|_1^2 = 1.\end{aligned}$$

It follows that $\|tf + (1-t)g\|_1 < 1$, proving that $\| \ \|_1$ is strictly convex. □

Given $x \in E$, set

$$\|x\|_2 = \left\{\|x\|_1^2 + \sum_{n=1}^{\infty}\frac{1}{2^n}|\langle b_n, x\rangle|^2\right\}^{\frac{1}{2}}.$$

where $\|x\|_1 = \sup_{\|f\|_1 \leq 1}\langle f, x\rangle$.

3. Prove that $\| \ \|_2$ is a strictly convex norm that is equivalent to $\| \ \|$.

Proof. Again, homogeneity and positive-definiteness of $\| \ \|_2$ are both obvious. Observe that my proof that $\| \ \|_1$ satisfies the triangle inequality only made use of the fact that $\| \ \|_1$ satisfies the strict convexity property. Thus, to complete our verification that $\| \ \|_2$ defines a norm on E, it suffices to prove that $\| \ \|_2$ satisfies the strict convexity property. To this end, fix $t \in (0,1)$ and $x, y \in E$ such that $\|x\|_2 = \|y\|_2 = 1$ and $x \neq y$. I claim that there must exist some n such that $\langle b_n, x\rangle \neq \langle b_n, y\rangle$. To see why this is the case, observe that if it weren't, so that $\langle b_n, x - y\rangle = 0$ for all n, then for any $f \in B_{E^*}$ and $\delta > 0$, since (b_n) is weak* dense in B_{E^*}, there must exist some n such that $\delta > |\langle b_n - f, x - y\rangle| = |\langle f, x - y\rangle|$. But then $\langle f, x - y\rangle = 0$ for all $f \in B_{E^*}$, which would force the contradiction that $x = y$. Thus, there exists some n such that $\langle b_n, x\rangle \neq \langle b_n, y\rangle$. By the strict convexity of $x \mapsto x^2$, $|\langle b_n, tx + (1-t)y\rangle|^2 < t|\langle b_n, x\rangle|^2 + (1-t)|\langle b_n, y\rangle|^2$. Applying the convexity of $x \mapsto x^2$, it follows that

$$\begin{aligned}\|tx + (1-t)y\|_2^2 &= \|tx + (1-t)y\|_1^2 + \sum_{n=1}^{\infty}\frac{1}{2^n}|\langle b_n, tx + (1-t)y\rangle|^2 \\ &< t\|x\|_1^2 + (1-t)\|y\|_1^2 + t\sum_{n=1}^{\infty}\frac{1}{2^n}|\langle b_n, x\rangle|^2 + (1-t)\sum_{n=1}^{\infty}\frac{1}{2^n}|\langle b_n, y\rangle|^2 \\ &= t\|x\|_2^2 + (1-t)\|y\|_2^2 = 1.\end{aligned}$$

Taking square-roots, it follows that $\|tx + (1-t)y\|_2 < 1$, proving that $\| \ \|_2$ satisfies the strict convexity property. Thus, from the comments above, $\| \ \|_2$ is a strictly convex norm on E.

Towards proving that $\| \ \|_2$ and $\| \ \|$ are equivalent norms, fix $x \in E$ and observe that for any $f \in E^*$ such that $\|f\|_1 \le 1$, since $\|f\| \le \|f\|_1 \le 1$, it follows that $\langle f, x\rangle \le \|f\|\|x\| \le \|x\|$. Hence, $\|x\|_1 \le \|x\|$. Thus,

$$\|x\|_2^2 \le \|x\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} |\langle b_n, x\rangle|^2 \le 2\|x\|^2,$$

so that $\|x\|_2 \le \sqrt{2}\|x\|$. Now applying Hahn-Banach, pick some $f \in E^*$ such that $\langle f, x\rangle = \|x\|^2$ and $\|f\| = \|x\|$. Then $\left\| \frac{1}{\sqrt{2}\|x\|} f \right\|_1 \le \frac{\|f\|}{\|x\|} = 1$ and $\langle \frac{1}{\sqrt{2}\|x\|} f, x\rangle = \frac{1}{\sqrt{2}}\|x\|$, so that $\frac{1}{\sqrt{2}}\|x\| \le \|x\|_1 \le \|x\|_2$, which proves that $\| \ \|_2$ and $\| \ \|$ are equivalent norms on E. ☐

3.28

Let E be a uniformly convex Banach space. Let F denote the (multivalued) duality map from E into E^*. Prove that for every $f \in E^*$ there exists a unique $x \in E$ such that $f \in Fx$.

Proof. Fix some $f \in E^*$. By Hahn-Banach, there exists some $\xi \in E^{**}$ such that $\langle \xi, f\rangle = \|f\|^2$ and $\|\xi\| = \|f\|$. Since E is uniformly convex, by the Milman-Pettis Theorem, E is reflexive. Thus, letting $J : E \to E^{**}$ be the canonical embedding of E in E^{**}, it follows that there exists some $x \in E$ such that $\xi = J(x)$. Hence, $\|x\| = \|J(x)\| = \|f\|$ and $\langle f, x\rangle = \langle J(x), f\rangle = \|f\|^2 = \|x\|^2$. It follows that $f \in Fx$. Towards showing that x is the unique element of E such that $f \in Fx$, fix some $y \in E$ such that $f \in Fy$. Then $\|y\| = \|f\| = \|x\|$ and $\langle f, y\rangle = \|y\|^2$. Clearly if $x = 0$, then y is forced to be 0, so we may assume WLOG that $x \ne 0$. Observe that $\left\langle \frac{f}{\|x\|}, \frac{x+y}{2\|x\|} \right\rangle = 1$, and since $\left\| \frac{f}{\|x\|} \right\| = 1$, it follows that $\left\| \frac{x+y}{2\|x\|} \right\| \ge 1$. Since $\frac{x}{\|x\|}, \frac{y}{\|x\|}$ both belong to B_E, by uniform convexity, there cannot exist any $\varepsilon > 0$ such that $\left\| \frac{x}{\|x\|} - \frac{y}{\|x\|} \right\| > \varepsilon$. That is, $x = y$. ☐

3.29

Let E be a uniformly convex Banach space.

1. Prove that $\forall M > 0, \forall \varepsilon > 0, \exists \delta > 0$ such that

$$\left\| \frac{x+y}{2} \right\|^2 \le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \delta$$

$$\forall x, y \in E \quad \text{with} \quad \|x\| \le M, \|y\| \le M \quad \text{and} \quad \|x - y\| > \varepsilon.$$

Proof. Suppose for a contradiction that there exists some $M > 0$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exists some $x, y \in E$ with $\|x\| \le M$, $\|y\| \le M$ and $\|x - y\| > \varepsilon$ but $\left\| \frac{x+y}{2} \right\|^2 > \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \delta$. Then for each $n \ge 1$, there exists $x_n, y_n \in E$ with $\|x_n\|, \|y_n\| \le M$, $\|x_n - y_n\| > \varepsilon$ and $\left\| \frac{x_n+y_n}{2} \right\|^2 > \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \frac{1}{n}$. Since $(\|x_n\|)$ is a bounded sequence in $\mathbb{R}$, there exists some subsequence $(\|x_{n_k}\|)$ that converges, and since $(\|y_{n_k}\|)$ is a bounded sequence, there exists a subsequence $(\|y_{n_{k_j}}\|)$ that converges. Note that $(\|x_{n_{k_j}}\|)$ also converges, and so we may assume WLOG that $(\|x_n\|)$ and $(\|y_n\|)$ are both convergent sequences with limits a and b, respectively. Then $\frac{1}{2}a^2 + \frac{1}{2}b^2 = \lim_{n\to\infty} \frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \frac{1}{n} \le \limsup_{n\to\infty} \left\| \frac{x_n+y_n}{2} \right\|^2 \le \limsup_{n\to\infty} \left(\frac{1}{2}\|x_n\| + \frac{1}{2}\|y_n\| \right)^2 = \left(\frac{1}{2}a + \frac{1}{2}b \right)^2$. By the strict convexity of $x \mapsto x^2$, it follows that $\lim_n \|x_n\| = a = b = \lim_n \|y_n\|$. Thus, we have that there exists some N such that for all $n \ge N$, $\left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| > \frac{\varepsilon}{a + \frac{1}{2}}$. Then by uniform convexity, there exists some $\delta' > 0$ such that $\left\| \frac{\frac{1}{\|x_n\|}x_n + \frac{1}{\|y_n\|}y_n}{2} \right\| < 1 - \delta'$ for all $n \ge N$. But then

$$\begin{aligned} \limsup_{n\to\infty} \left\| \frac{x_n + y_n}{2} \right\|^2 &= \limsup_{n\to\infty} \|x_n\|^2 \left\| \frac{\frac{1}{\|x_n\|}x_n + \frac{1}{\|y_n\|}y_n}{2} \right\|^2 \\ &< a^2(1 - \delta') \\ &< \frac{1}{2}a^2 + \frac{1}{2}b^2, \end{aligned}$$

contradicting what we found above. Hence, the statement is proven by contradiction. ☐

2. Same question when $\| \ \|^2$ is replaced by $\| \ \|^p$ with $1 < p < \infty$.

   Proof. Since $x \mapsto x^p$ is strictly convex on $(0, \infty)$ for all $1 < p < \infty$, the exact same proof above but replacing 2 by p works. $\square$

3.30

Let E be a Banach space with norm $\| \ \|$. Assume that there exists on E an equivalent norm, denoted by $| \ |$, that is uniformly convex.

Prove that given any $k > 1$, there exists a uniformly convex norm $[[\quad]]$ on E such that

$$\|x\| \le [[x]] \le k\|x\| \quad \forall x \in E.$$

Proof. Fix $k > 1$. Since $| \ |$ is equivalent to $\| \ \|$, there exist constants $c, C > 0$ such that $c\|x\| \le |x| \le C\|x\|$ for all $x \in E$. Set $\alpha = \frac{k^2-1}{C^2} > 0$ and define $[[\quad]] : E \to [0, \infty)$ by $[[x]] = \sqrt{\|x\|^2 + \alpha|x|^2}$. Observe that for all $x \in E$, $\|x\| \le [[x]]$ and $[[x]]^2 \le (1 + \alpha C^2)\|x\|^2 = k^2\|x\|^2$, so that $[[x]] \le k\|x\|$. Thus, if we can show that $[[\quad]]$ is a uniformly convex norm on E, then we are done. That $[[\quad]]$ satisfies homogeneity and positive-definiteness is obvious. To prove the triangle inequality, note that for all $t \in (0,1)$ and $x, y \in E$ such that $[[x]], [[y]] \le 1$, $[[tx + (1-t)y]]^2 = \|tx + (1-t)y\|^2 + \alpha|tx + (1-t)y|^2 \le t\|x\|^2 + (1-t)\|y\|^2 + t\alpha|x|^2 + (1-t)\alpha|y|^2 = t[[x]]^2 + (1-t)[[y]]^2 \le 1$, so that $[[tx + (1-t)y]] \le 1$. Thus, for all $x, y \in E$,

$$\begin{aligned}\frac{1}{([[x]] + [[y]])}[[x+y]] &= \left[\left[\frac{[[x]]}{[[x]] + [[y]]}\frac{x}{[[x]]} + \left(1 - \frac{[[x]]}{[[x]] + [[y]]}\right)\frac{y}{[[y]]}\right]\right] \\ &\le 1,\end{aligned}$$

proving that $[[\quad]]$ is a norm on E. It remains to prove that $[[\quad]]$ is uniformly convex. To this end, fix $\varepsilon > 0$. Define $\beta = \sqrt{\frac{1}{c^2} + \alpha}$ and $\gamma = \sqrt{\frac{1}{C^2} + \alpha}$ and observe that for all $z \in E$, $\gamma|z| \le [[z]] \le \beta|z|$. By Exercise 3.29, there exists some $\delta > 0$ such that for all $x, y \in E$ with $|x|, |y| \le \frac{1}{\gamma}$ and $|x - y| > \frac{\varepsilon}{\beta}$, we have the inequality $\left|\frac{x+y}{2}\right|^2 \le \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \delta$. Fix $x, y \in E$ such that $[[x]], [[y]] \le 1$ and $[[x - y]] > \varepsilon$. Thus, $|x|, |y| \le \frac{1}{\gamma}$ and $|x - y| > \frac{\varepsilon}{\beta}$. It follows that

$$\begin{aligned}\left[\left[\frac{x+y}{2}\right]\right]^2 &= \left\|\frac{x+y}{2}\right\|^2 + \alpha\left|\frac{x+y}{2}\right|^2 \\ &\le \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 + \alpha\left(\frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \delta\right) \\ &= \frac{1}{2}[[x]]^2 + \frac{1}{2}[[y]]^2 - \alpha\delta \\ &\le 1 - \alpha\delta.\end{aligned}$$

So $\left[\left[\frac{x+y}{2}\right]\right] \le \sqrt{1 - \alpha\delta} < 1$, and we can pick any δ_0 such that $1 - \delta_0 > \sqrt{1 - \alpha\delta}$, proving that $[[\quad]]$ is uniformly convex. $\square$

3.31

Let E be a uniformly convex Banach space.

1. Prove that

$$\forall \varepsilon > 0, \quad \forall \alpha \in \left(0, \frac{1}{2}\right), \quad \exists \delta > 0 \quad \text{such that}$$

$$\|tx + (1-t)y\| \le 1 - \delta$$

$$\forall t \in [\alpha, 1-\alpha], \quad \forall x, y \in E \quad \text{with } \|x\| \le 1, \|y\| \le 1 \text{ and } \|x - y\| \ge \varepsilon.$$

   Proof. Fix $\varepsilon > 0$, $\alpha \in \left(0, \frac{1}{2}\right)$. By the uniform convexity of E, there exists some $\delta > 0$ such that for all $x, y \in E$ with $\|x\|, \|y\| \le 1$ and $\|x - y\| \ge 2\alpha\varepsilon$ we have the inequality $\left\|\frac{x+y}{2}\right\| \le 1 - \delta$. Fix $x, y \in E$ with $\|x\|, \|y\| \le 1$ and

$\|x-y\| \geq \varepsilon$. Then for any $t \in \left[\alpha, \frac{1}{2}\right]$, define $z = 2tx + y - 2ty$ and observe that since $\|z\| \leq 2t\|x\| + (1-2t)\|y\| \leq 1$ and $\|y-z\| = 2t\|x-y\| \geq 2t\varepsilon \geq 2\alpha\varepsilon$, it follows that $\|tx+(1-t)y\| = \left\|\frac{y+z}{2}\right\| \leq 1-\delta$. Moreover, for any $t \in \left(\frac{1}{2}, 1-\alpha\right]$, define $z = 2tx - 2ty + 2y - x$ and observe that $\|z\| \leq (2t-1)\|x\| + (2-2t)\|y\| \leq 1$ and $\|z-x\| = (2-2t)\|x-y\| \geq (2-2t)\varepsilon \geq 2\alpha\varepsilon$, so that $\|tx + (1-t)y\| = \left\|\frac{x+z}{2}\right\| \leq 1-\delta$. Hence, $\|tx + (1-t)y\| \leq 1-\delta$ for all $t \in [\alpha, 1-\alpha]$. ☐

2. Deduce that E is strictly convex.

   Proof. Fix $x, y \in E$ such that $\|x\| = \|y\| = 1$ and suppose that $x \neq y$. Then there exists some $\varepsilon > 0$ such that $\|x-y\| > \varepsilon$. Fix $t \in (0,1)$ and pick $\alpha \in (0, \min(t, 1-t)) \subset (0, \frac{1}{2})$. Observe that $t \in [\alpha, 1-\alpha]$ and so by part 1. above, there exists some $\delta > 0$ such that $\|tx + (1-t)y\| \leq 1-\delta < 1$. Thus, $\|tx + (1-t)y\| < 1$ for all $t \in (0,1)$ and it follows that E is strictly convex. ☐

3.32 Projection on a closed convex set in a uniformly convex Banach space.

Let E be a uniformly convex Banach space and $C \subset E$ a nonempty closed convex set.

1. Prove that for every $x \in E$,
$$\inf_{y \in C} \|x-y\|$$
   is achieved by some unique point in C, denoted by $P_C x$.

   Proof. Observe that since E is uniformly convex, E is reflexive by the Milman-Pettis Theorem. Thus, by Theorem 3.18, every bounded sequence in E has a weakly convergent subsequence. Fix $x \in E$ and for every $n \geq 1$, pick some $y_n \in C$ such that $\|x - y_n\| < \inf_{y \in C} \|x-y\| + \frac{1}{n}$. Since for every n, $\|y_n\| \leq \inf_{y \in C} \|x-y\| + \|x\| + 1$, (y_n) is a bounded sequence in E and therefore there exists a subsequence (y_{n_k}) that converges weakly to a point $y \in E$. Since C is strongly closed and convex, by Theorem 3.7 C is weakly closed. Thus, the fact that $(y_{n_k}) \subset C$ and $y_{n_k} \rightharpoonup y$ implies that $y \in C$. Since $x - y_{n_k} \rightharpoonup x - y$, by Proposition 3.5, $\|x-y\| \leq \liminf_{k\to\infty} \|x - y_{n_k}\| \leq \liminf_{k\to\infty} (\inf_{z \in C} \|x-z\| + \frac{1}{n_k}) = \inf_{z\in C} \|x-z\|$. This proves that $\inf_{z \in C} \|x-z\|$ is achieved by y. Towards proving that y is the unique such point in C, suppose for a contradiction that $z \in C$ such that $\|x-z\| = \inf_{z \in C} \|x-z\|$ and $z \neq y$. Then there exists some $\varepsilon > 0$ such that $\|(x-z)-(x-y)\| = \|z-y\| > \varepsilon$, and by the convexity of C, $\frac{z+y}{2} \in C$. Moreover, since $\|x-z\|, \|x-y\| \leq \|x-y\|$, by Exercise 3.29, there exists some $\delta > 0$ such that $\left\|x - \frac{z+y}{2}\right\|^2 \leq \frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|x-z\|^2 - \delta = \inf_{z \in C}\|x-z\|^2 - \delta$, which is clearly absurd. Thus, y is the unique point in C that achieves the distance from x to C. ☐

2. Prove that every minimizing sequence (y_n) in C converges strongly to $P_C x$.

   Proof. I shall first prove that $y_n \rightharpoonup P_C x$. Suppose for a contradiction that (y_n) does not converge weakly to $P_C x$. Then there must exist a subsequence (y_{n_k}), $\varepsilon > 0$ and $f \in E^*$ such that $|\langle f, y_{n_k} - P_C x\rangle| > \varepsilon$. Since E is reflexive and (y_{n_k}) is a bounded sequence (as there exists some N such that for all $k \geq N$, $\|y_{n_k}\| \leq \inf_{y \in C} \|x-y\| + \|x\| + 1$), it follows that (y_{n_k}) has a weakly convergent subsequence $(y_{n_{k_j}})$, converging weakly to some point $y \in E$. Since $(y_{n_{k_j}}) \subset C$ and C is strongly closed and convex, so weakly closed, it follows that $y \in C$. Moreover, since $x - y_{n_{k_j}} \rightharpoonup x - y$, by Proposition 3.5, $\|x-y\| \leq \liminf \|x - y_{n_{k_j}}\| = \inf_{z \in C} \|x-z\|$. But then from part 1., we conclude that $y = P_C x$, and so $\langle f, y_{n_{k_j}} - P_C x\rangle \to 0$, a contradiction. Thus, by contradiction, $y_n \rightharpoonup P_C x$ weakly. To complete the proof, note that because $x - y_n \rightharpoonup x - P_C x$ weakly and $\|x - y_n\| \to \|x - P_C x\|$, it follows by Proposition 3.32 that $x - y_n \to x - P_C x$ strongly, and therefore $y_n \to P_C x$ strongly. ☐

3. Prove that the map $x \mapsto P_C x$ is continuous from E strong into E strong.

   Proof. See part 4. below. ☐

4. More precisely, prove that P_C is uniformly continuous on bounded subsets of E.

Proof. Towards a contradiction, suppose that there exists a bounded subset $B \subset E$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x, y \in B$ with $\|x-y\| < \delta$ and $\|P_C x - P_C y\| \geq \varepsilon$. Then we can construct sequences $(x_n), (y_n) \subset B$ such that $\|x_n - y_n\| \to 0$ as $n \to \infty$ and $\inf_n \|P_C x_n - P_C y_n\| \geq \varepsilon$. Moreover, since B is bounded, there exists a constant $M > 0$ such that $\|x_n\|, \|y_n\| \leq M$ for all n. Note that by the convexity of C and the definition of P_C,

$$\begin{aligned}\|x_n - P_C x_n\| &\leq \left\| x - \frac{P_C x_n + P_C y_n}{2} \right\| \\ &\leq \left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\| + \frac{1}{2}\|x_n - y_n\|.\end{aligned}$$

Similarly, $\|y_n - P_C y_n\| \leq \left\| \frac{x_n+y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\| + \frac{1}{2}\|x_n - y_n\|$. Thus, we have that $\frac{1}{2}\|x_n - P_C x_n\|^2 + \frac{1}{2}\|y_n - P_C y_n\|^2 \leq \left\| \frac{x_n+y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\|^2 + o(n)$. Now, with the intent of applying Exercise 3.29, I claim that $(\|x_n - P_C x_n\|)$ and $(\|y_n - P_C y_n\|)$ are bounded sequences. Indeed, for any $n \geq 1$, $\|x_n - P_C x_n\| \leq \|x_n - P_C x_1\| \leq \|x_n - x_1\| + \|x_1 - P_C x_1\| \leq 2M + \|x_1 - P_C x_1\|$. (The proof for $(\|y_n - P_C y_n\|)$ being bounded follows by the exact same argument.) Since $\|x_n - y_n\| \to 0$, there must exist some N such that for all $n \geq N$, $\|x_n - y_n\| < \frac{\varepsilon}{2}$ and it follows that for all $n \geq N$, $\|(x_n - P_C x_n) - (y_n - P_C y_n)\| \geq |\|P_C x_n - P_C y_n\| - \|x_n - y_n\|| \geq \frac{\varepsilon}{2}$. Thus, by Exercise 3.29, there exists some $\delta > 0$ such that for every $n \geq N$,

$$\left\| \frac{x_n + y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\|^2 \leq \frac{1}{2}\|x_n - P_C x_n\|^2 + \frac{1}{2}\|y_n - P_C y_n\|^2 - \delta.$$

This conclusion gives us the desired contradiction since, when combined with the above inequality, $\frac{1}{2}\|x_n - P_C x_n\|^2 + \frac{1}{2}\|y_n - P_C y_n\|^2 \leq \left\| \frac{x_n+y_n}{2} - \frac{P_C x_n + P_C y_n}{2} \right\|^2 + o(n)$, we get $0 < \delta \leq o(n)$, which is absurd. □

Let $\varphi : E \to (-\infty, +\infty]$ be a convex l.s.c. function, $\varphi \not\equiv +\infty$.

5. Prove that for every $x \in E$ and every integer $n \geq 1$,

$$\inf_{y \in E} \left\{ n\|x - y\|^2 + \varphi(y) \right\}$$

is achieved at some unique point, denoted by y_n.

Proof. Fix $x \in E$ and $n \geq 1$. Since $\varphi \not\equiv +\infty$, there exists some $y \in E$ such that $\alpha = n\|x - y\|^2 + \varphi(y) < \infty$. Since $y \mapsto n\|x - y\|^2$ is continuous and φ is l.s.c., $y \mapsto n\|x - y\|^2 + \varphi(y)$ is l.s.c. Moreover, for any $y_1, y_2 \in E$ and $t \in (0, 1)$,

$$\begin{aligned}n\|x - (ty_1 + (1-t)y_2)\|^2 + \varphi(ty_1 + (1-t)y_2) &\leq n\|t(x - y_1) + (1-t)(x - y_2)\|^2 + t\varphi(y_1) + (1-t)\varphi(y_2) \\ &\leq t(n\|x - y_1\|^2 + \varphi(y_1)) + (1-t)(n\|x - y_2\|^2 + \varphi(y_2)).\end{aligned}$$

It follows that $y \mapsto n\|x - y\|^2 + \varphi(y)$ is convex and l.s.c. and so $C = \{y \in E : n\|x - y\|^2 + \varphi(y) \leq \alpha\}$ is a nonempty, closed, convex subset of E. By Proposition 1.10, there exists some $f \in E^*$ such that for all $y \in E$, $\langle f, y\rangle \leq \varphi(y)$. Thus, for any $y \in E$ such that $\varphi(y) < 0$, $|\varphi(y)| \leq |\langle f, y\rangle| \leq \|f\|\|y\|$, and it follows that for all $y \in E$ such that $\varphi(y) < 0$,

$$n\|x - y\|^2 + \varphi(y) \geq n\|x - y\|^2 - \|f\|\|y\| \geq n\|y\|^2 - (2n\|x\| + \|f\|)\|y\| + n\|x\|^2,$$

which is a positive quadratic polynomial in $\|y\|$ and so is bounded below. Thus, $\beta = \inf_{y \in E}\{n\|x - y\|^2 + \varphi(y)\} > -\infty$. Now we argue exactly as we did in part 1: fix some sequence $(y_n) \subset C$ such that $n\|x - y_n\|^2 + \varphi(y_n) \to \beta$. Note that (y_n) is a bounded sequence since $n\|x - y_n\|^2 + \varphi(y_n)$ is a convergent sequence, so bounded, and $\varphi(y_n)$ is bounded, so that $\|x - y_n\|^2$ must be bounded. Thus, since E is reflexive, (y_n) has a weakly convergent subsequence (y_{n_k}) with weak limit point y. By Corollary 3.9 $y \mapsto n\|x - y\|^2 + \varphi(y)$ is l.s.c. in the weak topology $\sigma(E, E^*)$, and so $y_{n_k} \rightharpoonup y$ implies that $n\|x - y\|^2 + \varphi(y) \leq \liminf_{k \to \infty} n\|x - y_{n_k}\|^2 + \varphi(y_{n_k}) = \beta$. Thus, $y \in E$ achieves the desired infimum. To see that y is the unique point in E that achieves this minimum, suppose for a contradiction that $z \in E$ such that $n\|x - z\|^2 + \varphi(z) = n\|x - y\|^2 + \varphi(y)$ and $z \neq y$. Then $\|y - z\| \geq \varepsilon$ for some $\varepsilon > 0$ and so by Exercise 3.29 there exists some $\delta > 0$ such that $\left\|x - \frac{x+y}{2}\right\|^2 \leq \frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x - z\|^2 - \delta$. But then $n\left\|x - \frac{x+y}{2}\right\|^2 + \varphi(\frac{x+y}{2}) \leq \frac{1}{2}n\|x - y\|^2 + \frac{1}{2}n\|x - z\|^2 + \frac{1}{2}\varphi(y) + \frac{1}{2}\varphi(z) - \delta = \inf_{y \in E}\{n\|x - y\|^2 + \varphi(y)\} - \delta$, which is impossible. Thus, by contradiction y is the unique point in E that achieves this minimum. □

6. Prove that $y_n \xrightarrow[n\to\infty]{} P_C x$, where $C = \overline{D(\varphi)}$.

Proof. Note that for any $n \geq 1$, $n\|x - y_n\|^2 + \varphi(y_n) \leq n\|x - y\|^2 + \varphi(y)$ for all $y \in E$. In particular, for any $n \geq 1$, $n\|x-y_n\|^2+\varphi(y_n) \leq n\|x-y_{n+1}\|^2+\varphi(y_n)$ and $(n+1)\|x-y_{n+1}\|^2+\varphi(y_{n+1}) \leq (n+1)\|x-y_n\|^2+\varphi(y_n)$. Thus, by playing around with these two inequalities, we get that $\|x - y_{n+1}\|^2 \leq (n+1)\|x - y_n\|^2 - n\|x + y_{n+1}\|^2 + \varphi(y_{n+1}) - \varphi(y_n) \leq \|x - y_n\|^2$, and so $\|x - y_{n+1}\| \leq \|x - y_n\|$ for all $n \geq 1$. It follows that $(x - y_n)$ is a bounded sequence, so that (y_n) is a also a bounded sequence, and since E is reflexive, (y_n) has a weakly convergent subsequence (y_{n_k}) with weak limit $y \in E$. Moreover, since $x - y_{n_k} \rightharpoonup x - y$ weakly, it follows that $\|x - y\|^2 \leq \liminf_k \|x - y_{n_k}\|^2$. Since $(\|x - y_n\|)$ is a monotonically decreasing sequence, bounded below by 0, it follows that $\|x - y_n\|$ converges, and since $\|x - y\| \leq \liminf_k \|x - y_{n_k}\| = \lim_n \|x - y_n\|$, it follows that $\|x - y\| \leq \|x - y_n\|$ for all $n \geq 1$. Note that for any $n \geq 1$, $n\|x-y\|^2+\varphi(y_n) \leq n\|x-y_n\|^2+\varphi(y_n) \leq n\|x-z\|^2+\varphi(z)$ for all $z \in E$. From my reasoning in part 5. above, we know that $\varphi(y_n) \in D(\varphi)$ for all n, and so we have that for all $z \in E$ and $n \geq 1$, $\|x - y\|^2 - \|x - z\| \leq \frac{\varphi(z)-\varphi(y_n)}{n}$. Using Proposition 1.10 again, we have an $f \in E^*$ such that if $\varphi(z) < 0$, then $|\varphi(z)| \leq \|f\|\|z\|$. Combining this insight with the fact that (y_n) is a bounded sequence, say with bound $M > 0$, we have that for all $n \geq 1$ and $z \in E$, $\|x - y\|^2 - \|x - z\|^2 \leq \frac{\varphi(z)+\|f\|M}{n}$. Taking the limit over n, we get that for any $z \in D(\varphi)$, $\|x - y\|^2 \leq \|x - z\|^2$, and it follows that $y = P_C x$ (since each $y_n \in D(\varphi) \subset C$ and C is the strong closure of a convex space, so weakly closed, the fact that $y_{n_k} \rightharpoonup y$ implies that $y \in C$). Finally, since for any $z \in D(\varphi)$, $\|x-y_n\|-\|x-z\| \leq \frac{\varphi(z)+\|f\|M}{n} \to 0$ as $n \to \infty$, it follows that $\|x - y_n\| \to \|x - P_C x\|$, so that $(y_n) \subset C$ is a minimizing sequence of C. Since $C = \overline{D(\varphi)}$ is a nonempty closed convex set, by part 2. above, $y_n \to P_C x$ strongly. ☐

Except where otherwise stated, Ω denotes a σ-finite measure space.

4.1

Let $\alpha > 0$ and $\beta > 0$. Set

$$f(x) = \{1 + |x|^\alpha\}^{-1}\{1 + |\log|x||^\beta\}^{-1}, \quad x \in \mathbb{R}^N.$$

Under what conditions does f belong to $L^p(\mathbb{R}^N)$?

Solution

Observe that since $\frac{1}{1+|\log|x||^\beta} \leq 1$ and $\frac{1}{1+|x|^\alpha} \leq 1$ for all $\alpha, \beta > 0$, $f \in L^\infty(\mathbb{R}^N)$. Now, for $1 \leq p < \infty$, we perform a spherical change of coordinates to get that $\int_{\mathbb{R}^N} f(x)^p dx < \infty$ if and only if $\int_0^\infty \frac{r^{N-1}}{(1+r^\alpha)^p(1+|\log r|^\beta)^p} dr < \infty$, if and only if $\int_2^\infty \frac{r^{N-1}}{(1+r^\alpha)^p(1+|\log r|^\beta)^p} dr < \infty$. Observe that $\frac{r^{N-1}}{(1+r^\alpha)^p(1+|\log r|^\beta)^p} = \frac{1}{r^{\alpha p-N+1}|\log r|^{\beta p}(1+r^{-\alpha})^p(1+|\log r|^{-\beta})^p} \leq \frac{C}{r^{\alpha p-N+1}|\log r|^{\beta p}}$ for some constant $C > 0$, and it's therefore clear that $f \in L^p(\mathbb{R}^N)$ if and only if $\alpha > \frac{N}{p}$, or $\alpha = \frac{N}{p}$ and $\beta > \frac{1}{p}$.

4.3

1. Let $f, g \in L^p(\Omega)$ with $1 \leq p \leq \infty$. Prove that

$$h(x) = \max\{f(x), g(x)\} \in L^p(\Omega).$$

Proof. Observe that for all $x \in \Omega$, $|h(x)| \leq |f(x)|+|g(x)|$. Thus, if $p = 1$ then $\int_\Omega |h| d\mu \leq \int_\Omega |f|+|g| d\mu \leq \|f\|_1+\|g\|_1 < \infty$. If $p = \infty$, we have that $\{|h| > \|f\|_\infty + \|g\|_\infty\} \subset \{|f| > \|f\|_\infty\} \cup \{|g| > \|g\|_\infty\}$, which is a μ-null set. Finally if $1 < p < \infty$, we have that

$$\begin{aligned}\int_\Omega |h|^p d\mu &\leq \int_\Omega \frac{2^p}{2}|f|^p + \frac{2^p}{2}|g|^p d\mu \\ &\leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p) < \infty.\end{aligned}$$

Thus, in all cases, $h \in L^p(\Omega)$. ☐

2. Let (f_n) and (g_n) be two sequences in $L^p(\Omega)$ with $1 \leq p \leq \infty$ such that $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

Proof. We have

$$\|h - h_n\|_p = \|\frac{1}{2}(|f-g| + f + g) - \frac{1}{2}(|f_n - g_n| + f_n + g_n)\|_p$$
$$\leq \frac{1}{2}\||f-g| - |f_n - g_n|\|_p + \frac{1}{2}\|f - f_n\|_p + \frac{1}{2}\|g - g_n\|_p.$$

Thus, it suffices to prove that $|f_n - g_n| \to |f-g|$ in $L^p(\Omega)$. Since $||f-g| - |f_n - g_n|| \leq |(f-g) - (f_n - g_n)|$ over Ω, we get that $\||f-g| - |f_n-g_n|\|_p \leq \|(f-f_n) + (g_n - g)\|_p \leq \|f - f_n\|_p + \|g - g_n\|_p \to 0$ as $n \to \infty$, and the statement follows. □

3. Let (f_n) be a sequence in $L^p(\Omega)$ with $1 \leq p < \infty$ and let (g_n) be a bounded sequence in $L^\infty(\Omega)$. Assume $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

   Proof. Note that $\|fg - f_n g_n\|_p \leq \|f(g-g_n)\|_p + \|g_n(f-f_n)\|_p \leq \|f(g-g_n)\|_p + \sup_n \|g_n\|_\infty \|f - f_n\|_p$, and since $\sup_n \|g_n\|_\infty < \infty$ and $\|f - f_n\|_p \to 0$ as $n \to \infty$, it suffices to prove that $\|f(g-g_n)\|_p \to 0$ as $n \to \infty$. Because $g_n \to g$ a.e., it follows that $f(g-g_n) \to 0$ a.e. and, moreover, $|g| \leq \sup_n \|g_n\|$ a.e. so that $|f(g-g_n)|^p \leq (2\sup_n \|g_n\|_\infty)^p |f|^p$. Thus, we can apply the dominated convergence theorem to conclude that $\|f(g-g_n)\|_p \to 0$ as $n \to \infty$, and the statement follows. □

4.5

Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$.

1. Prove that $L^1(\Omega) \cap L^\infty(\Omega)$ is a dense subset of $L^p(\Omega)$.

   Proof. Note first that for any $f \in L^1(\Omega) \cap L^\infty(\Omega)$, $\int |f|^p d\mu = \int |f||f|^{p-1} d\mu \leq \|f\|_1 \|f\|_\infty^{p-1} < \infty$. Thus, $f \in L^p(\Omega)$, proving that $L^1(\Omega) \cap L^\infty(\Omega) \subset L^p(\Omega)$. Now fix $f \in L^p(\Omega)$, let (F_n) be a measurable sequence such that $\bigcup_{n=1}^\infty F_n = \Omega$ and $|F_n| < \infty$ for all n, and for each $n \geq 1$ let T_n be the truncation function on $\mathbb{R}$ defined in the proof of Theorem 4.12. Then for each $n \geq 1$, $T_n \circ (f\chi_{F_n})$ clearly belongs to $L^1(\Omega) \cap L^\infty(\Omega)$, $|T_n \circ (f\chi_{F_n}) - f|^p \to 0$ a.e. as $n \to \infty$ and $|T_n \circ (f\chi_{F_n}) - f|^p \leq |f|^p$. By the dominated convergence theorem, $T_n \circ (f\chi_{F_n}) \to f$ in $L^p(\Omega)$, which proves that $L^1(\Omega) \cap L^\infty(\Omega)$ is dense in $L^p(\Omega)$. □

2. Prove that the set
$$\left\{f \in L^p(\Omega) \cap L^q(\Omega) : \|f\|_q \leq 1\right\}$$
   is closed in $L^p(\Omega)$.

   Proof. Fix a sequence $(f_n) \subset \left\{f \in L^p(\Omega) \cap L^q(\Omega) : \|f\|_q \leq 1\right\}$ that converges to some point f in $L^p(\Omega)$. Since $f_n \to f$ in $L^p(\Omega)$, it follows that there exists some subsequence (f_{n_k}) that converges a.e. to f. Now, when $1 \leq q < \infty$, applying Fatou's lemma, we have that $\int |f|^q d\mu \leq \liminf_k \int |f_{n_k}|^q d\mu \leq 1$, which proves that $f \in L^q(\Omega)$ and $\|f\|_q \leq 1$. And when $q = \infty$, $f_{n_k} \to f$ a.e. and $\|f_{n_k}\|_\infty \leq 1$ for all k implies that $|f| \leq 1$ a.e. The statement follows. □

3. Let (f_n) be a sequence in $L^p(\Omega) \cap L^q(\Omega)$ and let $f \in L^p(\Omega)$. Assume that
$$f_n \to f \text{ in } L^p(\Omega) \text{ and } \|f_n\|_q \leq C.$$
   Prove that $f \in L^r(\Omega)$ and that $f_n \to f$ in $L^r(\Omega)$ for every r between p and q, $r \neq q$.

   Proof. The statement is trivial when $C = 0$, so we may assume WLOG that $C > 0$. From part 2 above, since $\frac{1}{C}f_n \to \frac{1}{C}f$ in $L^p(\Omega)$ and $\left\|\frac{1}{C}f_n\right\|_q \leq 1$, it follows that $\frac{1}{C}f \in L^q(\Omega)$ and $\left\|\frac{1}{C}f\right\|_q \leq 1$. Fix r between p and q with $r \neq q$. For convenience, assume $p \leq r < q$. Observe that for all $p \leq s \leq q$ $L^p(\Omega) \cap L^q(\Omega) \subset L^s(\Omega)$ since, when $q < \infty$, $\int_\Omega |f|^s d\mu = \int_{\{|f| \leq 1\}} |f|^s d\mu + \int_{\{|f|>1\}} |f|^s d\mu \leq \|f\|_p^p + \|f\|_q^q < \infty$, and if $q = \infty$ then $\int |f|^s d\mu = \int |f|^p |f|^{s-p} d\mu \leq \|f\|_\infty^{s-p} \|f\|_p^p < \infty$. Since $\frac{1}{q} < \frac{1}{r} \leq \frac{1}{p}$, there must exist some $t \in (0,1]$ such that $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$. Then since rt and $(1-t)r$ are both between p and q, we can apply the interpolation inequality to get that
$$\|f_n - f\|_r \leq \|f_n - f\|_p^t \|f_n - f\|_q^{1-t} \leq (2C)^{1-t} \|f_n - f\|_p \to 0,$$
   as $n \to \infty$. □

4.7

Let $1 \le q \le p \le \infty$. Let $a(x)$ be a measurable function on Ω. Assume that $au \in L^q(\Omega)$ for every function $u \in L^p(\Omega)$. Prove that $a \in L^r(\Omega)$ with

$$r = \begin{cases} \frac{pq}{p-q} & \text{if } p < \infty, \\ q & \text{if } p = \infty. \end{cases}$$

Proof. If $p = \infty$ then taking $u = \chi_\Omega \in L^\infty(\Omega)$, we have that $a = au \in L^q(\Omega) = L^r(\Omega)$. When $p < \infty$, define the map $T : L^p(\Omega) \to L^q(\Omega); u \mapsto au$. Note that T is linear. To see that T is a bounded linear operator, suppose that (u_n) is a convergent sequence in $L^p(\Omega)$ with limit u, and that $Tu_n \to f$ in $L^q(\Omega)$ as $n \to \infty$. Then there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e., and since $Tu_{n_k} \to f$ in $L^q(\Omega)$, there exists a subsequence $(u_{n_{k_l}})$ such that $u_{n_{k_l}} \to u$ a.e. and $Tu_{n_{k_l}} \to f$ a.e. For convenience, we shall write this subsequence as (u_l). Thus, we have that $au_l = Tu_l \to f$ a.e. and since $u_l \to u$ a.e., it follows that $au_l \to au$ a.e., so that $f = au = Tu$ a.e. This proves that the graph of T is closed and, thus, by the Closed Graph Theorem, T is a bounded linear operator. It follows that for all $u \in L^{\frac{p}{q}}(\Omega)$, since $|u|^{\frac{1}{q}} \in L^p(\Omega)$, $\int |a|^q |u| d\mu \le \|T\|^q \||u|^{1/q}\|_p^q = \|T\|^q \|u\|_{p/q}$, so that $\varphi : L^{\frac{p}{q}}(\Omega) \to \mathbb{R}; u \mapsto \int |a|^q u d\mu$ is a bounded linear functional. Observe that the conjugate of $\frac{p}{q}$ is $\frac{p/q}{p/q-1} = \frac{p}{p-q}$. By the Riesz Representation Theorem, there exists unique $f \in L^{\frac{p}{p-q}}(\Omega)$ such that $\int |a|^q u d\mu = \int f u d\mu$ for all $u \in L^{p/q}(\Omega)$. By the usual argument, we see that $\int_K \||a|^q - f| d\mu$ for any measurable subset K with finite measure, so that $|a|^q = f$ a.e. on Ω. Hence, $|a|^q \in L^{\frac{p}{p-q}}(\Omega)$, proving that $a \in L^r(\Omega)$. ☐

4.11(a) The spaces $L^\alpha(\Omega)$ with $0 < \alpha < 1$.

Let $0 < \alpha < 1$. Set

$$L^\alpha(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \quad u \text{ is measurable and } |u|^\alpha \in L^1(\Omega) \right\}$$

and

$$[u]_\alpha = \left(\int |u|^\alpha \right)^{1/\alpha}.$$

Check that L^α is a vector space but that $[\quad]_\alpha$ is not a norm. More precisely, prove that if $u, v \in L^\alpha(\Omega)$, $u \ge 0$ a.e. and $v \ge 0$ a.e., then

$$[u + v]_\alpha \ge [u]_\alpha + [v]_\alpha.$$

Proof. Fix $u, v \in L^\alpha$ and $\lambda \in \mathbb{R}$. Since u is measurable and $|u|^\alpha \in L^1(\Omega)$, λu is measurable and $|\lambda u|^\alpha = |\lambda|^\alpha |u|^\alpha \in L^1(\Omega)$, so that $\lambda u \in L^\alpha$. Moreover, $u+v$ is measurable, being the sum of measurable functions, and $|u+v|^\alpha \le 2^\alpha \max(|u|^\alpha, |v|^\alpha)$. Since $2^\alpha \max(|u|^\alpha, |v|^\alpha) \in L^1(\Omega)$ by Exercise 4.3 Part 1, it follows that $|u+v|^\alpha \in L^1(\Omega)$, so that $u + v \in L^\alpha$. Thus, L^α is a vector space. To see that $[\quad]_\alpha$ is not a norm, note that for all $u, v \in L^\alpha$ such that $u \ge 0$ a.e. and $v \ge 0$ a.e.,

$$\begin{aligned} [u]_\alpha + [v]_\alpha &= \int \left(\left(\int |u|^\alpha \right)^{1/\alpha - 1} |u|^\alpha + \left(\int |v|^\alpha \right)^{1/\alpha - 1} |v|^\alpha \right) \\ &\le \int \left([u]_\alpha + [v]_\alpha \right)^{1-\alpha} |u + v|^\alpha \\ &= \left([u]_\alpha + [v]_\alpha \right)^{1-\alpha} [u + v]_\alpha^\alpha. \end{aligned}$$

Rearranging, we have that $[u]_\alpha + [v]_\alpha \le [u+v]_\alpha$. Now taking any $u, v \in L^\alpha$ such that $u, v \ge 0$ a.e. and $[u]_\alpha + [v]_\alpha < [u+v]_\alpha$, we see that $[\quad]_\alpha$ cannot satisfy the triangle inequality, so it cannot be a norm. ☐

4.13(c)

Let (f_n) be a sequence in $L^1(\Omega)$ and let f be a function in $L^1(\Omega)$ such that

(i) $f_n(x) \to f(x)$ a.e.,

(ii) $\|f_n\|_1 \to \|f\|$.

Prove that $\|f_n - f\|_1 \to 0$.

Proof. Note that for all $n \geq 1$, $||f_n| - |f_n - f|| \leq |f|$ and $|f_n| - |f_n - f| \to |f|$ a.e. Thus, since $f \in L^1(\Omega)$, we can apply the Dominated Convergence Theorem to conclude that $\lim_{n\to\infty} \left(\|f\|_1 - \|f - f_n\|_1\right) = \lim_{n\to\infty} \int \left(|f_n| - |f - f_n|\right) = \|f\|_1$. Rearranging, we have that $\lim_{n\to\infty} \|f - f_n\|_1 = 0$. ☐

4.15

Let $\Omega = (0,1)$.

1. Consider the sequence (f_n) of functions defined by $f_n(x) = ne^{-nx}$. Prove that

   (i) $f_n \to 0$ a.e.

   Proof. Fix $x \in (0,1)$ and observe that by L'Hôpital's rule, $y \mapsto \frac{y}{e^{yx}}$ has limit equal to 0 as $y \to \infty$, proving that $f_n(x) = ne^{-nx} \to 0$ as $n \to \infty$. Thus, $f_n \to 0$ pointwise as $n \to \infty$. ☐

   (ii) f_n is bounded in $L^1(\Omega)$.

   Proof. $\|f_n\|_1 = \int_\Omega |f_n| = \int_0^1 ne^{-nx}dx = 1 - e^{-n} \leq 1$ for all $n \geq 1$. ☐

   (iii) $f_n \not\to 0$ in $L^1(\Omega)$ strongly.

   Proof. From above, we see that $\lim_{n\to\infty} \|f_n\|_1 = \lim_{n\to\infty} 1 - e^{-n} = 1 \neq \|0\|_1$. By the continuity of $\| \; \|_1$ on $L^1(\Omega)$, it follows that $f_n \not\to 0$ in $L^1(\Omega)$. ☐

   (iv) $f_n \not\rightharpoonup 0$ weakly $\sigma(L^1, L^\infty)$. More precisely, there is no subsequence that converges weakly $\sigma(L^1, L^\infty)$.

   Proof. Observe that for any $g \in C_c(\Omega)$, there exists $a < b \in (0,1)$ such that $|\int_0^1 gf_n| \leq \max_{x\in\Omega} |g(x)| \int_a^b |f_n| = \max_{x\in\Omega} |g(x)|(e^{-an} - e^{-bn}) \to 0$ as $n \to \infty$, so that $\langle g, f_n\rangle_{L^\infty, L^1} \to 0$ for all $g \in C_c(\Omega) \subset L^\infty(\Omega)$. Suppose for a contradiction that there exists some subsequence (f_{n_k}) and $f \in L^1(\Omega)$ such that $f_{n_k} \rightharpoonup f$ weakly. Then for all $g \in C_c(\Omega)$, $\int gf = \lim_{k\to\infty} \int gf_{n_k} = 0$. Applying Corollary 4.24, it follows that $f = 0$ a.e. on Ω. But then since $f_{n_k} \rightharpoonup f$ and $\chi_\Omega \in L^\infty(\Omega)$, we have that $0 = \int f = \int \chi_\Omega f = \lim_{k\to\infty} \int \chi_\Omega f_{n_k} = \lim_{k\to\infty} \|f_{n_k}\|_1 = 1$, a contradiction. Thus, no subsequence of (f_n) converges weakly. ☐

2. Let $1 < p < \infty$ and consider the sequence (g_n) of functions defined by $g_n(x) = n^{1/p}e^{-nx}$. Prove that

   (i) $g_n \to 0$ a.e.

   Proof. Observe that $0 \leq g_n \leq f_n$ on Ω. Since $f_n \to 0$ pointwise as $n \to \infty$, it follows that $g_n \to 0$ pointwise as $n \to \infty$. ☐

   (ii) (g_n) is bounded in $L^p(\Omega)$.

   Proof. $\|g_n\|_p^p = \int_0^1 ne^{-pnx}dx = \frac{1-e^{-pn}}{p} \leq 1$ for all $n \geq 1$. ☐

   (iii) $g_n \not\to 0$ in $L^p(\Omega)$ strongly.

   Proof. From above, we see that $\lim_{n\to\infty} \|g_n\|_p^p = \lim_{n\to\infty} \frac{1-e^{-pn}}{p} = \frac{1}{p} \neq \|0\|_p$. By the continuity of $\| \; \|_p$ on $L^p(\Omega)$, it follows that $f_n \not\to 0$ in $L^p(\Omega)$. ☐

   (iv) $g_n \rightharpoonup 0$ weakly $\sigma(L^p, L^{p'})$.

   Proof. Observe that for any $f \in C_c(\Omega)$, $|\int_0^1 fg_n| \leq \max_{x\in\Omega} |f(x)| \int_0^1 g_n = n^{1/p-1} \max_{x\in\Omega} |f(x)|(1 - e^{-n}) \to 0$ as $n \to \infty$, so that $\langle f, g_n\rangle_{L^{p'}, L^p} \to \langle f, 0\rangle_{L^{p'}, L^p}$ for all $f \in C_c(\Omega)$. Since $C_c(\Omega)$ is dense in $L^{p'}(\Omega)$, it follows that $g_n \rightharpoonup 0$ weakly $\sigma(L^p, L^{p'})$. ☐

4.18 Rademacher's functions.

Let $1 \le p \le \infty$ and let $f \in L^p_{\text{loc}}(\mathbb{R})$. Assume that f is T-periodic, i.e. $f(x+T) = f(x)$ a.e. $x \in \mathbb{R}$. Set

$$\overline{f} = \frac{1}{T}\int_0^T f(t)dt.$$

Consider the sequence (u_n) in $L^p(0,1)$ defined by

$$u_n(x) = f(nx), \quad x \in (0,1).$$

1. Prove that $u_n \rightharpoonup \overline{f}$ in $L^p(0,1)$ with respect to the topology $\sigma(L^p, L^{p'})$.

   Proof. Case $1 \le p < \infty$: Fix $g \in C_c^\infty(0,1)$ and observe that

$$\begin{aligned}
\int_0^1 gu_n &= \int_0^1 g(x)f(nx)dx \\
&= \int_0^1 g(x)f(\frac{n}{T}Tx)dx \\
&= \frac{1}{n}\int_0^{mT} g\Big(\frac{x}{n}\Big)f(x)dx + \frac{1}{n}\int_{mT}^n g\Big(\frac{x}{n}\Big)f(x)dx \\
&\approx \frac{1}{n}\sum_{k=0}^m g\Big(\frac{kT}{n}\Big)\int_0^T f(x)dx + \frac{1}{n}\int_{mT}^n g\Big(\frac{x}{n}\Big)f(x)dx \\
&= \frac{1}{m}\sum_{k=0}^m g\Big(\frac{kT}{n}\Big)\frac{1}{T+\frac{n-mT}{m}}\int_0^T f(x)dx + \frac{1}{n}\int_{mT}^n g\Big(\frac{x}{n}\Big)f(x)dx \\
&\to \int_0^1 g(x)\overline{f}dx,
\end{aligned}$$

   where m is the integer remainder of $\frac{T}{n}$, and the use of $\approx$ becomes exact in the limit by noting that if we take n large enough, since g is smooth, $g\Big(\frac{x}{n}\Big)$ will be equal to $g\Big(\frac{kT}{n}\Big) + o(n)$ on $\big[\frac{kT}{n}, \frac{(k+1)T}{n}\big]$, where $o(n)$ is a function that goes to 0 as $n \to \infty$. The limit in the final line is justified by observing that $\frac{n-mT}{m} \le \frac{T}{m} \to 0$ and $\frac{1}{n}\int_{mT}^n g\Big(\frac{x}{n}\Big)f(x)dx \le \frac{\max_{x\in(0,1)}|g(x)|}{n}\int_0^T |f| \to 0$, and noting that the sum on the left is just a Riemann sum, which is equal to the integral in the limit. Since $C_c^\infty(0,1)$ is dense in $L^p(0,1)$, the case $1 \le p < \infty$ follows.

   Observe that since we are taking the weak* limit for the case $p = \infty$, we can use the exact same argument, concluding by noting that $C_c^\infty(0,1)$ is dense in $L^1(0,1)$. □

2. Determine $\lim_{n\to\infty} \|u_n - \overline{f}\|_p$.

   Solution

   For $1 \le p < \infty$, we have

$$\begin{aligned}
\|u_n - \overline{f}\|_p^p &= \int_0^1 |f(nx) - \overline{f}|^p dx \\
&= \frac{1}{n}\int_0^n |f(x) - \overline{f}|^p dx \\
&= \frac{1}{n}\int_0^{mT} |f(x) - \overline{f}|^p dx + \frac{1}{n}\int_{mT}^n |f(x) - \overline{f}|^p dx \\
&= \frac{1}{T - \frac{n-mT}{m}}\int_0^T |f(x) - \overline{f}|^p dx + \frac{1}{n}\int_{mT}^n |f(x) - \overline{f}|^p dx \\
&\to \frac{1}{T}\int_0^T |f(x) - \overline{f}|^p dx,
\end{aligned}$$

where the limit follows from the fact that $\frac{1}{n}\int_{mT}^{n}|f(x)-\overline{f}|^p dx \le \frac{1}{n}\int_0^T |f(x)-\overline{f}|^p dx \to 0$. Thus,

$$\lim_{n\to\infty}\|u_n-\overline{f}\|_p = \Big(\frac{1}{T}\int_0^T |f(x)-\overline{f}|^p dx\Big)^{1/p}.$$

For $p=\infty$, clearly $\lim_{n\to\infty}\|u_n-\overline{f}\|_{L^\infty(0,1)} = \lim_{n\to\infty}\|f-\overline{f}\|_{L^\infty(0,n)} = \|f-\overline{f}\|_{L^\infty[0,T]}$.

3. Examine the following examples:

(i) $u_n(x)=\sin nx$

Solution

Observe that $T=2\pi$ and $\overline{f}=\frac{1}{2\pi}\int_0^{2\pi}\sin x\,dx = 0$. Thus, we conclude from our analysis above that $u_n=\sin nx \rightharpoonup 0$ in the topology $\sigma(L^p, L^{p'})$ on $L^p(0,1)$ for $1\le p\le\infty$. We also have that $\lim_{n\to\infty}\|\sin nx\|_p = \Big(\frac{1}{2\pi}\int_0^{2\pi}|\sin x|^p dx\Big)^{1/p}$ for $1\le p<\infty$ and $\lim_{n\to\infty}\|\sin nx\|_\infty = 1$.

(ii) $u_n(x)=f(nx)$ where f is 1-periodic and

$$f(x)=\begin{cases}\alpha & \text{for } x\in(0,1/2),\\ \beta & \text{for } x\in(1/2,1).\end{cases}$$

The functions of example (ii) are called *Rademacher's functions*.

Solution

We have $\overline{f}=\int_0^1 f(x)dx = \frac{1}{2}\alpha+\frac{1}{2}\beta$, so that $u_n \rightharpoonup \frac{1}{2}\alpha+\frac{1}{2}\beta$ in the topology $\sigma(L^p, L^{p'})$ on $L^p(0,1)$ for $1\le p\le\infty$. We also have that $\lim_{n\to\infty}\|u_n-\frac{1}{2}\alpha-\frac{1}{2}\beta\|_p = \Big(\int_0^1|f(x)-\frac{1}{2}\alpha-\frac{1}{2}\beta|^p dx\Big)^{1/p} = \frac{1}{2^{1/p}}|\alpha-\beta|$ for $1\le p<\infty$ and $\lim_{n\to\infty}\|u_n-\frac{1}{2}\alpha-\frac{1}{2}\beta\|_\infty = \frac{1}{2}|\alpha-\beta|$.

4.21

Given a function $u_0:\mathbb{R}\to\mathbb{R}$, set $u_n(x)=u_0(x+n)$.

1. Assume $u_0\in L^p(\mathbb{R})$ with $1<p<\infty$. Prove that $u_n\rightharpoonup 0$ in $L^p(\mathbb{R})$ with respect to the weak topology $\sigma(L^p, L^{p'})$.

Proof. Fix nonzero $g\in C_c(\mathbb{R})$. Note that there exists N_1 such that $g(x)=0$ for all $|x|>N_1$. Moreover, since $u_0\in L^p(\mathbb{R})$, for any $\varepsilon>0$, there exists N_2 such that $|u_0(x)|<\frac{\varepsilon}{2N_1\|g\|_\infty}$ for almost all $|x|\ge N_2$. Thus, we have that for all $n\ge N_1+N_2$

$$\begin{aligned}\Big|\int g u_n\Big| &\le \int_{-N_1}^{N_1}|g(x)u_0(x+n)|dx\\ &\le \|g\|_\infty\int_{-N_1}^{N_1}|u_0(x+n)|dx < \varepsilon.\end{aligned}$$

Thus, $\lim_{n\to\infty}\int g u_n = 0$ for all $g\in C_c(\mathbb{R})$, and since $C_c(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$, it follows that $u_n\rightharpoonup 0$ weakly in $\sigma(L^p, L^{p'})$. ☐

2. Assume $u_0\in L^\infty(\mathbb{R})$ and that $u_0(x)\to 0$ as $|x|\to\infty$ in the following weak sense:

for every $\delta>0$ the set $[|u_0|>\delta]$ has finite measure.

Prove that $u_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\mathbb{R})$ weak* $\sigma(L^\infty, L^1)$.

Proof. Fix nonzero $g \in C_c(\mathbb{R})$ and $\varepsilon > 0$. Again, let N_1 be such that $g(x) = 0$ for all $|x| > N_1$. By assumption $\left[|u_0| > \frac{\varepsilon}{2N_1\|g\|_\infty}\right]$ has finite measure. In particular, there must exist a finite interval $(-N_2, N_2)$ such that $\left|\left[|u_0| > \frac{\varepsilon}{2N_1\|g\|_\infty}\right] \setminus (-N_2, N_2)\right| < \frac{\varepsilon}{\|g\|_\infty \|u_0\|_\infty}$ (obviously we may assume WLOG that $\|u_0\|_\infty > 0$). Thus, we have that for all $n \geq N_1 + N_2$

$$\begin{aligned}\left|\int g u_n\right| &\leq \int_{-N_1}^{N_1} |g(x)u_0(x+n)|dx \\ &\leq \int_{[-N_1,N_1]\cap[u_0>\varepsilon/(2N_1\|g\|_\infty)]} |g(x)u_0(x+n)|dx + \varepsilon \\ &\leq \|g\|_\infty \|u_0\|_\infty \left|\left[|u_0| > \frac{\varepsilon}{2N_1\|g\|_\infty}\right] \setminus (-N_2, N_2)\right| + \varepsilon < 2\varepsilon.\end{aligned}$$

It follows that $\lim_{n\to\infty} \int g u_n = 0$ for all $g \in C_c(\mathbb{R})$, and since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, this proves that $u_n \overset{*}{\rightharpoonup} 0$ in $\sigma(L^\infty, L^1)$. □

3. Take $u_0 = \chi_{(0,1)}$. Prove that there exists no subsequence (u_{n_k}) that converges in $L^1(\mathbb{R})$ with respect to $\sigma(L^1, L^\infty)$.

   Proof. Towards a contradiction, suppose that there exists some $u \in L^1(\mathbb{R})$ and a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ in $\sigma(L^1, L^\infty)$. Then we must have that $0 = \lim_{k\to\infty} \int \chi_{\mathbb{R}}(u_{n_k} - u) = 1 - \int u$, so that $\int u = 1$. Thus, there must exist some finite interval $(a,b) \subset \mathbb{R}$ such that $\int \chi_{(a,b)} u = \int_a^b u > \frac{1}{2}$. Since $u_{n_k} \rightharpoonup u$, it follows that $\lim_{k\to\infty} \int_a^b \chi_{(-n_k, 1-n_k)} = \lim_{k\to\infty} \int \chi_{(a,b)} u_{n_k} = \int \chi_{(a,b)} u > \frac{1}{2}$, which is absurd since (a,b) is a finite interval. By contradiction, it follows that there exists no subsequence (u_{n_k}) that converges in $L^1(\mathbb{R})$ with respect to $\sigma(L^1, L^\infty)$. □

4.23

Let $f : \Omega \to \mathbb{R}$ be a measurable function and let $1 \leq p \leq \infty$. The purpose of this exercise is to show that the set

$$C = \left\{u \in L^p(\Omega) : \quad u \geq f \quad \text{a.e.}\right\}$$

is closed in $L^p(\Omega)$ with respect to the topology $\sigma(L^p, L^{p'})$.

1. Assume first that $1 \leq p < \infty$. Prove that C is convex and closed in the strong L^p topology. Deduce that C is closed in $\sigma(L^p, L^{p'})$.

   Proof. First, to see that C is convex, observe that for any $u_1, u_2 \in C$ and $t \in (0,1)$, $tu_1 + (1-t)u_2 \geq tf + (1-t)f = f$ a.e. so that $tu_1 + (1-t)u_2 \in C$. Now suppose that $(u_n) \subset C$ is a sequence such that $u_n \to u$ in $L^p(\Omega)$ for some $u \in L^p(\Omega)$. Then there exists a subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e., and since $u_{n_k} \geq f$ a.e. for all $k \geq 1$, it follows that $u \in C$. Thus, C is convex and strongly closed in L^p. It follows by Theorem 3.7 that C is weakly closed in $\sigma(L^p, L^{p'})$. □

2. Taking $p = \infty$, prove that

$$C = \left\{u \in L^\infty(\Omega) : \int u\varphi \geq \int f\varphi \quad \forall \varphi \in L^1(\Omega) \text{ with } f\varphi \in L^1(\Omega) \text{ and } \varphi \geq 0 \text{ a.e.}\right\}.$$

   Proof. Clearly if $u \in C$ then for all $\varphi \in L^1(\Omega)$ with $f\varphi \in L^1(\Omega)$ and $\varphi \geq 0$ a.e., since $u \geq f$ a.e., it follows that $u\varphi \geq f\varphi$ a.e. so that $\int u\varphi \geq \int f\varphi$. Thus, one direction is clear. Towards proving the other direction, suppose that $u \in L^\infty(\Omega)$ has the property that for all $\varphi \in L^1(\Omega)$ such that $f\varphi \in L^1(\Omega)$ and $\varphi \geq 0$ a.e., $\int u\varphi \geq \int f\varphi$. Consider first the case where $f \in L^\infty(\Omega)$. Then for all measurable subsets $F \subset \Omega$ with finite measure, we have that $\int_F (u - f) = \int \chi_F(u - f) \geq 0$. Thus, applying the fact that Ω is σ-finite, pick an increasing sequence of subsets (F_n) of finite measure such that $\bigcup_n F_n = \Omega$ and observe that since for all n and $k \geq 1$ $\int_{F_n \cap [u-f<-\frac{1}{k}]} (u - f) \geq 0$, it follows that for all $k \geq 1$, $[u - f < -\frac{1}{k}] = \lim_{n\to\infty} F_n \cap [u - f < -\frac{1}{k}] = 0$. Thus, $u \geq f$ proving that $u \in C$, and the statement follows in the case where $f \in L^\infty(\Omega)$.

If $f \notin L^\infty(\Omega)$, for each $n \geq 1$ define $\omega_n = [|f| \leq n]$. Clearly $\Omega = \bigcup_n \omega_n$. Since for any n, $f|_{\omega_n} \in L^\infty(\omega_n)$ and $u|_{\omega_n} \in L^\infty(\omega_n)$ and for any $\varphi \in L^1(\omega_n)$ with $\varphi \geq 0$ a.e., we have that $\int_{\omega_n} u|_{\omega_n}\varphi \geq \int_{\omega_n} f|_{\omega_n}\varphi$, the situation above applies and we can conclude that $u \geq f$ a.e. on ω_n for all n. It follows that $u \geq f$ a.e. on Ω so that $u \in C$, and the statement follows. ☐

3. Deduce that when $p = \infty$, C is closed in $\sigma(L^\infty, L^1)$.

   Proof. Observe that since $L^1(\Omega)$ is separable, $B_{L^\infty(\Omega)}$ is metrizable with respect to the weak* topology $\sigma(L^\infty, L^1)$. Thus, since C is convex, by the Krein-Šmulian Theorem, to prove that C is closed in $\sigma(L^\infty, L^1)$, it suffices to prove that every bounded weak* convergent sequence in C converges in $\sigma(L^\infty, L^1)$ to some point in C with the same bound. To this end, fix a sequence $(u_n) \subset C \cap nB_{L^\infty(\Omega)}$ and suppose that there exists some $u \in L^\infty(\Omega)$ such that $u_n \overset{*}{\rightharpoonup} u$ in $\sigma(L^\infty, L^1)$. Then $\|u\|_\infty \leq \liminf_n \|u_n\|_\infty \leq n$ and for any $\varphi \in L^1(\Omega)$ with $f\varphi \in L^1(\Omega)$ and $\varphi \geq 0$ a.e., we have that $\int u\varphi = \lim_{n\to\infty} \int u_n\varphi \geq \int f\varphi$, proving that $u \in C \cap nB_{L^\infty(\Omega)}$. It follows that C is closed in $\sigma(L^\infty, L^1)$. ☐

4. Let $f_1, f_2 \in L^\infty(\Omega)$ with $f_1 \leq f_2$ a.e. Prove that the set

$$C = \left\{u \in L^\infty(\Omega) : \quad f_1 \leq u \leq f_2 \quad \text{a.e.}\right\}$$

   is compact in $L^\infty(\Omega)$ with respect to the topology $\sigma(L^\infty, L^1)$.

   Proof. To see that C is closed in $\sigma(L^\infty, L^1)$, observe that $C = \{u \in L^\infty(\Omega) : u \geq f_1 \quad \text{a.e.}\} \cap \{-u \in L^\infty(\Omega) : u \geq -f_2 \quad \text{a.e.}\}$. From part 3 above and the fact that $u \mapsto -u$ is continuous on $L^\infty(\Omega)$, it follows that C is the intersection of weak* closed subsets, so is closed in $\sigma(L^\infty, L^1)$. Moreover, note that for any $u \in C$, $|u| \leq \max(|f_1|, |f_2|)$ a.e. so that $\|u\|_\infty \leq \|f_1\|_\infty + \|f_2\|_\infty$, proving that C is a $\sigma(L^\infty, L^1)$ closed bounded subset of $L^\infty(\Omega)$. Thus, by the Banach-Alaoglu Theorem, C is compact in $\sigma(L^\infty, L^1)$. ☐

4.25 Regularization of functions in $L^\infty(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be open.

1. Let $u \in L^\infty(\Omega)$. Prove that there exists a sequence (u_n) in $C_c^\infty(\Omega)$ such that

   (a) $\|u_n\|_\infty \leq \|u\|_\infty \quad \forall n$,

   (b) $u_n \to u$ a.e. on Ω,

   (c) $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ weak* $\sigma(L^\infty, L^1)$.

   Proof. Extend u to a function $\overline{u} \in L^\infty(\mathbb{R})$ by defining $\overline{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$ For each n define

$$K_n = \left\{x \in \Omega : \operatorname{dist}(x, \Omega^c) \geq \frac{2}{n} \quad \text{and} \quad |x| \leq n\right\},$$

   so that $\bigcup_{n=1}^\infty K_n = \Omega$ and each K_n is a compact subset of $\mathbb{R}^N$. Set $g_n = \chi_{K_n}\overline{u}$ and $\overline{u}_n = \rho_n \star g_n$, where (ρ_n) is a sequence of mollifiers. Observe that by Proposition 4.18, $\operatorname{supp} \overline{u}_n \subset \overline{\operatorname{supp} g_n + \operatorname{supp} \rho_n} \subset K_n + \overline{B(0, 1/n)}$ and since each $g_n \in L^1_{\text{loc}}(\mathbb{R})$, applying Proposition 4.20, we have that each $\overline{u}_n \in C_c^\infty(\mathbb{R})$. Moreover, observe that $K_n + \overline{B(0, 1/n)} \subset \Omega$ so that for each n, $u_n := \overline{u}_n|_\Omega \in C_c^\infty(\Omega)$. Note that for any $x \in \Omega$, $|u_n(x)| = |\int_{K_n} \rho_n(x -$

$y)u(y)dy| \le \|u\|_\infty \int_{K_n} \rho_n \le \|u\|_\infty$, which verifies that $\|u_n\|_\infty \le \|u\|_\infty$ for all n. Moreover, for any $x \in \Omega$, we have that for all n large enough,

$$\begin{aligned}|u_n(x) - u(x)| &\le \int_{B(0,1/n)} |u(x-y) - u(y)|\rho_n(y)dy \\ &\le \|\rho_n\|_\infty \int_{B(0,1/n)} |u(x-y) - u(y)|dy \\ &= n^N \int_{B(0,1/n)} |u(x-y) - u(y)|dy \to 0 \quad \text{a.e.}\end{aligned}$$

by Lebesgue's Differentiation Theorem. Thus, $u_n \to u$ a.e. on Ω. Finally, toward proving that $u_n \overset{*}{\rightharpoonup} u$, fix $\varphi \in C_c^\infty(\Omega)$ and note that

$$\begin{aligned}\int_\Omega u_n \varphi &= \int_{K_n} (\rho_n \star u)\varphi \\ &= \int_{K_n} u(\check{\rho}_n \star \varphi) \\ &= \int_\Omega u(\check{\rho}_n \star \varphi) - \int_{\Omega \setminus K_n} u(\check{\rho}_n \star \varphi) \\ &\xrightarrow[n\to\infty]{} \int_\Omega u\varphi,\end{aligned}$$

where $\check{\rho}_n(x) = \rho_n(-x)$, and the final line above is justified by observing that since $\check{\rho}_n$ is again a sequence of mollifiers, by Theorem 4.22 $\check{\rho}_n \star \varphi \to \varphi$ in $L^1(\Omega)$ as $n \to \infty$. Thus, $\int_\Omega u(\check{\rho}_n \star \varphi) \to \int_\Omega u\varphi$ as $n \to \infty$, and the second integral can be bounded by $\left|\int_{\Omega \setminus K_n} u(\check{\rho}_n \star \varphi)\right| \le \|u\|_\infty \int_{\Omega\setminus K_n} \varphi \to 0$ as $n \to \infty$ since φ is compactly supported on Ω. Since $C_c^\infty(\Omega)$ is dense in $L^1(\Omega)$ by Theorem 4.23, it follows that $u_n \overset{*}{\rightharpoonup} u$ in $\sigma(L^\infty, L^1)$. □

2. If $u \ge 0$ a.e. on Ω, show that one can also take

   (d) $u_n \ge 0$ on $\Omega \quad \forall n$.

   Proof. If $u \ge 0$ a.e. on Ω, then for each $n \in \mathbb{N}$ and $x \in \Omega$, we have that $u_n(x) = \int_{K_n} u(x-y)\rho_n(y)dy \ge 0$ since $u(x-y)\rho_n(y) \ge 0$ a.e. on K_n. □

3. Deduce that $C_c^\infty(\Omega)$ is dense in $L^\infty(\Omega)$ with respect to the topology $\sigma(L^\infty, L^1)$.

   Proof. Fix $u \in L^\infty(\Omega)$ and a weak* open neighborhood $V \subset L^\infty(\Omega)$ of u. By part 1 above, there exists a sequence $(u_n) \subset C_c^\infty(\Omega)$ such that $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ weak* $\sigma(L^\infty, L^1)$. Thus, there must exist some N such that $u_n \in V$ for all $n \ge N$. It follows that every nonempty weak* open neighborhood $V \subset L^\infty(\Omega)$ contains a point in $C_c^\infty(\Omega)$, and so $C_c^\infty(\Omega)$ is dense in $L^\infty(\Omega)$ with respect to $\sigma(L^\infty, L^1)$. □

4.33

Fix a function $\varphi \in C_c(\mathbb{R})$, $\varphi \not\equiv 0$, and consider the family of functions

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \{\varphi_n\},$$

where $\varphi_n(x) = \varphi(x+n)$, $x \in \mathbb{R}$.

1. Assume $1 \le p < \infty$. Prove that $\forall \varepsilon > 0\, \exists \delta > 0$ such that

$$\|\tau_h f - f\|_p < \varepsilon \quad \forall f \in \mathcal{F} \text{ and } \forall h \in \mathbb{R} \text{ with } |h| < \delta.$$

Proof. Fix $\varepsilon > 0$. Since $\varphi \in C_c(\mathbb{R})$, φ is uniformly continuous on $\mathbb{R}$ and so there exists some $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{|K|^{1/p}}$ for all $|x-y| < \delta$, where $K \subset \mathbb{R}$ is a compact subset containing supp $\varphi + B(0,\delta)$. Thus, for all $f \in \mathcal{F}$, there exists some $n \in \mathbb{N}$ such that $f = \varphi_n$, and so for all $|h| < \delta$

$$\begin{aligned}\|\tau_h f - f\|_p^p &= \int_{\mathbb{R}} |\varphi(x+n+h) - \varphi(x+n)|^p dx \\ &= \int_K |\varphi(x+h) - \varphi(x)|^p dx \\ &< \varepsilon^p.\end{aligned}$$

The statement follows. □

2. Prove that $\mathcal{F}$ does *not* have compact closure in $L^p(\mathbb{R})$.

Proof. Let $N \in \mathbb{N}$ be such that $\varphi(x) = 0$ for all $|x| > N$. Consider the sequence $(\varphi_{2kN})_{k\geq 1} \subset \mathcal{F}$. Observe that for any $j \neq k$,

$$\begin{aligned}\|\varphi_{2kN} - \varphi_{2jN}\|_p^p &= \int_{\mathbb{R}} |\varphi(x+2kN) - \varphi(x+2jN)|^p dx \\ &= \int_{-(2k+1)N}^{-(2k-1)N} |\varphi(x+2kN)|^p dx + \int_{-(2j+1)N}^{-(2j-1)N} |\varphi(x+2jN)|^p dx \\ &= 2\|\varphi\|_p^p,\end{aligned}$$

and so (φ_{2kN}) has no convergent subsequence. Since a subset of a metric space is compact if and only if it is sequentially compact, it follows that the closure of $\mathcal{F}$ is not compact in $L^p(\mathbb{R})$. □

In what follows, H will always denote a Hilbert space equipped with the scalar product $(\ ,\ )$ and the corresponding norm $|\ \ |$.

5.2 L^p is not a Hilbert space for $p \neq 2$.

Let Ω be a measure space and assume that there exists a measurable set $A \subset \Omega$ such that $0 < |A| < |\Omega|$. Prove that the $\|\ \|_p$ norm does not satisfy the parallelogram law for any $1 \leq p \leq \infty$, $p \neq 2$.

Proof. Clearly we're going to need more than just the assumption that there exists $0 < |A| < |\Omega|$. For example, if $\Omega = \{1,2\}$, $|\{1\}| = 1$ and $|\{2\}| = \infty$, then we can pick $A = \{1\}$ and $0 < |A| < |\Omega|$ is satisfied and it's obvious that for any $1 \leq p \leq \infty$, $L^p(\Omega) = \{f : f(2) = 0\}$, so that $\|f\|_p = |f(1)|$ for all $f \in L^p(\Omega)$. But then for any $f, g \in L^p(\Omega)$, $\left\|\frac{f-g}{2}\right\|_p^2 + \left\|\frac{f+g}{2}\right\|_p^2 = \frac{|f(1)-g(1)|^2}{4} + \frac{|f(1)+g(1)|^2}{4} = \frac{1}{2}\Big(|f(1)|^2 + |g(1)|^2\Big) = \frac{1}{2}\Big(\|f\|_p^2 + \|g\|_p^2\Big)$. I think the most general condition required to prove the result is this: In addition to some measurable A with $0 < |A| < |\Omega|$, we also have measurable $B \subset \Omega$ with $A \cap B = \varnothing$ and $0 < |B| < |\Omega|$. Then observe that for any $p \neq 2$ with $1 \leq p < \infty$, we have that for any $x > 0$

$$\left\|\frac{x\chi_A - \frac{1}{|B|^{1/p}}\chi_B}{2}\right\|_p^2 + \left\|\frac{x\chi_A + \frac{1}{|B|^{1/p}}\chi_B}{2}\right\|_p^2 = \frac{1}{2}\Big(x^p|A| + 1\Big)^{2/p} \tag{1}$$

and

$$\frac{1}{2}\|x\chi_1\|_p^2 + \frac{1}{2}\left\|\frac{1}{|B|^{1/p}}\chi_2\right\|_p^2 = \frac{1}{2}x^2|A|^{2/p} + \frac{1}{2}. \tag{2}$$

Differentiating the RHS of (1) with respect to x, we get $x^{p-1}|A|(x^p|A|+1)^{(2-p)/p}$ and then differentiating the RHS of (2) with respect to x, we get $x|A|^{2/p}$. Now picking $x = \frac{1}{|A|^{1/p}}$, we see that the derivative of (1) simplifies to $2^{(2-p)/p}|A|^{1/p}$ and the derivative of (2) simplifies to $|A|^{1/p}$. Since these two terms are equal if and only if $p = 2$, it follows that (1) and (2) cannot be equal for all $x > 0$, proving that the parallelogram law does not hold for $p \neq 2$ with $1 \leq p < \infty$.

For $p = \infty$, observe that $\left\|\frac{2\chi_A - \chi_B}{2}\right\|_\infty^2 + \left\|\frac{2\chi_A + \chi_B}{2}\right\|_\infty^2 = 2 \neq \frac{5}{2} = \frac{1}{2}\|2\chi_A\|_\infty^2 + \frac{1}{2}\|\chi_B\|_\infty^2$. □

5.3

Let (u_n) be a sequence in H and let (t_n) be a sequence in $(0,\infty)$ such that

$$(t_n u_n - t_m u_m,\, u_n - u_m) \le 0 \quad \forall m,n.$$

1. Assume that the sequence (t_n) is *nondecreasing* (possibly unbounded). Prove that the sequence (u_n) converges.

   Proof. Observe that

$$\begin{aligned}
2(t_n u_n - t_m u_m,\, u_n - u_m) &= 2t_n|u_n|^2 - 2(t_n + t_m)(u_n, u_m) + 2t_m|u_m|^2 \\
&= 2t_n|u_n|^2 - t_m|u_n|^2 + t_m|u_n|^2 - t_n|u_m|^2 + t_n|u_m|^2 + 2t_m|u_m|^2 - 2(t_n + t_n)(u_n, u_m) \\
&= (t_n + t_m)(|u_n|^2 + |u_m|^2 - 2(u_n, u_m)) + (t_n - t_m)(|u_n|^2 - |u_m|^2) \\
&= (t_n + t_m)|u_n - u_m|^2 + (t_n - t_m)(|u_n|^2 - |u_m|^2).
\end{aligned}$$

   Thus, for all $m \le n$ we have $(t_m + t_n)|u_n - u_m|^2 \le (t_m - t_n)(|u_n|^2 - |u_m|^2)$. Since (t_n) is nondecreasing, it follows that if $u_n \ne u_m$, then $t_m < t_n$. Since the LHS of the inequality is strictly positive, this forces that $|u_n|^2 - |u_m|^2 < 0$, so that $|u_n| < |u_m|$. Thus, $(|u_n|)$ is a nonincreasing sequence in $\mathbb{R}$, bounded below by 0, and therefore converges to a limit. Finally, observe that for all n,m we have that

$$|u_n - u_m|^2 \le \frac{t_n - t_m}{t_n + t_m}(|u_m|^2 - |u_n|^2) \le |u_m|^2 - |u_n|^2,$$

   and since $(|u_n|^2)$ is a Cauchy sequence in $\mathbb{R}$, it follows that (u_n) is a Cauchy sequence in H and therefore converges. □

2. Assume that the sequence (t_n) is *nonincreasing*. Prove that the following alternative holds:

   (i) either $|u_n| \to \infty$,

   (ii) or (u_n) converges.

   If $t_n \to t > 0$, prove that (u_n) converges, and if $t_n \to 0$, prove that both cases (i) and (ii) may occur.

   Proof. Observe that for all $m \le n$

$$0 \le |u_n - u_m|^2 \le \frac{t_m - t_n}{t_m + t_n}(|u_n|^2 - |u_m|^2) \le |u_n|^2 - |u_m|^2,$$

   and it follows that $(|u_n|)$ is a nondecreasing sequence. Thus, either $(|u_n|)$ has some finite limit and is therefore a Cauchy sequence in $\mathbb{R}$, which forces (u_n) to be a Cauchy sequence in H by the same inequality, and therefore converge, or $(|u_n|)$ diverges to infinity. Thus, the first part of the question is proven. Towards proving the second part, assume first that $t_n \to t > 0$. Then observe that $(h_n) := (\frac{1}{t_n})$ is a nondecreasing sequence and we have that

$$\begin{aligned}
0 &\ge (t_n u_n - t_m u_m, u_n - u_m) \\
&= ((t_n u_n) - (t_m u_m),\, h_n(t_n u_n) - h_m(t_m u_m)) \\
&= (h_n v_n - h_m v_m,\, v_n - v_m),
\end{aligned}$$

   for all n,m where $v_n := t_n u_n$. It follows by part 1 above that (v_n) converges to some limit $v \in H$, and since for all n we have that $t|u_n| \le t_n|u_n| = |v_n| \to |v|$, it follows that $(|u_n|)$ is bounded, proving from our reasoning above that (u_n) converges. When $t_n \to 0$, observe that the constant sequence $(u_n) = (u)$ always converges and obviously $(t_n u - t_m u, u - u) \le 0$, so case (ii) can definitely occur. Moreover, for $u \ne 0$, the sequence $(u_n) = (h_n u)$ obviously has the property that $|u_n| \to \infty$ since $h_n = \frac{1}{t_n} \to \infty$, and we have that $(t_n u_n - t_m u_m,\, u_n - u_m) = (u - u,\, h_n u - h_m u) \le 0$. Thus, case (i) is also possible. □

5.5

1. Let (K_n) be a *nonincreasing* sequence of closed convex sets in H such that $\cap_n K_n \neq \varnothing$. Prove that for every $f \in H$ the sequence $u_n = P_{K_n} f$ converges (strongly) to a limit and identify the limit.

   Proof. Fix $f \in H$ and for each $n \geq 1$, set $u_n = P_{K_n} f$. By assumption $\cap_n K_n$ is nonempty and closed, being the intersection of closed sets. Moreover, since for any $u, v \in \cap_n K_n$ and $t \in (0,1)$, $tu + (1-t)v \in K_n$ $\forall n$ by the convexity of each K_n, so that $tu + (1-t)v \in \cap_n K_n$, it follows that $\cap_n K_n$ is a nonempty, closed and convex subset of K_m for all m. Define $u = P_{\cap_n K_n} f$. I claim that $u_n \to u$. Since $K_m \supset K_n \supset \cap_j K_j$ for all $m \leq n$, from the definition of the projection, it follows that $|f - u_m| \leq |f - u_n| \leq |f - u|$. Thus, $(|f - u_n|)$ is a upper-bounded, nondecreasing sequence in $\mathbb{R}$ and therefore converges to some limit. Observe that for any $m \leq n$, by the convexity of $K_m \supset K_n$ and the definition of the projection, we have that $|f - u_m| \leq \left| f - \frac{u_m + u_n}{2} \right|$. It follows by the parallelogram law that for all $m \leq n$,

$$\frac{1}{4}|u_n - u_m|^2 + |f - u_m|^2 \leq \left| f - \frac{u_n + u_m}{2} \right|^2 + \left| \frac{u_n - u_m}{2} \right|^2$$
$$= \frac{1}{2}|f - u_n|^2 + \frac{1}{2}|f - u_m|^2.$$

   Thus, for all $m \leq n$, we have that $|u_n - u_m|^2 \leq 2\big(|f - u_n|^2 - |f - u_m|^2\big)$. Taking the lim sup with respect to $n \geq m$, and then with respect to $m \geq 1$, it follows that (u_n) is a Cauchy sequence and therefore converges to some point u'. Since (u_n) is eventually entirely contained in K_m for each m, and each K_m is closed, it follows that $u' \in \cap_n K_n$. But because $|f - u'| = \lim_{n\to\infty} |f - u_n| \leq |f - u| = \min_{v \in \cap_n K_n} |f - v|$, and u is the unique element of $\cap_n K_n$ that minimizes the distance to f, it follows that $\lim_{n\to\infty} u_n = u = P_{\cap_n K_n} f$, as claimed. □

2. Let (K_n) be a *nondecreasing* sequence of nonempty closed sets in H. Prove that for every $f \in H$ the sequence $u_n = P_{K_n} f$ converges (strongly) to a limit and identify the limit.

   Proof. Fix $f \in H$ and for each n set $u_n = P_{K_n} f$. Since for all $m \leq n$, we have that $K_m \subset K_n$, it follows by the definition of the projection function that $|f - u_n| \leq |f - u_m|$, and so $(|f - u_n|)$ is a lower-bounded, nonincreasing sequence in $\mathbb{R}$ and therefore converges to some limit. Observe that by the convexity of each K_n and the fact that $K_m \subset K_n$ for each $m \leq n$, applying the definition of the projection we have that $|f - u_n| \leq \left| f - \frac{u_n + u_m}{2} \right|$. By the same parallelogram law argument as above, we therefore have that for all $m \leq n$, $|u_n - u_m|^2 \leq 2\big(|f - u_m|^2 - |f - u_n|^2\big)$. Taking the lim sup with respect to $n \geq m$ and then with respect to m, we see that (u_n) is a Cauchy sequence and therefore converges to some limit $u \in H$. Define $F = \overline{\bigcup_n K_n}$. Since $K_n \subset K_{n+1} \subset \cdots$, it's clear that $\bigcup_n K_n$ is convex by virtue of each K_n being convex, and since the closure of a convex subset is convex, it follows that F is a nonempty, closed convex set. I claim that $u = P_F f$. Indeed, observe that for all m, $u_m \in K_m \subset \bigcup_n K_n \subset F$, and it follows that $u = \lim_{n\to\infty} u_n \in F$. Moreover, we have that for any $v \in \bigcup_n K_n$, there exists some N such that $v \in K_N$ so that $|f - u| \leq |f - u_N| \leq |f - v|$. Since $\bigcup_n K_n$ is dense in F, the continuity of $|\ \ |$ implies that $|f - u| \leq |f - v|$ for all $v \in F$, proving that $u = P_F f$ as claimed.

   □

   Let $\varphi : H \to \mathbb{R}$ be a continuous function that is bounded from below. Prove that the sequence $\alpha_n = \inf_{K_n} \varphi$ converges and identify the limit.

   Proof. Let C be a finite lower bound for φ. Observe that for any $m \leq n$ and $u \in K_m \subset K_n$, we have $C \leq \alpha_n = \inf_{K_n} \varphi \leq \varphi(u)$. Taking the inf over all $u \in K_m$, we have that $C \leq \alpha_n \leq \alpha_m$. Thus, (α_n) is a lower-bounded, nonincreasing sequence in $\mathbb{R}$ and therefore converges to some limit. Let F be as defined above. I claim that $\lim_{n\to\infty} \alpha_n = \inf_F \varphi$. Since $K_n \subset F$ for each n, it's clear by the same argument as before that $\inf_F \varphi \leq \inf_{K_n} \varphi = \alpha_n \quad \forall n$. On the other hand, for any $v \in \bigcup_n K_n$, there must exist some N such that $v \in K_N$ and so $\lim_{n\to\infty} \alpha_n \leq \alpha_N = \inf_{K_N} \varphi \leq \varphi(v)$. Since $\bigcup_n K_n$ is dense in F, applying the continuity of φ, it follows that $\lim_{n\to\infty} \alpha_n \leq \varphi(v)$ for all $v \in F$, and so $\lim_{n\to\infty} \alpha_n \leq \inf_F \varphi$. The claim follows. □

5.8

Let Ω be a measure space and let $h : \Omega \to [0, +\infty)$ be a measurable function. Let

$$K = \{u \in L^2(\Omega) : \quad |u(x)| \le h(x) \text{ a.e. on } \Omega\}.$$

Check that K is a nonempty closed convex set in $H = L^2(\Omega)$. Determine P_K.

Proof. Clearly we have that $0(x) = 0 \le h(x) \quad \forall x \in \Omega$, and since $0 \in L^2(\Omega)$, it follows that $0 \in K$, confirming that K is nonempty. Moreover, for any $u_1, u_2 \in K$ and $t \in (0,1)$, we have that $t|u_1| \le th$ a.e. on Ω and $(1-t)|u_2| \le (1-t)h$ a.e. on Ω. Thus, $|tu_1 + (1-t)u_2| \le t|u_1| + (1-t)|u_2| \le th + (1-t)h = h$ a.e. on Ω, which confirms that K is convex. Now towards proving that K is closed in $L^2(\Omega)$, suppose that a sequence $(u_n) \subset K$ converges (strongly in $L^2(\Omega)$) to some point $u \in L^2(\Omega)$. Then there exists some subsequence (u_{n_k}) such that $u_{n_k} \to u$ a.e. on Ω. Let $N_u \subset \Omega$ be the null set where $u_{n_k} \not\to u$, and for each k, let $N_k \subset \Omega$ be the null set where $|u_{n_k}| > h$. Then $N_u \cup (\bigcup_k N_k)$ is a null set and for all $x \notin N_u \cup (\bigcup_k N_k)$, we have that $|u(x)| = \lim_{k\to\infty} |u_{n_k}(x)| \le h(x)$, proving that $u \in K$ and therefore that K is a nonempty closed convex subset of $L^2(\Omega)$.

Observe that for any $f \in L^2(\Omega)$, $P_K f$ is the unique element $u \in K$ satisfying $\int_\Omega (f-u)(v-u)\,dx \le 0$ for all $v \in K$. Define $u = sgn(f)\min(|f|, h)$. Then u is measurable and we have that $|u| \le |f|$ on Ω so that $u \in L^2(\Omega)$. Moreover, $|u| \le h$ on Ω so that $u \in K$. Fix $v \in K$ and define $A = [(f-u)(v-u) > 0]$. Observe that for all $x \in A$, either we have $u(x) < f(x)$, so that $u(x) = h(x)$, and $h(x) = u(x) < v(x)$, or we have $f(x) < u(x)$, so that $u(x) = -h(x)$, and $v(x) < u(x) = -h(x)$. In either case, $|v(x)| > h(x)$ and it follows that A is a null set. Thus, $(f-u)(v-u) \le 0$ a.e. on Ω, so that $\int (f-u)(v-u)\,dx \le 0$. Since this inequality holds for all $v \in K$, it follows that $u = P_K f$. ☐

5.10

Let $F : H \to \mathbb{R}$ be a convex function of class C^1. Let $K \subset H$ be convex and let $u \in H$. Show that the following properties are equivalent:

(i) $F(u) \le F(v) \quad \forall v \in K$,

(ii) $(F'(u), v-u) \ge 0 \quad \forall v \in K$.

Example: $F(v) = |v-f|^2$ with $f \in H$ given.

Proof. This statement is not true in general: take $H = \mathbb{R}$, $u = -1$, $K = \{2\}$ and $f = x^2$. I'm going to assume here Brezis meant to write "let $u \in K$". With this assumption in mind, fix $F : H \to \mathbb{R} \in C^1(H)$, K convex and $u \in K$. Assume (i) holds. Then by the convexity of K, for any $t \in (0,1)$ and $v \in K$, we have that $t(v-u) + u = tv + (1-t)u \in K$, so that $F(u) \le F(t(v-u)+u)$. It follows that for all $t \in (0,1)$ and $v \in K$, $(F'(u), v-u) + o(t) = F(t(v-u)+u) - F(u) \ge 0$. Taking the limit as $t \to 0$, we get $(F'(u), v-u) \ge 0$, proving that (i) $\implies$ (ii). Now assume (ii) holds. Fix $v \in K$ and observe that by the convexity of F, for all $t \in (0,1)$ we have $t(F(v) - F(u)) \ge F(t(v-u)+u) - F(u) \ge F(t(v-u)+u) - F(u) - (F'(u), v-u) = o(t)$. Thus, $F(v) - F(u) \ge \frac{1}{t}o(t)$ for all $t \in (0,1)$, and taking the limit as $t \to 0$, we see that $F(v) - F(u) \ge 0$, proving that (ii) $\implies$ (i). ☐

5.11

Let $M \subset H$ be a closed linear subspace that is not reduced to $\{0\}$. Let $f \in H$, $f \notin M^\perp$.

1. Prove that

$$m = \inf_{\substack{u\in M\\|u|=1}} (f, u)$$

is uniquely achieved.

Proof. Set $K = M \cap B_H$ and note that since M and B_H are both closed and convex, K is closed and convex. Define $\varphi : H \to \mathbb{R};\ u \mapsto (f, u)$. Clearly φ is a bounded linear functional on H and so is convex and l.s.c. Since H is reflexive, $\varphi \not\equiv \infty$ and K is bounded, we can apply Corollary 3.23 to conclude that φ achieves its minimum on K. Let u be a minimum for φ. Towards proving that $|u| = 1$, observe that because $f \notin M^\perp$, it follows that there exists some $v \in M$ such that $(f, v) < 0$, and by potentially scaling v, we may assume WLOG that $v \in K$, so that $(f,u) \le (f,v) < 0$. Now we have that $\frac{u}{|u|} \in K$, and so if it were the case that $|u| \ne 1$, then since $u \in B_H$, we would have that $|u| < 1$, so that $(f, \frac{u}{|u|}) = \frac{1}{|u|}(f,u) < (f,u)$, which is absurd. Thus, since $\{v \in M : |v| = 1\} \subset K$, it follows that u achieves the minimum $\inf_{\substack{v\in M\\|v|=1}}(f,v)$. Finally, to see that u uniquely achieves this minimum, suppose towards a contradiction that there exists some $u' \in K$ such that $(f,u') = (f,u)$ and $u' \ne u$. We showed above that every minimum of K must have norm 1. However, we have that $\frac{1}{2}u + \frac{1}{2}u' \in K$ and this element achieves the minimum since $(f,u) = (f, \frac{1}{2}u + \frac{1}{2}u')$, which produces a contradiction as H is uniformly convex, so strictly convex, implying that $|\frac{1}{2}u + \frac{1}{2}u'| < 1$. Thus, u is the unique element that achieves the minimum $m = \inf_{\substack{v\in M\\|v|=1}}(f,v)$. □

2. Let $\varphi_1, \varphi_2, \varphi_3 \in H$ be given and let E denote the linear space spanned by $\{\varphi_1, \varphi_2, \varphi_3\}$. Determine m in the following cases:

   (i) $M = E$,

   Solution

   By possibly performing Gram-Schmidt, we may assume WLOG that $\varphi_1, \varphi_2, \varphi_3$ is an orthonormal basis for E. Observe that $f = P_E f + P_{E^\perp} f$, and so for all $u \in E$, we have $(f,u) = (P_E f + P_{E^\perp} f, u) = (P_E f, u)$. Thus, for all $u \in E$ with $|u| = 1$, we have $(f,u) = (P_E f, u) \ge -|P_E f|$. Set $u_0 = -\frac{P_E f}{|P_E f|} \in E$ and observe that $|u_0| = 1$ and $(f,u) = -|P_E f|$. It follows that $m = -|P_E f| = -|\sum_i (\varphi_i, f)\varphi_i| = -\sqrt{\sum_i |(\varphi_i, f)|^2}$.

   (ii) $M = E^\perp$

   Solution

   Regurgitating the exact same argument as above but replacing E with $E^\perp$, we see that $m = -|P_{E^\perp}|$. Since $|f|^2 = |P_E f|^2 + |P_{E^\perp} f|^2$, it follows that $m = -\sqrt{|f|^2 - |P_E f|^2} = -\sqrt{|f|^2 - \sum_i |(f, \varphi_i)|^2}$.

3. Examine the case in which $H = L^2(0,1), \varphi_1(t) = t, \varphi_2(t) = t^2, \varphi_3(t) = t^3$.

   Solution

   After performing Gram-Schmidt on $\varphi_1, \varphi_2, \varphi_3$, we get the orthonormal basis $e_1(t) = \sqrt{3}t$, $e_2(t) = \sqrt{5}(4t^2 - 3t), e_3(t) = \sqrt{7}(15t^3 - 20t^2 + 6t)$. From our results in part 2 above, it follows that

$$m_E = -\sqrt{3\left|\int_0^1 f(t)t\,dt\right|^2 + 5\left|\int_0^1 f(t)(4t^2-3t)dt\right|^2 + 7\left|\int_0^1 f(t)(15t^3-20t^2+6t)dt\right|^2},$$

   and $m_{E^\perp} = \sqrt{|f|^2 - m_E^2}$.

5.14

Let $a : H \times H \to \mathbb{R}$ be a bilinear continuous form such that

$$a(v,v) \ge 0 \quad \forall v \in H.$$

Prove the the function $v \mapsto F(v) = a(v,v)$ is convex, of class C^1, and determine its differential.

Proof. Since a is a continuous bilinear form, there exists $C \ge 0$ such that $|a(u,v)| \le C|u||v|$ for all $u,v \in H$. Thus, if $u_n \to u$, then $|F(u_n) - F(u)| = |a(u_n - u, u_n) - a(u - u_n, u)| \le C|u_n - u|(|u_n| + |u|) \to 0$. Since H is a metric space, this proves that F is continuous. Fix $v,u \in H$ and $t \in (0,1)$. Since $a(u-v,u-v) \geq 0$, it follows that $a(u,v) + a(v,u) \leq F(u) + F(v)$, and therefore

$$\begin{aligned}
F(tv + (1-t)u) &\leq t^2 F(v) + (1-t)^2 F(u) + t(1-t)(F(u) + F(v)) \\
&= t(tF(v) + (1-t)(F(u) + F(v))) + (1-t)^2 F(u) \\
&= t((1-t)F(v) + (tF(v) + (1-t)F(u)) + (1-t)^2 F(u) \\
&= (1-t)((1-t)F(u) + tF(v)) + t(tF(v) + (1-t)F(u)) \\
&= tF(v) + (1-t)F(u),
\end{aligned}$$

proving that F is convex. Towards proving that $F \in C^1$, observe that for all $u,h \in H$,

$$\begin{aligned}
\frac{|F(u+h) - F(u) - a(u,h) - a(h,u)|}{|h|} &= \frac{|a(u+h,u+h) - a(u,u+h) - a(h,u)|}{|h|} \\
&= \frac{|a(h,h)|}{|h|} \leq C|h|.
\end{aligned}$$

Thus, the LHS goes to 0 as $|h| \to 0$, and since the map $\varphi_u : h \in H \mapsto a(u,h) + a(h,u)$ is a continuous linear functional on H, it follows that $F \in C^1$ and $F'(u) = \varphi_u$. □

5.15

Let $G \subset H$ be a linear subspace of a Hilbert space H; G is equipped with the norm of H. Let F be a Banach space. Let $S : G \to F$ be a bounded linear operator. Prove that there exists a bounded linear operator $T : H \to F$ that extends S and such that

$$\|T\|_{\mathcal{L}(H,F)} = \|S\|_{\mathcal{L}(G,F)}.$$

Proof. By Exercise 1.6, G is either dense or closed in H. If G is dense in H then S extends uniquely to some bounded linear operator T on H since S is bounded and so uniformly continuous on a dense subset of H. That is, for any $u \in H$, we pick any sequence $(u_n) \subset G$ that converges to u in H. Since $|S(u_n - u_m)|_F \leq \|S\||u_n - u_m|_H$, (Su_n) is a Cauchy sequence in F and therefore converges to some point $Tu \in F$. Observe that if $(v_n) \subset G$ is another sequence converging to u, then $|Tu - Sv_n| \leq |Tu - Su_m| + \|S\||u_m - v_n| \to 0$, and so T is a well defined function from H into F. That T is linear and extends S is clear. To see that T is bounded, fix $v \in H$ and pick a sequence $(u_n) \subset G$ such that $u_n \to v$. Then $|Tv| = \lim_{n\to\infty} |Su_n| \leq \lim_{n\to\infty} \|S\||u_n| = \|S\||v|$. Hence, T is bounded and $\|T\|_{\mathcal{L}(H,F)} \leq \|S\|_{\mathcal{L}(G,F)}$. Since T extends S, it's clear that $\|S\|_{\mathcal{L}(G,F)} \leq \|T\|_{\mathcal{L}(H,F)}$, and the case where G is dense in H follows.

Suppose now that G is not dense in H so that G is a closed linear subspace. Define $T : H \to F$ by $Tv = S \circ P_G v$. Since $P_G u = u$ for all $u \in G$, T extends S. Moreover, because $S \in \mathcal{L}(G,H)$ and $P_G \in \mathcal{L}(H,G)$, it follows that $T \in \mathcal{L}(H,F)$. For any $v \in H$, we have $\|Tv\| \leq \|S\|\|P_G\||v| = \|S\||v|$, proving that $\|T\|_{\mathcal{L}(H,F)} \leq \|S\|_{\mathcal{L}(G,F)}$. Again, since T is an extension of S, the inequality in the other direction follows, and so $\|T\|_{\mathcal{L}(H,F)} = \|S\|_{\mathcal{L}(G,F)}$, as required. □

5.16 The triplet $V \subset H \subset V^\star$.

Let H be a Hilbert space equipped with the scalar product $(\ ,\ )$ and the corresponding norm $|\ \ |$. Let $V \subset H$ be a linear subspace that is dense in H. Assume that V has its own norm $\|\ \ \|$ and that V is a Banach space for $\|\ \ \|$. Assume also that the injection $V \subset H$ is continuous, i.e., $|v| \leq C\|v\|\ \forall v \in V$. Consider the operator $T : H \to V^\star$ defined by

$$\langle Tu, v\rangle_{V^\star,V} = (u,v) \quad \forall u \in H, \quad \forall v \in V.$$

1. Prove that $\|Tu\|_{V^\star} \leq C|u|\ \forall u \in H$.

   Proof. Fix $u \in H$. Note that for all $v \in V$, $|\langle Tu, v\rangle| = |(u,v)| \leq |u||v| \leq C|u|\|v\|$, and so $\|Tu\|_{V^\star} \leq C|u|$ for all $u \in H$. □

2. Prove that T is injective.

Proof. Fix $u, u' \in H$ and suppose that $Tu = Tu'$. Then for all $v \in V$, $(u - u', v) = \langle T(u - u'), v\rangle = 0$. Since V is dense in H, the continuity of $(\quad , \quad)$ implies that $(u - u', v) = 0$ for all $v \in H$, and so $u - u' = 0$, proving that T is injective. □

3. Prove that $R(T)$ is dense in $V^\star$ if V is reflexive.

Proof. Towards a contradiction, suppose that V is reflexive but $R(T)$ is not dense in $V^\star$. Since $R(T)$ is a linear subspace of $V^\star$, the fact that $R(T)$ is not dense in $V^\star$ implies that $R(T)$ is closed (by Exercise 1.6). Pick some $f \notin R(T)$ and apply the second geometric form of the Hahn-Banach theorem to get a bounded linear functional $\xi \in V^{\star\star}$ and some $\varepsilon > 0$ such that $\langle \xi, Tu\rangle < \langle \xi, f\rangle - \varepsilon$ for all $u \in H$. It follows that $\langle \xi, Tu\rangle = 0$ for all $u \in H$ and $\xi \not\equiv 0$. But since V is reflexive, there exists some $u \in V$ such that $\langle \xi, v^*\rangle_{V^{\star\star},V^\star} = \langle v^*, u\rangle_{V^\star,V}$ for all $v^* \in V^\star$ and it follows that $0 = \langle \xi, Tu\rangle = \langle Tu, u\rangle = |u|^2$, so that $u = 0$ which is absurd because we then have that $\langle f, 0\rangle = \langle \xi, f\rangle > 0$. By contradiction, if V is reflexive then $R(T)$ is dense in $V^\star$. □

4. Given $f \in V^\star$, prove that $f \in R(T)$ iff there is a constant $a \geq 0$ such that $|\langle f, v\rangle_{V^\star,V}| \leq a|v| \; \forall v \in V$.

Proof. Fix $f \in V^\star$. Clearly if $f \in R(T)$, then we have some $u \in H$ such that for all $v \in V$, $|\langle f, v\rangle| = |\langle Tu, v\rangle| = |(u, v)| \leq |u||v|$, and so we can choose $a = |u| \geq 0$. Now suppose that there exists some $a \geq 0$ such that $|\langle f, v\rangle| \leq a|v|$ for all $v \in V$. Then $f|_V$ is a bounded linear functional on the subspace $V \subset H$ with respect to the norm $| \quad |$, and so by the analytic form of the Hahn-Banach theorem, there exists an extension $F \in H^\star$ of f. By the Riesz Representation Theorem for Hilbert spaces, there exists some $u \in H$ such that $\langle F, v\rangle = (u, v)$ for all $v \in H$. It follows that for all $v \in V$, $\langle Tu, v\rangle = (u, v) = \langle F, v\rangle = \langle f, v\rangle$, and so $f \in R(T)$. □

5.20

Assume that $S \in \mathcal{L}(H)$ satisfies $(Su, u) \geq 0 \; \forall u \in H$.

1. Prove that $N(S) = R(S)^\perp$.

Proof. Fix $u \in N(S)$. Observe that for all $v \in H$, since $0 \leq (S(v-u), v-u) = (Sv, v-u)$, it follows that $((Sv, u) \leq (Sv, v)$. Now fix $v \in H$. For every $t > 0$ we have that $(S(tv), u) \leq (S(tv), tv)$, so that $(Sv, u) \leq t(Sv, v)$. Taking the limit as $t \to 0$, we have that $(Sv, u) \leq 0$. We also have that for all $t < 0$, $(S(tv), u) \leq (S(tv), tv)$, and so $(Sv, u) \geq t(Sv, v)$. Taking the limit as $t \to 0$, we get $(Sv, u) \geq 0$, showing that $u \perp Sv$. Since this holds for all $v \in H$, we have that $u \in R(S)^\perp$. Thus, $N(S) \subset R(S)^\perp = N(S^\star)$. Observe that $(Su, u) \geq 0 \quad \forall u \in H$ implies that $(S^\star u, u) = (u, Su) \geq 0 \quad \forall u \in H$. Thus, $N(S^\star) \subset R(S^\star)^\perp = N(S)$, proving that $N(S) = N(S^\star) = R(S)^\perp$. □

2. Prove that $I + tS$ is bijective for every $t > 0$.

Proof. Fix $t > 0$. Suppose that $u \in N(I + tS)$, so that $u = -tSu$. Then $0 \leq (Su, u) = -t(Su, Su) = -t|Su|^2$, which implies that $u \in N(S)$. Thus, $u = -tSu = 0$, proving that $I + tS$ is injective. Towards proving that $I + tS$ is surjective, fix $v \in H$ and define the bilinear form $a : H \times H \to \mathbb{R}$; $(x, y) \mapsto (x + tSx, y)$. Observe that for all $x, y \in H$, $|a(x, y)| \leq |x + tSx||y| \leq (1 + t\|S\|)|x||y|$ and $a(x, x) = |x|^2 + t(Sx, x) \geq |x|^2$, proving that a is continuous and coercive. Thus, by Lax-Milgram, there exists a unique element $u \in H$ such that $a(u, x) = \langle \varphi, x\rangle$ for all $x \in H$, where $\varphi : x \in H \mapsto (v, x)$. That is, there exists unique $u \in H$ such that $(u + tSu, x) = (v, x)$ for all $x \in H$, proving that $u + tSu = v$. It follows that $I + tS$ is a bijection. □

3. Prove that

$$\lim_{t \to +\infty} (I + tS)^{-1} f = P_{N(S)} f \quad \forall f \in H.$$

Proof. Suppose first that $f \in N(S)$. Fix $t > 0$ and set $u_t = (I + tS)^{-1} f$. Observe that since $N(S) = R(S)^\perp$, we have $(Su_t, u_t) + t|Su_t|^2 = (Su_t, f) = 0$. Since (Su_t, u_t) and $t|Su_t|^2$ are both at nonnegative, it follows that $|Su_t| = 0$ so that $f = u_t + tSu_t = u_t$. Thus, $\lim_{t\to\infty}(I + tS)^{-1} f = f$ for all $f \in N(S)$.

Now suppose that $f \in R(S)$. Then there exists $v \in H$ such that $f = Sv$. Fix $t > 0$ and set $u_t = (I + tS)^{-1} f$, so that $u_t + tSu_t = Sv$. Since $u_t + S(tu_t - v) = 0$, it follows that $(u_t, tu_t - v) = -(S(tu_t - v), tu_t - v) \leq 0$. Thus, $t|u_t|^2 \leq (u_t, v) \leq |u_t||v|$, and so $|u_t| \leq \frac{1}{t}|v|$. Taking the limit as $t \to \infty$, we have $\lim_{t\to\infty}(I + tS)^{-1} f = \lim_{t\to\infty} u_t = 0$.

Applying Exercise 5.15 with $G = R(S)$, it follows that $\lim_{t\to\infty}(I + tS)^{-1}|_{\overline{R(S)}} = 0$.

Since $H = N(S) \oplus N(S)^{\perp} = N(S) \oplus (R(S)^{\perp})^{\perp} = N(S) \oplus \overline{R(S)}$, for any $f \in H$, we have that

$$\lim_{t\to\infty}(I + tS)^{-1}f = \lim_{t\to\infty}(I + tS)^{-1}(P_{N(S)}f + P_{\overline{R(S)}}f) = P_{N(S)}f.$$

□

5.22

Let $C \subset H$ be a nonempty closed convex set and let $T : C \to C$ be a nonlinear contraction, i.e.,

$$|Tu - Tv| \le |u - v| \quad \forall u, v \in C.$$

1. Let (u_n) be a sequence in C such that

$$u_n \rightharpoonup u \text{ weakly and } (u_n - Tu_n) \to f \text{ strongly.}$$

Prove that $u - Tu = f$.

Proof. Note that since C is convex and strongly closed, C is weakly closed and so $u \in C$. Moreover, since T is a contraction, we have the following chain of inequalities

$$\begin{aligned}|u_n - u|^2 &\ge |Tu_n - Tu|^2\\ &= |(u - Tu) - (u_n - Tu_n) - (u - u_n)|^2\\ &= |(u - Tu) - (u_n - Tu_n)|^2 + |u_n - u|^2 - 2((u - Tu) - (u_n - Tu_n), u - u_n).\end{aligned}$$

Thus, it follows that $|(u-Tu)-(u_n-Tu_n)|^2 \le 2((u-Tu)-(u_n-Tu_n), u-u_n)$ for all n. Since $(u-Tu)-(u_n-Tu_n) \to u - Tu - f$ strongly and $u - u_n \rightharpoonup 0$ weakly, it follows that $|(u - Tu) - f|^2 = \lim_{n\to\infty} |(u - Tu) - (u_n - Tu_n)|^2 = 0$. Thus, $u - Tu = f$, as required. □

2. Deduce that if C is bounded and $T(C) \subset C$, then T has a fixed point.

Proof. Fix $a \in C$ and $\varepsilon \in (0, 1)$. Observe that for any $u \in C$, $(1 - \varepsilon)Tu + \varepsilon a \in C$ by the convexity of C. Define $F_\varepsilon : C \to C$; $u \mapsto (1 - \varepsilon)Tu + \varepsilon a$. Note that F_ε is a strict contraction since for any $u, v \in C$, $|F_\varepsilon u - F_\varepsilon v| = (1 - \varepsilon)|Tu - Tv| \le (1 - \varepsilon)|u - v|$. Since C is closed subspace of H, C is a complete metric space with respect to the metric induced by the norm on C, and so by the Banach-fixed point theorem, there exists unique $u_\varepsilon \in C$ such that $(1 - \varepsilon)Tu_\varepsilon + \varepsilon a = F_\varepsilon u_\varepsilon = u_\varepsilon$. For each $n \ge 1$, define $u_n := u_{\frac{1}{n}}$. Since C is bounded, (u_n) is a bounded sequence in H and therefore there exists subsequence (u_{n_k}) that converges weakly to some $u \in H$. Moreover, $u_{n_k} - Tu_{n_k} = \frac{1}{n_k}(a - Tu_{n_k}) \to 0$ strongly as $k \to \infty$ (since $(Tu_{n_k}) \subset C$ is bounded). Thus, by part 1 above, $u - Tu = 0$, and so T has a fixed point. □

5.26

Assume that (e_n) is an orthonormal basis of H.

1. Check that $e_n \rightharpoonup 0$ weakly.

Proof. Fix $u \in H$ and observe that by Parseval's identity, $\sum_n |(e_n, u)|^2 = |u|^2 < \infty$. Thus, $|(e_n, u)|^2 \to 0$ as $n \to \infty$, and it follows that $(e_n, u) \to 0$ as $n \to \infty$ for all $u \in H$, so that $e_n \rightharpoonup 0$ weakly. □

Let (a_n) be a bounded sequence in $\mathbb{R}$ and set $u_n = \frac{1}{n}\sum_{i=1}^{n} a_i e_i$.

2. Prove that $|u_n| \to 0$.

Proof. By assumption, there exists $C \ge 0$ such that $|a_n| \le C$ for all n. Observe that $|u_n|^2 = \frac{1}{n^2}\sum_{i=1}^{n} |a_i|^2 \le \frac{1}{n^2}\sum_{i=1}^{n} C^2 = \frac{1}{n}C^2$. It follows that for all n, $|u_n| \le \frac{1}{\sqrt{n}}C$, which goes to 0 as $n \to \infty$. □

3. Prove that $\sqrt{n}u_n \rightharpoonup 0$ weakly.

Proof. Fix $v \in H$. Observe that for any $m \geq 1$ and for all $n \geq m$, we have that $(\sqrt{n}u_n, e_m) = \frac{a_m}{\sqrt{n}} \to 0$ as $n \to \infty$. It follows that for all $v \in \operatorname{span}(e_i)_{i\geq 1}$, $(\sqrt{n}u_n, v) \to 0$ as $n \to \infty$. Since $\operatorname{span}(e_i)_{i\geq 1}$ is dense in H, there exists a sequence $(v_n) \subset \operatorname{span}(e_i)_{i\geq 1}$ such that $v_n \to v$ strongly. Fix $\varepsilon > 0$ and pick M such that $|v_m - v| < \frac{\varepsilon}{2C}$ for all $m \geq M$. Choose N such that $|(\sqrt{n}u_n, v_M)| < \frac{\varepsilon}{2}$. Then for all $n \geq N$

$$\begin{aligned} |(\sqrt{n}u_n, v)| &\leq |(\sqrt{n}u_n, v_M)| + |(\sqrt{n}u_n, v - v_M)| \\ &\leq \frac{\varepsilon}{2} + C|v - v_M| < \varepsilon. \end{aligned}$$

It follows that $\lim_{n\to\infty}(\sqrt{n}u_n, v) = 0$ for all $v \in H$, proving that $\sqrt{n}u_n \rightharpoonup 0$ weakly. □

5.28

Assume that H is separable.

1. Let $V \subset H$ be a linear subspace that is dense in H. Prove that V contains an orthonormal basis of H.

Proof. Since H is separable, so is V. Let (v_n) be a countable dense subset of V. Let F_k denote the linear subspace of V spanned by $\{v_1, \dots, v_k\}$. The sequence (F_k) is a nondecreasing sequence of finite dimensional subspaces of V such that $\bigcup_{k=1}^{\infty}\{v_n\} \subset \bigcup_{k=1}^{\infty} F_k$ is dense in V, and therefore in H. Now pick any unit vector $e_1 \in F_1$ and assume that we have picked a nondecreasing sequence of orthonormal bases for $F_1, \dots, F_{k-1}$, which we shall denote by $\{e_1\} \subset \cdots \subset \{e_1, \dots, e_{n_{k-1}}\}$. Then we can construct an orthonormal basis of F_k that includes $\{e_1, \dots, e_{n_{k-1}}\}$ as follows: if $\operatorname{span}(e_1, \dots, e_{n_{k-1}}) = F_k$ then choose $\{e_1, \dots, e_{n_{k-1}}\}$. Otherwise pick any vector $v_k \in F_k \setminus \operatorname{span}(e_i)_{i=1}^{k-1}$ and perform Gram-Schmidt to get the orthonormal basis $\{e_1, \dots, e_{n_{k-1}}, e_k\} \supset \{e_1, \dots, e_{n_{k-1}}\}$ of F_k. Repeating this process for each $k \geq 1$, we get an orthonormal sequence $(e_n) \subset V$ whose span is equal to $\bigcup_{k=1}^{\infty} F_k$, which is dense in H. Thus, $(e_n) \subset V$ is an orthonormal basis of H, as required. □

2. Let $(e_n)_{n\geq 1}$ be an orthonormal sequence in H, i.e., $(e_i, e_j) = \delta_{ij}$. Prove that there exists an orthonormal basis of H that contains $\bigcup_{n=1}^{\infty}\{e_n\}$.

Proof. Let $E = \overline{\operatorname{span}(e_n)}$. If $E = H$, then we're done so suppose WLOG that $E \subsetneq H$. Clearly (e_n) is an orthonormal basis for E. Since $E^{\perp}$ is a closed linear subspace of H, $E^{\perp}$ is a separable Hilbert space with respect to the inner product $(\ ,\ )$, and so by Theorem 5.11, $E^{\perp}$ has an orthonormal basis $(v_n)_{n=1}^{\infty} \subset E^{\perp}$. Define $(u_n)_{n=1}^{\infty}$ by $u_{2n} = e_n$ and $u_{2n-1} = v_n$. Since $e_n \perp v_m$ for all n, m, it's clear that (u_n) is an orthonormal sequence. Moreover, $\overline{\operatorname{span}(u_n)} \supset \overline{\operatorname{span}(e_n)} = E$ and $\overline{\operatorname{span}(u_n)} \supset \overline{\operatorname{span}(v_n)} = E^{\perp}$. Since $\overline{\operatorname{span}(u_n)}$ is a linear subspace of H, it follows that $H = E \oplus E^{\perp} \subset \overline{\operatorname{span}(y_n)}$, proving that (u_n) is an orthonormal basis of H containing $\bigcup_{n=1}^{\infty}\{e_n\}$. □

5.30

Let $(e_n)_{n\geq 1}$ be an orthonormal sequence in $H = L^2(0,1)$. Let $p(t)$ be a given function in H.

1. Prove that for every $t \in [0,1]$, one has

$$\sum_{n=1}^{\infty} \left| \int_0^t p(s)e_n(s)ds \right|^2 \leq \int_0^t |p(s)|^2 ds. \tag{1}$$

Proof. Since $L^2(0,1)$ is separable, we can apply Exercise 5.28 to extend (e_n) to an orthonormal basis (u_n) of $L^2(0,1)$. Observe that $|p\chi_{[0,1]}| \leq |p|$ for all $x \in (0,1)$, and so $p\chi_{[0,1]} \in L^2(0,1)$. Applying Parseval's identity, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\int_0^t p(s)e_n(s)ds\right|^2 &= \sum_{n=1}^{\infty}\left|\int_0^1 p(s)\chi_{[0,t]}(s)e_n(s)ds\right|^2\\ &\le \sum_{n=1}^{\infty}\left|\int_0^1 p(s)\chi_{[0,t]}(s)u_n(s)ds\right|^2\\ &= \int_0^1 |p(s)\chi_{[0,t]}(s)|^2 ds\\ &= \int_0^t |p(s)|^2 ds.\end{aligned}$$

□

2. Deduce that

$$\sum_{n=1}^{\infty}\int_0^1\left|\int_0^t p(s)e_n(s)ds\right|^2 dt \le \int_0^1 |p(t)|^2(1-t)dt. \tag{2}$$

Proof. Applying the inequality from part 1 above, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\int_0^1\left|\int_0^t p(s)e_n(s)ds\right|^2 dt &= \int_0^1\sum_{n=1}^{\infty}\left|\int_0^t p(s)e_n(s)ds\right|^2 dt \quad \text{(Monotone Convergence Theorem)}\\ &\le \int_0^1\int_0^t |p(s)|^2 ds dt\\ &= \int_0^1\int_0^1 |p(s)|^2\chi_{[s\le t]}(s,t)dsdt\\ &= \int_0^1\int_0^1 |p(s)|^2\chi_{[s\le t]}(s,t)dtds \quad \text{(Fubini's Theorem)}\\ &= \int_0^1 |p(s)|^2(1-s)ds.\end{aligned}$$

□

3. Assume now that $(e_n)_{n\ge1}$ is an orthonormal basis of H. Prove that (1) and (2) become equalities.

Proof. Since (e_n) is an orthonormal basis, we don't need to extend (e_n) to the basis (u_n) in part 1 above, and so the line with the inequality is removed, giving equality. E.g., we apply Parseval's identity with (e_n) to get $\sum_{n=1}^{\infty}|(e_n, p\chi_{[0,t]})|^2 = \|p\chi_{[0,t]}\|_2^2$. Observe that the only inequality in my proof of part 2 is now an equality, and so (2) also becomes an equality. □

4. Conversely, assume that equality holds in (2) and that $p(t) \ne 0$ a.e. Prove that $(e_n)_{n\ge1}$ is an orthonormal basis.

Proof. Observe from the chain of (in)equalities in part 2 that equality in (2) forces that

$$\begin{aligned}0 &= \int_0^1\int_0^t |p(s)|^2 ds - \sum_{n=1}^{\infty}\left|\int_0^t p(s)e_n(s)ds\right|^2 dt\\ &= \int_0^1\left|\int_0^t |p(s)|^2 ds - \sum_{n=1}^{\infty}\left|\int_0^t p(s)e_n(s)ds\right|^2\right|dt,\end{aligned}$$

and it follows that $\sum_{n=1}^{\infty}\left|\int_0^t p(s)e_n(s)ds\right|^2 = \int_0^t |p(s)|^2 ds$ for almost all $t \in [0,1]$. Thus, equality in (2) implies equality for almost all $t \in [0,1]$ in (1). Extend (e_n) to an orthonormal basis (u_n) of $L^2(0,1)$. Towards a contradiction, suppose that there exists some m such that $u_m \notin (e_n)$. Then we have that for t a.e. $\sum_{n=1}^{\infty}|(u_n, p\chi_{[0,t]})|^2 = \|p\chi_{[0,1]}\|_2^2 = \sum_{n=1}^{\infty}|(e_n, p\chi_{[0,t]})|^2$, which forces that $|(u_m, p\chi_{[0,t]})| = 0$ for almost all $t \in [0,1]$. That is, $\int_0^t u_m(s)p(s)ds = 0$ for almost all $t \in [0,1]$, and since $t \mapsto \int_0^t u_m(s)p(s)ds$ is a continuous function, it follows that $\int_0^t u_m(s)p(s)ds = 0$ for all $t \in [0,1]$. Thus, $\int_{t_1}^{t_2} u_m(s)p(s)ds = 0$ for all $(t_1,t_2) \subset [0,1]$, which implies that $u_m p = 0$ a.e. on $[0,1]$. Since $p \neq 0$ a.e. on $[0,1]$, we must have that $u_m = 0$ a.e., so that $\|u_m\|_2^2 = \int_0^1 |u_m|^2 = 0$, which contradicts the fact that u_m is a unit vector. Thus, by contradiction, $\bigcup_n\{e_n\} = \bigcup_n\{u_n\}$, proving that (e_n) is an orthonormal basis for $L^2(0,1)$. ☐

6.1

Let $E = \ell^p$ with $1 \leq p \leq \infty$. Let (λ_n) be a bounded sequence in $\mathbb{R}$ and consider the operator $T \in \mathcal{L}(E)$ defined by

$$Tx = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots),$$

where

$$x = (x_1, x_2, \ldots, x_n, \ldots).$$

Prove that T is a compact operator from E into E iff $\lambda_n \to 0$.

Proof. It's clear that $T \in \mathcal{L}(\ell^p)$ with $\|T\| = \sup_n |\lambda_n|$. Observe that for any n with $\lambda_n \neq 0$, $\delta_{nm} \in N(T - \lambda_n I)$, where δ_{nm} is the sequence with 1 in the nth position and zeroes elsewhere. Hence, $N(T-\lambda_n I) \supsetneq \{0\}$, proving that $\{\lambda_n : \lambda_n \neq 0\} \subset \text{EV}(T) \setminus \{0\} \subset \sigma(T) \setminus \{0\}$. Thus, by Lemma 6.2, if $T \in K(\ell^p)$, then either $\{\lambda_n : \lambda_n \neq 0\}$ is finite or $\{\lambda_n : \lambda_n \neq 0\}$ is a subset in $\mathbb{R}$ with limit point 0. In either case, the sequence (λ_n) converges to 0, proving the "only if" direction.

For the "if" direction, suppose that $\lambda_n \to 0$. Then for each n, define $T_n \in \mathcal{L}(\ell^p)$; $x \mapsto (\lambda_1 x_1, \ldots, \lambda_n x_n, 0, \ldots)$. Clearly each T_n has finite rank and is therefore compact. Observe that for any $x \in \ell^p$

$$\|(T - T_n)x\|_p = \|(0, \ldots, 0, \lambda_{n+1}x_{n+1}, \lambda_{n+2}x_{n+2}, \ldots)\|_p \leq \max_{m \geq n} |\lambda_m| \|x\|_p,$$

proving that $\|T - T_n\| \leq \max_{m\geq n} |\lambda_m| \to 0$ as $n \to \infty$. Thus, T is a compact operator, being the limit of compact operators. ☐

6.2

Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$.

1. Assume that E is reflexive. Prove that $T(B_E)$ is closed (strongly).

   Proof. By Kakutani's Theorem, E reflexive implies that B_E is weakly compact. Since T is continuous from E with the weak topology $\sigma(E, E^\star)$ into F with the weak topology $\sigma(F, F^\star)$ by Theorem 3.10, it follows that $T(B_E)$ is a weakly compact subset of F. In particular, $T(B_E)$ is weakly closed and therefore strongly closed in F. ☐

2. Assume that E is reflexive and that $T \in K(E, F)$. prove that $T(B_E)$ is compact.

   Proof. By part 1 above, $T(B_E)$ is (strongly) closed. Thus, since T is a compact operator, $T(B_E) = \overline{T(B_E)}$ is (strongly) compact. ☐

3. Let $E = F = C([0,1])$ and $Tu(t) = \int_0^t u(s)ds$. Check that $T \in K(E)$. Prove that $T(B_E)$ is not closed.

   Proof. It's clear that T is a bounded linear operator (linearity is obvious, the range of T being contained in $C([0,1])$ follows by the continuity of $t \mapsto \int_0^t u$ for any (locally integrable) u, and boundedness follows by the fact that $\|Tu\|_{C([0,1])} = \max_{t\in[0,1]} |Tu(t)| \leq \max_{t\in[0,1]} \int_0^t |u| = \|u\|_{C([0,1])}$). Now towards proving that T is a compact operator, observe that for any $u \in B_E$ and $t \in [0,1]$

$$\begin{aligned} |Tu(t+h) - Tu(t)| &= \left| \int_t^{t+h} u(s)ds \right| \\ &\leq |h|. \end{aligned}$$

Thus, clearly $T(B_E)$ is an equicontininuous subset of $C([0,1])$. Moreover, for any $u \in B_E$, $\|Tu\|_{C([0,1])} \leq \|T\| = 1$, and so $T(B_E)$ is equibounded. By Arzelà-Ascoli, it follows that $\overline{T(B_E)}$ is a compact subset of $C([0,1])$, and so $T \in K(E)$.

Finally, towards proving that $T(B_E)$ is not closed, observe that by the Fundamental Theorem of Calculus Part I, $T(B_E) \subset C^1([0,1])$ and so it suffices to construct a sequence $(u_n) \subset B_E$ such that Tu_n converges to a function not belonging to $C^1([0,1])$. Define the sequence $(u_n) \subset B_E$ by

$$u_n(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}], \\ n(t-1/2), & t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}], \\ 1, & t \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

Clearly $\lim_{n\to\infty} Tu_n(t) = (t - \frac{1}{2})\chi_{[\frac{1}{2},1]}(t)$, and since this function is not differentiable at $t = \frac{1}{2}$, it follows that $\lim_{n\to\infty} Tu_n \notin C^1([0,1])$. Thus, $T(B_E)$ is not closed. □

6.3

Let E and F be two Banach spaces, and let $T \in K(E, F)$. Assume $\dim E = \infty$. Prove that there exists a sequence (u_n) in E such that $\|u_n\|_E = 1$ and $\|Tu_n\|_F \to 0$.

Proof. Towards a contradiction, suppose that there exists no sequence $(u_n) \subset S_E$ such that $\|Tu_n\|_F \to 0$. Then there must exist some $\varepsilon > 0$ such that $\|Tu\| \geq \varepsilon$ for all $u \in S_E$ (or else we could clearly construct a sequence $(u_n) \subset S_E$ such that $\|Tu_n\|_F \to 0$). It follows that for all $u \in E$, $\|Tu\| \geq \varepsilon\|u\|$. Since $\dim E = \infty$, applying Riesz's Lemma, there exists a sequence $(u_n) \subset S_E$ such that $\|u_n - u_m\|_E \geq \frac{1}{2}$ for all $n \neq m$. But then for all $n \neq m$,

$$\|Tu_n - Tu_m\| \geq \varepsilon\|u_n - u_m\| \geq \frac{\varepsilon}{2},$$

and so (Tu_n) is a sequence in $T(B_E)$ without any convergent subsequence, which is absurd since T is a compact operator. Thus, by contradiction, there must exist a sequence $(u_n) \subset B_E$ such that $\|Tu_n\|_F \to 0$. □

6.5

Let (λ_n) be a sequence of positve numbers such that $\lim_{n\to\infty} \lambda_n = +\infty$. Let V be the space of sequences $(u_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} \lambda_n |u_n|^2 < \infty.$$

The space V is equipped with the scalar product

$$((u, v)) = \sum_{n=1}^{\infty} \lambda_n u_n v_n.$$

Prove that V is a Hilbert space and the $V \subset \ell^2$ with compact injection.

Proof. Observe that for any $(u_n), (v_n) \in V$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, by the convexity of $x \mapsto x^2$

$$\sum_{n=1}^{\infty} \lambda_n |\alpha_1 u_n + \alpha_2 v_n|^2 \leq |\alpha_1| \sum_{n=1}^{\infty} \lambda_n |u_n|^2 + |\alpha_2| \sum_{n=1}^{\infty} \lambda_n |v_n|^2 < \infty,$$

and since the zero sequence clearly belongs to V, it follows that V is a vector space. Towards proving that $V \subset \ell^2$, suppose that $(u_n) \notin \ell^2$, so that $\sum_{n=1}^{\infty} |u_n|^2 = \infty$. Then for any $N \geq 1$, $\sum_{n=N}^{\infty} |u_n|^2 = \infty$, and since $\lambda_n \to +\infty$, there exists some N_0 such that $\lambda_n \geq 1$ for all $n \geq N_0$. Thus, $\sum_{n=1}^{\infty} \lambda_n |u_n|^2 \geq \sum_{n=N_0}^{\infty} \lambda_n |u_n|^2 \geq \sum_{n=N_0}^{\infty} |u_n|^2 = \infty$, proving that $(u_n) \notin V$. It follows that $V \subset \ell^2$.

Towards proving that V is a Hilbert space with respect to $((\quad,\quad))$, note that $((\quad,\quad))$ is clearly a symmetric, positive-definite bilinear form on V, and so it suffices to verify that V is a Banach space with respect to the norm induced by $((\quad,\quad))$. To this end, fix a Cauchy sequence $(u^n) \subset V$. Then we have that $\sum_{n=1}^{\infty} \lambda_n |u_n^{m_1} - u_n^{m_2}|^2 \to 0$

as $m_1, m_2 \to \infty$. It follows that $(\sqrt{\lambda_n} u_n^m)_{m\geq 1}$ is a Cauchy sequence in ℓ^2 and therefore converges to some $(u_n) \in \ell^2$. Observe that $\sum_{n=1}^{\infty} \lambda_n \left| \frac{1}{\sqrt{\lambda_n}} u_n \right|^2 = \|u\|_2^2 < \infty$, so that $(\frac{1}{\sqrt{\lambda_n}} u_n) \in V$. Moreover,

$$\sum_{n=1}^{\infty} \lambda_n \left| u_n^m - \frac{1}{\sqrt{\lambda_n}} u_n \right|^2 = \sum_{n=1}^{\infty} \left| \sqrt{\lambda_n} u_n^m - u_n \right|^2 \to 0 \qquad \text{as } m \to \infty.$$

It follows that (u^n) converges to $u \in V$ with respect to $((\quad , \quad))$, and so V is a Hilbert space.

Finally, to see that the injection $\iota : V \to \ell^2$ is a compact operator, note first that $\iota \in \mathcal{L}(V, \ell^2)$. Indeed, it's obvious that ι is linear and if $(u^n) \subset V$ converges in V to $(u_n) \in V$, then let N be such that $\lambda_n \geq 1$ for all $n \geq N$ and note that

$$\sum_{n=1}^{\infty} |u_n^m - u_n|^2 \leq \frac{1}{\min_{i\in\{1,\ldots,N\}} \lambda_i} \sum_{n=1}^{\infty} \lambda_n |u_n^m - u_n|^2 \to 0 \qquad \text{as } m \to \infty.$$

It follows that $\iota(u^n) \to \iota(u)$ in ℓ^2, which verifies that ι is a bounded linear operator. For each n, define $\iota_n \in \mathcal{L}(V, \ell^2)$; $(u_1, \ldots, u_n, u_{n+1}, \ldots) \mapsto (u_1, \ldots, u_n, 0, \ldots)$. Clearly each ι_n has finite rank and for any $u \in V$,

$$\|\iota(u) - \iota_n(u)\|_2 = \sum_{k=n+1}^{\infty} |u_k|^2 \leq \frac{1}{\min_{i\geq n+1} \lambda_i} \sum_{n=1}^{\infty} \lambda_n |u_n|^2 = \frac{1}{\min_{i\geq n+1} \lambda_i} \|u\|_V.$$

Since $\min_{i\geq n} \lambda_n \to \infty$, it follows that $\iota_n \to \iota$ in $\mathcal{L}(V, \ell^2)$, and so ι is a compact operator. □

6.7

Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$. Consider the following properties:

$$\begin{cases} \text{For every weakly convergent sequence } (u_n) \text{ in } E \\ u_n \rightharpoonup u, \text{ then } Tu_n \to Tu \text{ strongly in } F. \end{cases} \tag{P}$$

$$\begin{cases} T \text{ is continuous from } E \text{ equipped with the weak topology} \\ \sigma(E, E^\star) \text{ into } F \text{ equipped with the strong topology.} \end{cases} \tag{Q}$$

1. Prove that
$$(Q) \iff T \text{ is a finite-rank operator.}$$

   Proof. Suppose that T is a finite-rank operator. Then define $T' \in \mathcal{L}(E, R(T))$ by $T'(u) = T(u)$, and let $\iota : R(T) \to F$ be the inclusion. Since $\dim R(T) < \infty$, it follows that the weak and strong topologies on $R(T)$ are equivalent, and since T' is continuous from E weak into $R(T)$ weak, it follows that T' is continuous from E weak into $R(T)$ strong. Since $T = \iota \circ T'$ and ι is continuous (with respect to the strong topologies), it follows that T is continuous from the weak topology on E into the strong topology on F.

   Finally, suppose that (Q) holds. Then there exists $u_1^\star, \ldots, u_n^\star \in E^\star$ and $\delta > 0$ such that $\{u \in E : |\langle u_i^\star, u\rangle| < \delta \quad \forall i \in \{1, \ldots, n\}\} \subset T^{-1}(B_E)$. Since $\bigcap_{i=1}^n N(u_i^\star)$ has finite codimension, it follows that there exists a finite dimensional subspace $G \subset E$ such that $E = G + \bigcap_{i=1}^n N(u_i^\star)$ and $G \cap \bigcap_{i=1}^n N(u_i^\star) = \{0\}$. Moreover, for any $u \in \bigcap_{i=1}^n N(u_i^\star)$, we have that $\langle u_i^\star, \lambda u\rangle = 0$ for all $\lambda \in \mathbb{R}$ so that $\|T(\lambda u)\|_F \leq 1$ for all $\lambda \in \mathbb{R}$. It follows that $Tu = 0$ and so $\dim R(T) = \dim T(G \oplus \bigcap_{i=1}^n N(u_i^\star)) = \dim T(G) < \infty$. □

2. Prove that $T \in K(E, F) \implies (P)$.

   Proof. Suppose that $T \in K(E, F)$ and fix a sequence $(u_n) \subset E$ that converges weakly to some point $u \in E$. Then since T is continuous from E weak into F weak, it follows that $Tu_n \rightharpoonup Tu$ weakly in F and so $(\|Tu_n\|)$ is a bounded sequence. Thus, there exists some $M > 0$ such that $(Tu_n) \subset \overline{T(MB_E)} = M\overline{T(B_E)}$. Since $M\overline{T(B_E)}$ is a (strongly) compact subset of F, we can apply Exercise 3.5 to conclude that $Tu_n \to Tu$ strongly. Property (P) follows. □

3. Assume that either $E = \ell^1$ or $F = \ell^1$. Prove that *every* operator $T \in \mathcal{L}(E, F)$ satisfies (P).

   Proof. Fix $T \in \mathcal{L}(E, F)$. By Schur's Theorem, a sequence in ℓ^1 converges strongly to some point iff the sequence converges weakly to that point. Thus, if $E = \ell^1$ and $u_n \rightharpoonup u \in E$, then $u_n \to u$ strongly and so $Tu_n \to Tu$ strongly. And if $F = \ell^1$ and $u_n \rightharpoonup u \in E$ then $Tu_n \rightharpoonup Tu$ weakly in $F = \ell^1$ and so $Tu_n \to Tu$ strongly. Thus, property (P) holds for all $T \in \mathcal{L}(E, F)$. □

   In what follows we assume that E is *reflexive.*

4. Prove that $T \in K(E, F) \iff (P)$.

   Proof. The left direction follows from part 3. Suppose (P). Fix a sequence $(Tu_n) \subset T(B_E)$. Since $(u_n) \subset E$ is a bounded sequence and E is reflexive, there exists a subsequence (u_{n_k}) that converges weakly to some point $u \in E$. Applying property (P), we have that $Tu_{n_k} \to Tu$ strongly. Since every sequence in $T(B_E)$ has a (strongly) convergent subsequence, it follows that the closure $\overline{T(B_E)}$ is sequentially compact, and therefore compact since the strong topology on E is metrizable. Thus, $T \in K(E, F)$, as required. □

5. Deduce that *every* operator $T \in \mathcal{L}(E, \ell^1)$ is compact.

   Proof. By part 3, every $T \in \mathcal{L}(E, \ell^1)$ satisfies (P). By part 4, it follows that every $T \in \mathcal{L}(E, \ell^1)$ is compact. □

6. Prove that *every* operator $T \in \mathcal{L}(c_0, E)$ is compact.

   Proof. Fix $T \in \mathcal{L}(E, c_0)$. Observe that $T^\star \in \mathcal{L}(E^\star, c_0^\star) = \mathcal{L}(E^\star, \ell^1)$. Thus, by part 5, $T^\star$ is a compact operator, and by Schauder's theorem, it follows that T is a compact. Hence, every $T \in \mathcal{L}(E, c_0)$ is compact. □

6.8

Let E and F be two Banach spaces, and let $T \in K(E, F)$. Assume that $R(T)$ is closed.

1. Prove that T is a finite-rank operator.

   Proof. Since $R(T)$ is closed, it follows that $R(T)$ is a Banach space with respect to the subspace topology. Then $T : E \to R(T)$ is a surjective bounded linear operator and so by the Open Mapping Theorem, there exists some $c > 0$ such that $T(B_E(0,1)) \supset B_F(0,c) \cap R(T)$, where $B_F(0,c)$ and $B_E(0,1)$ are open balls. It follows that $B_{R(T)} = B_F \cap R(T) \subset \frac{1}{c}\overline{T(B_E)}$. Since $B_{R(T)}$ is a closed subset of a compact set, $B_{R(T)}$ is compact, and it follows that $R(T)$ must be finite dimensional. □

2. Assume, in addition, that $\dim N(T) < \infty$. Prove that $\dim E < \infty$.

   Proof. Since $\dim N(T) < \infty$, $N(T)$ has a complement $G \subset E$. That is, there exists some closed subspace $G \subset E$ such that $E = N(T) \oplus G$. Since G is closed, G is a Banach space and clearly $T|_G : G \to R(T)$ is bijective. It follows by the Open Mapping Theorem that $T|_G$ is a bounded linear isomorphism between G and $R(T)$. Thus, $\dim E = \dim(N(T) \oplus G) = \dim N(T) + \dim R(T) < \infty$. □

6.10

Let $Q(t) = \sum_{k=1}^{p} a_k t^k$ be a polynomial such that $Q(1) \neq 0$. Let E be a Banach space, and let $T \in \mathcal{L}(E)$. Assume that $Q(T) \in K(E)$.

1. Prove that $\dim N(I - T) < \infty$, and that $R(I - T)$ is closed. More generally, prove that $(I - T)(E_0)$ is closed for every closed subspace $E_0 \subset E$.

Proof. Define the polynomial $\tilde{Q}(t) = \sum_{k=1}^{p}(\sum_{j=k}^{p} a_j)t^{k-1}$ and observe that $\tilde{Q}(t)(1-t) = Q(1) - Q(t)$. Thus, if $u \in N(I-T)$, then $0 = \tilde{Q}(T) \circ (I-T)(u) = Q(I)u - Q(T)u$, which implies that $u \in N(I - \frac{1}{Q(1)}Q(T))$. Since $Q(T)$ is compact, so is $\frac{1}{Q(1)}Q(T)$, and so applying the Fredholm Alternative Theorem, we have that $\dim N(I-T) \leq N(I - \frac{1}{Q(1)}Q(T)) < \infty$.

Towards proving the latter statement, consider a convergent sequence $(u_n - Tu_n) \subset R(I-T)$ with limit $f \in F$. Then observe that for all n,

$$\Big(I - \frac{1}{Q(1)}Q(T)\Big)(u_n) = \frac{1}{Q(1)}\tilde{Q}(T)(u_n - Tu_n) \to \frac{1}{Q(1)}\tilde{Q}(T)(f) \quad \text{as } n \to \infty.$$

Again, applying the Fredholm Alternative Theorem, it follows that $f \in R(I - \frac{1}{Q(1)}Q(T))$, and so there exists some $u \in E$ such that

$$\begin{aligned} f &= \frac{1}{Q(1)}(Q(1)I - Q(T))(u) \\ &= \frac{1}{Q(1)}\tilde{Q}(T) \circ (I-T)(u) \\ &= (I-T)\Bigg(\tilde{Q}(T)\Big(\frac{1}{Q(1)}u\Big)\Bigg). \end{aligned}$$

Thus, $f \in R(I-T)$, proving that $R(I-T)$ is closed. Thus, $I-T$ satisfies property (A) of Exercise 6.9, and by the equivalence of property (C), it follows that $(I-T)(E_0)$ is closed for every closed subspace $E_0 \subset E$. ☐

2. Prove that $N(I-T) = \{0\} \iff R(I-T) = E$.

Proof. Suppose that $N(I-T) = \{0\}$ and assume for a contradiction that $E_1 = R(I-T) \neq E$. Then since E_1 is closed by part 1, E_1 is a Banach space and we have that $T(E_1) \subset E_1$. Thus $E_2 = (I-T)(E_1) \subset E_1$ is a closed subspace of E_1 (by part 1). Since $I-T$ is injective, $E_2 \neq E_1$. Letting $E_n = (I-T)^n(E)$, we obtain a (strictly) decreasing sequence of closed subspaces. Using Riesz's lemma we may construct a sequence (u_n) such that $u_n \in E_n$, $\|u_n\| = 1$ and $\operatorname{dist}(u_n, E_{n+1}) \geq \frac{1}{2}$. We have

$$Q(T)u_n - Q(T)u_m = -(Q(1)u_n - Q(T)u_n) + (Q(1)u_m - Q(T)u_m) + (Q(1)u_n - Q(1)u_m).$$

Note that for any n, $Q(1)u_n - Q(T)u_n = \tilde{Q}(T) \circ (I-T)(u_n) = \tilde{Q}(T) \circ (I-T)((I-T)^n(u)) = (I-T)^{n+1}(\tilde{Q}(T)(u))$ for some $u \in E$, and so $Q(1)u_n - Q(T)u_n \in E_{n+1}$. Thus, if $n > m$, then since $E_{n+1} \subset E_n \subset E_{m+1} \subset E_m$, we have that

$$-(Q(1)u_n - Q(T)u_n) + (Q(1)u_m - Q(T)Tu_m) + Q(1)u_n \in E_{m+1}.$$

It follows that $\|Q(T)u_n - Q(T)u_m\| \geq \operatorname{dist}(Q(1)u_m, E_{m+1}) \geq \frac{|Q(1)|}{2}$, which contradicts the fact that $\overline{Q(T)(B_E)}$ is compact. Thus, by contradiction, $R(I-T) = E$.

Conversely, suppose that $R(I-T) = E$. Then by Corollary 2.18, we have that $N(I-T^\star) = R(I-T)^\perp = \{0\}$. Since $Q(T^\star) = (Q(T))^\star \in K(E^\star)$, we can apply the preceding step to conclude that $R(I-T^\star) = E^\star$, so that $N(I-T) = R(I-T^\star)^\perp = E^{\star\perp} = \{0\}$. ☐

3. Prove that $\dim N(I-T) = \dim N(I-T^\star)$.

Proof. Set $d = \dim N(I-T)$ and $d^\star = \dim N(I-T^\star)$. Towards a contradiction, suppose that $d < d^\star$. Since $N(I-T)$ is finite dimensional, there exists a closed complement $G \subset E$ such that $E = G \oplus N(I-T)$. It follows that there exists a continuous projection P from E onto $N(I-T)$. Since $R(I-T) = N(I-T^\star)^\perp$ (since $R(I-T)$ is closed) and $N(I-T^\star)$ is finite dimensional by applying part 1 to $Q(T^\star)$, it follows that $R(I-T)$ has finite codimension $d^\star$ in E and so there exists a complement F in E such that $E = R(I-T) \oplus F$ and $\dim F = d^\star$. Since $d < d^\star$, there exists an injection that is not surjective $\Lambda : N(I-T) \to F$. Set $S = T + \Lambda \circ P$. Observe that since $\Lambda \circ P$ has finite rank, $(\Lambda \circ P)^m \circ T^n$ and $T^n \circ (\Lambda \circ P)^m$ are finite rank operators for any $n, m \geq 1$. It follows that $Q(S) = \sum_{k=1}^{p} a_k(T + \Lambda \circ P)^k = Q(T) + \sum\{\text{finite-rank operators}\}$ is a compact operator. Since $N(I-S) = \{0\}$, it follows by part 2 that $R(I-S) = E$, which is absurd since $\Lambda \circ P$ is not surjective, so there is some $f \in F \setminus \Lambda \circ P(E)$, and noting that $F \cap R(I-T) = \{0\}$, therefore $f \notin R(I-T) + \Lambda \circ P(E) = R(I-S)$. Thus, by contradiction, $\dim N(I-T) \geq \dim N(I-T^\star)$. Applying this fact to $T^\star$, it follows that $\dim N(I-T^{\star\star}) \leq \dim N(I-T^\star) \leq \dim N(I-T)$. But $N(I-T^{\star\star}) \supset N(I-T)$ and so $N(I-T) = N(I-T^\star)$. ☐

6.11

Let K be a compact metric space, and let $E = C(K;\mathbb{R})$ equipped with the usual norm $\|u\| = \max_{x\in K}|u(x)|$. Let $F \subset E$ be a *closed* subspace. Assume that every function $u \in F$ is Hölder continuous, i.e.,

$$\begin{cases} \forall u \in F \quad \exists \alpha \in (0,1] \quad \text{and } \exists L \quad \text{such that} \\ |u(x) - u(y)| \leq L d(x,y)^\alpha \quad \forall x, y \in K. \end{cases}$$

The purpose of this exercise is to show that F is finite-dimensional.

1. Prove that there exist constants $\gamma \in (0,1]$ and $C \geq 0$ (both independent of u) such that

$$|u(x) - u(y)| \leq C\|u\| d(x,y)^\gamma \quad \forall u \in F, \quad \forall x, y \in K.$$

Proof. For each $n \geq 1$ define $F_n = \{u \in F : |u(x) - u(y)| \leq n d(x,y)^{1/n} \quad \forall x, y \in K\}$. Observe that each F_n is closed in F. Indeed, if $(u_k) \subset F_n$ is a convergent sequence with limit $u \in F$, fix $\varepsilon > 0$ and pick k such that $\|u_k - u\| < \varepsilon$. Then, for all $x, y \in K$, we have that $|u(x)-u(y)| \leq |u(x)-u_k(x)|+|u_k(x)-u_k(y)|+|u_k(y)-u(y)| \leq 2\varepsilon + n d(x,y)^{1/n}$. Thus, $u \in F_n$ verifying that F_n is a closed subset of F. Moreover, since every $u \in F$ is Hölder continuous, it's clear that for every $u \in F$, there exists some $n \geq 1$ such that $u \in F_n$. Thus, $F = \bigcup_{n=1}^\infty F_n$. Since F is a closed subspace of a complete metric space, F is a complete metric space and so by the Baire category theorem, there must exist some n such that $\text{Int}(F_n) \neq \varnothing$. It follows that there exists some $\varepsilon > 0$ and $u \in F_n$ such that $B(u,\varepsilon) \cap F \subset F_n$. Fix nonzero $v \in F$ and pick $\delta = \frac{\varepsilon}{2\|v\|}$. Since $u + \delta v \in B(u,\varepsilon) \cap F$, we have that $|u(x)+\delta v(x) - u(y) - \delta v(y)| \leq n d(x,y)^{1/n}$ for all $x, y \in K$. Thus,

$$\begin{aligned} \delta |v(x) - v(y)| &\leq |u(x) - u(y)| + n d(x,y)^{1/n} \\ &\leq 2n d(x,y)^{1/n}. \end{aligned}$$

The result follows with $C = 4n/\varepsilon$ and $\gamma = 1/n$. ☐

2. Prove that B_F is compact and conclude.

Proof. For any $u \in B_F$, we have that $|u(x) - u(y)| \leq C d(x,y)^\gamma$, and so clearly B_F is equicontininuous. Since B_F is also equibounded, being a subset of the closed unit ball in $C(K)$, $\overline{B_F}$ is a compact subset of $C(K)$ by Arzelà-Ascoli. Since $B_F = F \cap B_E$, and both F and B_E is closed, it follows that $B_F = \overline{B_F}$ is compact. Because the closed unit ball in a Banach space is compact iff the Banach space is finite dimensional, it follows that F is finite dimensional. ☐

6.12 A lemma of J.-L. Lions

Let X, Y, and Z be three Banach spaces with norms $\| \quad \|_X$, $\| \quad \|_Y$, and $\| \quad \|_Z$. Assume that $X \subset Y$ with *compact* injection and that $Y \subset Z$ with *continuous* injection. Prove that

$$\forall \varepsilon > 0 \, \exists C_\varepsilon \geq 0 \text{ satisfying } \|u\|_Y \leq \varepsilon \|u\|_X + C_\varepsilon \|u\|_Z \quad \forall u \in X.$$

Proof. Towards a contradiction, suppose not. Then there exists some $\varepsilon > 0$ such that for all $t \geq 0$, $\|u_t\|_Y > \varepsilon\|u_t\|_X + t\|u_t\|_Z$ for some $u_t \in X$. Thus, we can construct a sequence $(u_n) \subset X$ such that $\|u_n\|_Y > \varepsilon\|u_n\|_X + n\|u_n\|_Z$ for all n. Moreover, by possibly rescaling, we may assume WLOG that $\|u_n\|_Y = 1$ for all n. Since for all n, we have that $\varepsilon\|u_n\|_X < 1 - n\|u_n\|_Z \leq 1$, by the compactness of the injection $X \subset Y$, there exists a subsequence (u_{n_k}) and some $u \in \frac{1}{\varepsilon}\overline{B_X}^Y$ such that $\|u_{n_k} - u\|_Y \to 0$ as $k \to \infty$. Moreover, since the injection $Y \subset Z$ is continuous, it follows that $\|u_{n_k} - u\|_Z \to 0$ as $k \to \infty$. Because $\|u_n\|_Z < \frac{1-\varepsilon\|u_n\|_X}{n}$ for all n, $u_n \to 0$ in Z so that $u = 0$. But then $1 = \|u_{n_k}\|_Y = \|u_{n_k} - u\|_Y \to 0$, which is absurd. Thus, the statement follows by contradiction. ☐

6.14

Let E be a Banach space, and let $T \in \mathcal{L}(E)$ with $\|T\| < 1$.

1. Prove that $(I - T)$ is bijective and that

$$\|(I-T)^{-1}\| \leq 1/(1 - \|T\|).$$

Proof. By Proposition 6.7, $\sigma(T) \subset [-\|T\|, +\|T\|] \subset (-1, 1)$. Thus, $1 \in \mathbb{R} \setminus \sigma(T) = \rho(T)$. By the definition of the resolvent set $\rho(T)$ of T, it follows that $T - I$ is bijective from E onto itself, and so $I - T = -(T - I)$ is bijective. Moreover, for any $u \in E$,

$$\begin{aligned}\|(I-T)^{-1}u\| &\leq \|(I-T)^{-1}(u - Tu)\| + \|(I-T)^{-1}Tu\| \\ &\leq \|u\| + \|(I-T)^{-1}\|\|T\|\|u\| \\ &= (1 + \|(I-T)^{-1}\|\|T\|)\|u\|.\end{aligned}$$

It follows that $\|(I-T)^{-1}\| \leq 1 + \|(I-T)^{-1}\|\|T\|$. The desired inequality follows after rearranging. □

2. Set $S_n = I + T + \cdots + T^{n-1}$. Prove that

$$\|S_n - (I-T)^{-1}\| \leq \|T\|^n / (1 - \|T\|).$$

Proof. Observe that $(S_n - (I-T)^{-1}) \circ (I - T) = (I - T^n) - I = T^n$. Thus, for all $u \in E$ we have that

$$\begin{aligned}\|(S_n - (I-T)^{-1})u\| &\leq \|(S_n - (I-T)^{-1})(I-T)u\| + \|S_n - (I-T)^{-1}\|\|T\|\|u\| \\ &= \|T^n u\| + \|S_n - (I-T)^{-1}\|\|T\|\|u\| \\ &\leq \|T^n\|\|u\| + \|S_n - (I-T)^{-1}\|\|T\|\|u\| \\ &\leq \|T\|^n\|u\| + \|S_n - (I-T)^{-1}\|\|T\|\|u\| \\ &= (\|T\|^n + \|S_n - (I-T)^{-1}\|\|T\|)\|u\|,\end{aligned}$$

where the third line is justified inductively by noting that for any $u \in E$, $\|T^n u\| \leq \|T\|\|T^{n-1}u\|$. It follows that $\|S_n - (I-T)^{-1}\| \leq= \|T\|^n + \|S_n - (I-T)^{-1}\|\|T\|$. The desired inequality follows after rearranging. □

6.15

Let E be a Banach space and let $T \in \mathcal{L}(E)$.

1. Let $\lambda \in \mathbb{R}$ be such that $|\lambda| > \|T\|$. Prove that

$$\|I + \lambda(T - \lambda I)^{-1}\| \leq \|T\| / (|\lambda| - \|T\|).$$

Proof. For all $u \in E$, we have that

$$\begin{aligned}\|(I + \lambda(T - \lambda I)^{-1})u\| &= \|(I + \lambda(T - \lambda I)^{-1})((u - \frac{1}{\lambda}Tu) + \frac{1}{\lambda}Tu)\| \\ &\leq \|(u - \frac{1}{\lambda}Tu) - u\| + \frac{1}{|\lambda|}\|I + \lambda(T - \lambda I)^{-1}\|\|T\|\|u\| \\ &\leq \frac{1}{|\lambda|}(\|T\| + \|I + \lambda(T - \lambda I)^{-1}\|\|T\|)\|u\|.\end{aligned}$$

It follows that $|\lambda|\|I + \lambda(T - \lambda I)^{-1}\| \leq \|T\| + \|I + \lambda(T - \lambda I)^{-1}\|\|T\|$. The desired inequality follows after rearranging. □

2. Let $\lambda \in \rho(T)$. Check that

$$(T - \lambda I)^{-1}T = T(T - \lambda I)^{-1},$$

and prove that

$$\operatorname{dist}(\lambda, \sigma(T)) \geq 1 / \|(T - \lambda I)^{-1}\|.$$

Proof. Fix $u \in E$ and set $f = (T - \lambda I)^{-1}u$ so that $Tf - \lambda f = u$. Then

$$\begin{aligned} T(T-\lambda I)^{-1}u &= Tf = u + \lambda f = u + \lambda(T-\lambda I)^{-1}u \\ &= (T-\lambda I)^{-1}(Tu - \lambda u) + \lambda(T-\lambda I)^{-1}u \\ &= (T-\lambda I)^{-1}Tu. \end{aligned}$$

This gives the first part of the problem.

Towards proving the second part, fix $\gamma \in \mathbb{R}$ with $|\lambda - \gamma| < 1/\|(T-\lambda I)^{-1}\|$. Fix $f \in E$. To show that $\gamma \notin \sigma(T)$, we want to show that the equation $Tu - \gamma u = f$ has a unique solution for some $u \in E$. Write $Tu - \lambda u = f + (\lambda u - \gamma u)$, so that $u = (T-\lambda I)^{-1}(f + (\lambda - \gamma)u)$. Define $K_f : E \to E$ by $K_f(u) = (T-\lambda I)^{-1}(f+(\lambda-\gamma)u)$. Clearly it suffices to prove that K_f has a unique fixed point. Observe that $\|K_f(u_1) - K_f(u_2)\| \le |\lambda - \gamma| \|(T-\lambda I)^{-1}\| \|u_1 - u_2\| < \|u_1 - u_2\|$. By the Banach Fixed Point Theorem, it follows that K_f has a unique fixed point and so $T - \gamma I$ is bijective, proving that $\gamma \notin \sigma(T)$. It follows that $\operatorname{dist}(\lambda, \sigma(T)) \ge 1/\|(T-\lambda I)^{-1}\|$. ☐

3. Assume that $0 \in \rho(T)$. Prove that

$$\sigma(T^{-1}) = 1/\sigma(T).$$

Proof. Fix $\lambda \in \sigma(T)$. Since $T - \lambda I$ is not bijective, either there exists $f \in E \setminus R(T - \lambda I)$, or there exists $u_1 \neq u_2$ such that $Tu_1 - \lambda u_1 = Tu_2 - \lambda u_2$. In the first case, we have that for all $u \in E$, $Tu - \lambda u \neq f$, so that for all $u \in E$, $T^{-1}u - \frac{1}{\lambda}u \neq -\frac{1}{\lambda}T^{-1}f$. Thus, $-\frac{1}{\lambda}T^{-1}f \notin R(T^{-1} - \frac{1}{\lambda}I)$, proving that $\frac{1}{\lambda} \in \sigma(T^{-1})$. In the second case, we have that $T^{-1}u_1 - \frac{1}{\lambda}u_1 = T^{-1}u_2 - \frac{1}{\lambda}u_2$, so that $T^{-1} - \frac{1}{\lambda}I$ is not injective and therefore $\frac{1}{\lambda} \in \sigma(T^{-1})$. This proves that $1/\sigma(T) \subset \sigma(T^{-1})$. Applying the preceding reasoning to T^{-1} in place of T, we have that $1/\sigma(T^{-1}) \subset \sigma(T)$, which is equivalent to saying that $\sigma(T^{-1}) \subset 1/\sigma(T)$. The statement follows. ☐

In what follows assume that $1 \in \rho(T)$; set

$$U = (T+I)(T-I)^{-1} = (T-I)^{-1}(T+I).$$

4. Check that $1 \in \rho(U)$ and give a simple expression for $(U-I)^{-1}$ in terms of T.

Proof. Observe that

$$\begin{aligned} U - I &= T(T-I)^{-1} + (T-I)^{-1} - I \\ &= T(T-I)^{-1} + (T-I)^{-1} - (T-I)(T-I)^{-1} \\ &= 2(T-I)^{-1}. \end{aligned}$$

Thus, $(U-I)^{-1} = \frac{1}{2}(T-I)$ and $1 \in \rho(U)$. ☐

5. Prove that $T = (U+I)(U-I)^{-1}$.

Proof. $(U+I)(U-I)^{-1} = \frac{1}{2}((T+I)(T-I)^{-1} + I)(T-I) = T$. ☐

6. Consider the function $f(t) = (t+1)/(t-1), t \in \mathbb{R}$. Prove that

$$\sigma(U) = f(\sigma(T)).$$

Proof. Fix $\lambda \in \mathbb{R}$. Observe that $\lambda \in \sigma(U)$ iff it is not the case that the equation $Uu - \lambda u = f$ has a unique solution $u \in E$ for every $f \in E$. Moreover, we have that

$$\begin{aligned} &Uu - \lambda u = f \\ &\iff -\frac{2}{1-\lambda}Uu + \frac{2\lambda}{1-\lambda}u = -\frac{2}{1-\lambda}f \\ &\iff (U+I)u - \frac{\lambda+1}{\lambda-1}(U-I)u = \frac{1}{1-\lambda}\Big((U-I)f - (U+I)f\Big) \\ &\iff (U-I)^{-1}(U+I)u - \frac{\lambda+1}{\lambda-1}u = \frac{1}{1-\lambda}\Big(f - (U-I)^{-1}(U+I)f\Big) \\ &\iff Tu - \frac{\lambda+1}{\lambda-1}u = \frac{1}{\lambda-1}(Tf - f). \end{aligned}$$

Thus, $f \notin R(U - \lambda I)$ iff $\frac{1}{\lambda - 1}(Tf - f) \notin R(T - \frac{\lambda+1}{\lambda-1} I)$, and similary $U - \lambda I$ is not injective iff $T - \frac{\lambda+1}{\lambda-1} I$ is not injective. It follows that $\sigma(T) = f(\sigma(U))$. Rearranging, we have that $\sigma(U) = f(\sigma(T))$. □

6.16

Let E be a Banach space and let $T \in \mathcal{L}(E)$.

1. Assume that $T^2 = I$. Prove that $\sigma(T) \subset \{-1, +1\}$ and determine $(T - \lambda I)^{-1}$ for $\lambda \neq \pm 1$.

   Proof. Since $T^2 = I$, T is bijective and so $0 \in \rho(T)$. By Exercise 6.15, $\sigma(T) = \sigma(T^{-1}) = 1/\sigma(T)$, so that $\sigma(T)^2 = 1$. It follows that $\sigma(T) \subset \{-1, +1\}$. Fix $\lambda \neq \pm 1$ and $u \in E$. Observe that

   $$(T + \lambda I)(T - \lambda I)u = T^2 u - \lambda^2 u = (1 - \lambda^2)u,$$

   and so $(T - \lambda I)^{-1} = \frac{1}{1-\lambda^2}(T + \lambda I)$. □

2. More generally, assume that there is an integer $n \geq 2$ such that $T^n = I$. Prove that $\sigma(T) \subset \{-1, +1\}$ and determine $(T - \lambda I)^{-1}$ for $\lambda \neq \pm 1$.

   Proof. Fix $\lambda \neq \pm 1$. Observe that

   $$\begin{aligned} \left(\sum_{k=0}^{n-1} \lambda^{n-k-1} T^k \right)(T - \lambda I) &= \sum_{k=1}^{n} \lambda^{n-k} T^k - \sum_{k=0}^{n-1} \lambda^{n-k} T^k \\ &= T^n - \lambda^n I \\ &= (1 - \lambda^n) I. \end{aligned}$$

   It follows that $\sigma(T) \subset \{-1, +1\}$ and $(T - \lambda I)^{-1} = \frac{1}{1-\lambda^n} \sum_{k=0}^{n-1} \lambda^{n-k-1} T^k$. □

3. Assume that there is an integer $n \geq 2$ such that $T^n = 0$. Prove that $\sigma(T) = \{0\}$ and determine $(T - \lambda I)^{-1}$ for $\lambda \neq 0$.

   Proof. Fix $\lambda \neq 0$ and observe that

   $$\left(\sum_{k=0}^{n-1} \lambda^{n-k-1} T^k \right)(T - \lambda I) = T^n - \lambda^n I = -\lambda^n I.$$

   Thus, $\sigma(T) \subset \{0\}$ and $(T - \lambda I)^{-1} = -\sum_{k=0}^{n-1} \lambda^{-k-1} T^k$. Since $T^n = 0$, T cannot be injective (since if n is the least integer such that $T^n = 0$, then there exists $u \in E$ with $T^{n-1}u \neq 0$ and $T(T^{n-1}u) = 0$) and so $\sigma(T) = \{0\}$. □

4. Assume that there is an integer $n \geq 2$ such that $\|T^n\| < 1$. Prove that $I - T$ is bijective and give an expression for $(I - T)^{-1}$ in terms of $(I - T^n)^{-1}$ and the iterates of T.

   Proof. Since $\|T^n\| < 1$ and $\sigma(T^n) \subset [-\|T^n\|, \|T^n\|] \subset (-1, 1)$ by Proposition 6.7, it follows that $1 \in \rho(T^n)$ and therefore that $I - T^n$ is invertible. Observe that

   $$(I - T)\left(\sum_{k=0}^{n-1} T^k \right)(I - T^n)^{-1} = (I - T^n)(I - T^n)^{-1} = I.$$

   Thus, $(I - T)$ is bijective and $(I - T)^{-1} = \left(\sum_{k=0}^{n-1} T^k \right)(I - T^n)^{-1}$. □

6.17

Let $E = \ell^p$ with $1 \leq p \leq \infty$ and let (λ_n) be a bounded sequence in $\mathbb{R}$. Consider the multiplication operator $M \in \mathcal{L}(E)$ defined by

$$Mx = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots), \quad \text{where } x = (x_1, x_2, \ldots, x_n, \ldots).$$

Determine $EV(M)$ and $\sigma(M)$.

Solution

Observe that for any n, we have that $Mx - \lambda_n x = ((\lambda_1 - \lambda_n)x_1, \dots, (\lambda_{n-1} - \lambda_n)x_{n-1}, 0, (\lambda_{n+1} - \lambda_n)x_{n+1}, \dots)$ and so $e_n \in N(M - \lambda_n I)$. Thus, $\bigcup_n \{\lambda_n\} \subset EV(M)$. Moreover, if $\lambda \notin \bigcup_n \{\lambda_n\}$, then $Mx - \lambda x = 0$ implies that for each n, $(\lambda_n - \lambda)x_n = 0$, and since $\lambda \neq \lambda_n$, we must have that $x_n = 0$. Thus, $N(M - \lambda I) = \{0\}$, proving that $\bigcup_n \{\lambda_n\} = EV(M)$.

I claim that $\sigma(M) = \overline{\bigcup_n \{\lambda_n\}}$. Since $EV(M) \subset \sigma(M)$ and $\sigma(M)$ is compact by Proposition 6.7, it follows that $\overline{\bigcup_n \{\lambda_n\}} \subset \sigma(M)$. Fix $\lambda \notin \overline{\bigcup_n \{\lambda_n\}}$ and observe that there exists some $C > 0$ such that $|\lambda - \lambda_n| \geq C$. Thus, if $1 \leq p < \infty$ then for any $x \in E$, we have that $\sum_{n=1}^{\infty} \left| \frac{1}{\lambda_n - \lambda} x_n \right|^p \leq \frac{1}{C^p} \|x\|_p^p < \infty$, so that $(\frac{1}{\lambda_n - \lambda} x_n)_{n \geq 1} \in \ell^p$. And if $p = \infty$ then for all n, $\left| \frac{1}{\lambda_n - \lambda} x_n \right| \leq \frac{1}{C} \|x\|_\infty$, so that $(\frac{1}{\lambda_n - \lambda} x_n)_{n \geq 1} \in \ell^\infty$. Since $(M - \lambda I)(\frac{1}{\lambda_n - \lambda} x_n)_{n \geq 1} = x$, it follows that $M - \lambda I$ is surjective. Since $\lambda \notin EV(M)$, $M - \lambda I$ is also injective and so $\lambda \in \rho(M)$, proving that $\sigma(M) \subset \overline{\bigcup_n \{\lambda_n\}}$, and the claim follows.

6.18 Spectral properties of the shifts.

An element $x \in E = \ell^2$ is denoted by $x = (x_1, x_2, \dots, x_n, \dots)$. Consider the operators

$$S_r x = (0, x_1, x_2, \dots, x_{n-1}, \dots),$$

and

$$S_\ell x = (x_2, x_3, \dots, x_{n+1}, \dots),$$

respectively called the *right shift* and *left shift*.

1. Determine $\|S_r\|$ and $\|S_\ell\|$. Does S_r or S_ℓ belong to $K(E)$?

   Solution

   Fix $x \in \ell^2$ and observe that $\|S_r x\|_2^2 = 0^2 + \sum_n x_n^2 = \|x\|_2^2$, and so S_r is an isometry and has operator norm $\|S_r\| = 1$. Moreover, $\|S_\ell x\|_2^2 = \sum_{n \geq 2} x_n^2 \leq \|x\|_2^2$, so that $\|S_\ell\| \leq 1$. Since $\|S_\ell e_2\| = 1$, it follows that $\|S_\ell\| = 1$. It's clear that $S_\ell(B_E) = B_E$ and since $\dim E = \infty$, B_E is not compact and so S_ℓ is not a compact operator. Moreover, since S_r is an isometry, its image is a closed subspace of ℓ^2 which includes the linearly independent subset $\bigcup_{n \geq 2} \{e_n\}$. Thus, $S_r(B_E) = B_{S_r(E)}$ and since $S_r(E)$ is an infinite dimensional Banach space, $B_{S_r(E)}$ is not compact so that S_r is not a compact operator.

2. Prove that $EV(S_r) = \varnothing$.

   Proof. Fix $\lambda \in \mathbb{R}$ and suppose that $x \in N(S_r - \lambda I)$. Then for all $n \geq 2$, $x_{n-1} - \lambda x_n = 0$ and $-\lambda x_1 = 0$. If $\lambda = 0$, we immediately get that $x = 0$, and if $\lambda \neq 0$, it follows by an obvious inductive argument that $x_n = 0$ for all n and so $N(S_r - \lambda I) = \{0\}$ for all $\lambda \in \mathbb{R}$. Thus, $EV(S_r) = \varnothing$. □

3. Prove that $\sigma(S_r) = [-1, +1]$.

   Proof. Since $\|S_r\| = 1$, it follows by Proposition 6.7 that $\sigma(S_r) \subset [-1, +1]$. Fix $\lambda \in [-1, +1]$. To show that $\lambda \in \sigma(S_r)$, it suffices to construct some $x \in \ell^2$ with $x \notin (S_r - \lambda I)(E)$. Clearly $(-\lambda, 0, 0, \dots) \in \ell^2$. I claim that $(-\lambda, 0, 0, \dots) \notin S_r(E)$. Suppose for a contradiction that there existed $x \in \ell^2$ such that $(S_r - \lambda I)(x) = (-\lambda, 0, 0, \dots)$. Then $x_1 = 1$ and $x_n = \lambda x_{n+1}$ for $n \geq 1$. Solving this relation recursively (and noting that $1 = \lambda x_2$ forces that $\lambda \neq 0$), we get that $x = (\frac{1}{\lambda^{n-1}})_{n \geq 1}$. But then $x \notin \ell^2$ since $\frac{1}{\lambda^{n-1}} \not\to 0$ as $n \to \infty$. Thus, $\lambda \in \sigma(S_r)$, proving that $\sigma(S_r) = [-1, +1]$. □

4. Prove that $EV(S_\ell) = (-1, +1)$. Determine the corresponding eigenspaces.

   Proof. Fix $\lambda \in \mathbb{R}$ and suppose that $\lambda \in EV(S_\ell)$. Then there exists nonzero $x \in \ell^2$ such that $x_{n+1} = \lambda x_n$ for all $n \geq 1$. Thus, $x_n = \lambda^{n-1} x_1$ for all $n \geq 1$, and since $x_n \to 0$ as $n \to \infty$, it follows that $\lambda \in (-1, +1)$, proving that $EV(S_\ell) \subset (-1, +1)$. Now fix $\lambda \in (-1, +1)$ and observe that $x = (\lambda^{n-1})_{n \geq 1} \in \ell^2$ (by, for example, the ratio test. If $\lambda = 0$, pick $x = e_1$) and $S_\ell x = 0$ so that $N(S_\ell - \lambda I) \supsetneqq \{0\}$. It follows that $\lambda \in EV(S_\ell)$, proving that $EV(S_\ell) = (-1, +1)$. Moreover, from our analysis, we see that for any $\lambda \in EV(S_\ell)$, the eigenspace E_λ associated to λ is given by $E_\lambda = \text{span}\{(\lambda^{n-1})_{n \geq 1}\}$. □

5. Prove that $\sigma(S_\ell) = [-1, +1]$.

Proof. We know by Proposition 6.7 and the fact that $\|S_\ell\| = 1$ that $\sigma(S_\ell) \subset [-1, +1]$. From part 4, we see that $(-1, +1) = EV(S_\ell) \subset \sigma(S_\ell)$. Thus, we only need to check that $\pm 1 \in \sigma(S_\ell)$. Observe that $x = (\frac{1}{n})_{n\geq 1} \in \ell^2$. I claim that $x \notin (S_\ell - I)(E)$. Indeed, if there existed some $y \in \ell^2$ such that $(S_\ell - I)(y) = x$, then we would have that $y_{n+1} = y_n + \frac{1}{n} = y_1 + \sum_{k=1}^n \frac{1}{k}$. But then $y_n \to +\infty$ as $n \to \infty$ so that $y \notin \ell^2$. Thus, $1 \in \sigma(S_\ell)$. To see that $-1 \in \sigma(S_\ell)$, observe that $x = ((-1)^n \frac{1}{n})_{n\geq 1} \in \ell^2$ and if $(S_\ell + I)(y) = x$, then we would have that $y_{n+1} = -y_n + (-1)^n \frac{1}{n} = (-1)^n (y_1 + \sum_{k=1}^n \frac{1}{k})$, which does not converge to 0 for any choice of y_1. Thus, $((-1)^n \frac{1}{n})_{n\geq 1} \notin (S_\ell + I)(E)$, proving that $-1 \in \sigma(S_\ell)$. It follows that $\sigma(S_\ell) = [-1, +1]$. ☐

6. Determine $S_r^\star$ and $S_\ell^\star$.

Solution

Observe that for any $x, y \in \ell^2$, we have that $(S_r x, y) = \sum_{n=1}^\infty x_n y_{n+1} = (x, S_\ell y)$, so that $S_r^\star = S_\ell$ and $S_\ell^\star = S_r$.

7. Prove that for every $\lambda \in (-1, +1)$, the spaces $R(S_r - \lambda I)$ and $R(S_\ell - \lambda I)$ are closed. Give an explicit representation of these spaces.

Proof. Fix $\lambda \in (-1, +1)$. Observe that $\|S_r x - \lambda x\|_2 \geq |\|S_r x\|_2 - |\lambda| \|x\|_2| = |\|x\|_2 - |\lambda| \|x\|_2| = (1 - |\lambda|)\|x\|_2$ Thus, if $(S_r x^n - \lambda x^n) \subset R(S_r - \lambda I)$ converges in ℓ^2 to $x \in \ell^2$, then x^n converges in ℓ^2 to some limit $y \in \ell^2$ and by the continuity of $(S_r - \lambda I)$, it follows that $(S_r - \lambda I)(y) = \lim_{n\to\infty} (S_r - \lambda I)(x^n) = x$, proving that $x \in R(S_r - \lambda I)$. Moreover, since $\operatorname{span}\{(\lambda^{n-1})_{n\geq 1}\} = E(S_\ell, \lambda) = N(S_\ell - \lambda I) = R(S_r - \lambda I)^\perp$, it follows that $R(S_r - \lambda I) = \operatorname{span}\{(\lambda^{n-1})_{n\geq 1}\}^\perp$. Finally, since $R(S_r - \lambda I)$ is closed, we can apply Theorem 2.19 to conclude that $R(S_\ell - \lambda I)$ is closed and $R(S_\ell - \lambda I) = N(S_r - \lambda I)^\perp = \varnothing^\perp = E$. ☐

8. Prove that the spaces $R(S_r \pm I)$ and $R(S_\ell \pm I)$ are dense and that they are not closed.

Proof. Since $R(S_r \pm I)$ and $R(S_\ell \pm I)$ are all subspaces of ℓ^2 and all subspaces of an n.v.s. are either closed or dense by Exercise 1.6, it suffices to prove that neither $R(S_r \pm I)$ nor $R(S_\ell \pm I)$ are closed. Observe that $R(S_r \pm I)$ are closed iff $R(S_\ell \pm I)$ are closed by Theorem 2.19, and so it suffices to prove that $R(S_r \pm I)$ are not closed. Applying Theorem 2.19 again, we see that if it were the case that $R(S_r \pm I)$ were closed, then we would have that $R(S_\ell \pm I) = N(S_r \pm I)^\perp = \varnothing^\perp = \ell^2$. But by our proof of part 5, we know that $R(S_\ell \pm I) \subsetneq \ell^2$, so that $R(S_r \pm I)$ cannot possibly be closed. The statement follows. ☐

Consider the multiplication operator M defined by

$$Mx = (\alpha_1 x_1, \alpha_2, x_2, \ldots, \alpha_n x_n, \ldots),$$

where (α_n) is a bounded sequence in $\mathbb{R}$.

9. Determine $EV(S_r \circ M)$.

Solution

Suppose for some $\lambda \in \mathbb{R}$ and $x \in \ell^2$, $(S_r \circ M - \lambda I)x = 0$. Then $-\lambda x_1 = 0$ and for all $n \geq 1$ $\alpha_n x_n = \lambda x_{n+1}$. If $\lambda \neq 0$ then $x_1 = 0$ and we get inductively that $x_n = 0$ for all n so that $x = 0$. Thus, $EV(S_r \circ M) \subset \{0\}$. If $\lambda = 0$, the only way we can satisfy $\alpha_n x_n = \lambda x_{n+1}$ for some nonzero x_n is if $\alpha_n = 0$, and so $EV(S_r \circ M) = \begin{cases} \varnothing & \text{if } 0 \notin (\alpha_n) \\ \{0\} & \text{otherwise.} \end{cases}$

10. Assume that $\alpha_n \to \alpha$ as $n \to \infty$. Prove that

$$\sigma(S_r \circ M) = [-|\alpha|, +|\alpha|].$$

Proof. Observe that if $\alpha = 0$, then by Exercise 6.1, M is compact so that $S_r \circ M$ is compact, and so by Theorem 6.8, $0 \in \sigma(S_r \circ M)$ and $\sigma(S_r \circ M) \setminus \{0\} = EV(S_r \circ M) \setminus \{0\} = \varnothing$, so the statement follows. Thus, we may assume WLOG that $\alpha \neq 0$. Now suppose that $\lambda \in \mathbb{R}$ and $|\lambda| > |\alpha|$. Observe that since $\alpha_n - \alpha \to 0$, by Exercise 6.1, $M - \alpha I$ is compact. Let $K = M - \alpha I$ and observe that $S_r \circ M - \lambda I = S_r \circ (K + \alpha I) - \lambda I = S_r \circ K + \alpha S_r - \lambda I$. By part 3, $\alpha S_r - \lambda I$ is bijective and so we can define the compact function $K_1 = (\alpha S_r - \lambda I)^{-1} \circ S_r \circ K$ to get that $S_r \circ M - \lambda I = (\alpha S_r - \lambda I) \circ (I + K_2)$. Since $\alpha S_r - \lambda I$ is bijective, it suffices to check that $I + K_2$ is surjective. By the Fredholm Alternative Theorem, $I + K_2$ is surjective iff $N(I + K_2) = \{0\}$. Again using the fact that $\alpha S_r - \lambda I$ is bijective, this holds iff $N(S_r \circ M - \lambda I) = N((\alpha S_r - \lambda I) \circ (I + K_2)) = \{0\}$, which is true by part 9. Thus, I have shown that $\sigma(S_r \circ M) \subset [-|\alpha|, +|\alpha|]$. Towards proving the opposite direction, fix $\lambda \in [-|\alpha|, +|\alpha|]$. For a contradiction, suppose that $S_r \circ M - \lambda I$ is bijective. Applying essentially the same trick as above, we can write $S_r - \frac{\lambda}{\alpha} I = \frac{1}{\alpha}(S_r \circ M - \lambda I) - \frac{1}{\alpha} S_r \circ K = \frac{1}{\alpha}(S_r \circ M - \lambda I)^{-1}(I - (S \circ M - \lambda I) \circ S_r \circ K) = J \circ (I + K_2)$ where J is bijective and K_2 is compact. By part 3, $S_r - \frac{\lambda}{\alpha} I$. Applying Theorem 6.6, we have that $N(I + K_2) = \{0\}$ iff $R(I + K_2) = E$, so that $S_r - \frac{\lambda}{\alpha} I$ is injective iff it is surjective. However, from part 2 and 3, we have that $S_r - \frac{\lambda}{\alpha} I$ is injective but not surjective. By contradiction, $\lambda \in \sigma(S_r \circ M)$, and the statement follows. ☐

11. Assume that for every integer n, $\alpha_{2n} = a$ and $\alpha_{2n+1} = b$ with $a \neq b$. Determine $\sigma(S_r \circ M)$.

Solution

Observe that $(S_r \circ M)^2 = (ab) S_r^2$ so that $\|(S_r \circ M)^2\| = |ab| \|S_r^2\| = |ab|$. Thus, if $|\lambda| > \sqrt{|ab|}$, then $\|(\frac{1}{\lambda} S_r \circ M)^2\| = \frac{|ab|}{\lambda^2} < 1$. By Exercise 6.16 part 4, it follows that $I - \frac{1}{\lambda} S \circ M$ is bijective so that $\lambda \notin \sigma(S_r \circ M)$. Thus, $\sigma(S_r \circ M) \subset [-\sqrt{|ab|}, +\sqrt{|ab|}]$. Conversely, if $\lambda \in [-\sqrt{|ab|}, +\sqrt{|ab|}]$, then observe that $(-1, 0, 0, \dots) \notin R(S_r \circ M - \lambda I)$ since writing $(-\lambda x_1, b x_2 - \lambda x_1, \dots) = (-1, 0, \dots)$, we see that $x_{2n+1} = \frac{(ab)^n}{\lambda^{2n}} = \left(\frac{ab}{\lambda^2}\right)^n$ which does not converge to 0. Thus, $\sigma(S_r \circ M) = [-\sqrt{|ab|}, +\sqrt{|ab|}]$.

6.19

Let E be a Banach space and let $T \in \mathcal{L}(E)$.

1. Prove that $\sigma(T^\star) = \sigma(T)$.

   Proof. Observe that if $\lambda \in \rho(T)$, applying Corollary 2.18, we have that $N(T^\star - \lambda I) = R(T - \lambda I)^\perp = E^\perp = \{0\}$ and since $R(T - \lambda I) = E$ is closed, by Theorem 2.19 $R(T^\star - \lambda I) = N(T - \lambda I)^\perp = \{0\}^\perp = E^\star$. Thus, $\lambda \in \rho(T^\star)$. Moreover, if $\lambda \in \sigma(T)$ then either $N(T^\star - \lambda I) = R(T - \lambda I)^\perp \supsetneq \{0\}$ or $\overline{R(T^\star - \lambda I)} \subset N(T - \lambda I)^\perp \subsetneq E^\star$, so that $\lambda \in \sigma(T^\star)$. Thus, $\sigma(T) = \sigma(T^\star)$. ☐

2. Give examples showing that there is no general inclusion relation between $EV(T)$ and $EV(T^\star)$,

   Proof. From Exercise 6.18, we have that $EV(S_r) = \varnothing \subsetneq (-1, +1) = EV(S_\ell) = EV(S_r^\star)$, and since $S_\ell^\star = S_r$, we also have that $EV(S_\ell^\star) \subsetneq EV(S_\ell)$, which verifies that there exist no general inclusion relations between $EV(T)$ and $EV(T^\star)$. ☐

6.20

Let $E = L^p(0,1)$ with $1 \le p < \infty$. Given $u \in E$, set

$$Tu(x) = \int_0^x u(t) dt.$$

1. Prove that $T \in K(E)$.

Proof. For any $u \in B_E$, extend Tu to $L^p(\mathbb{R})$ by setting $Tu(x) = 0$ for $x \notin [0,1]$. With this extension in mind, set $\mathcal{F} = T(B_E)$ to be a subset of $L^p(\mathbb{R})$. By Jensen's inequality, we have that for any $Tu \in \mathcal{F}$

$$\begin{aligned}\int_{\mathbb{R}} |Tu|^p dx &\leq \int_0^1 \left(\int_0^x |u(t)| dt \right)^p dx \\ &\leq \int_0^1 \int_0^x |u(t)|^p dt\, dx \\ &\leq \|u\|_p^p = 1.\end{aligned}$$

Thus, $\|Tu\|_p \leq 1$ for all $Tu \in \mathcal{F}$, proving that $\mathcal{F}$ is a bounded subset of $L^p(0,1)$. Observe that the above analysis also shows that $\|T\| \leq 1$. Towards proving that $\mathcal{F}$ is equicontinuous, fix $\varepsilon > 0$, $u \in B_E$, and a sequence of mollifiers (ρ_n). Then for $\delta > 0$ and $|h| < \delta$, we have that for all $n > \frac{1}{\delta}$, $\|(\rho_n \star u) - u\|_p < \varepsilon$ so that

$$\begin{aligned}\|\tau_h Tu - Tu\|_p^p &\leq \|\tau_h Tu - \tau_h T(\rho_n \star u)\|_p^p + \|\tau_h T(\rho_n \star u) - T(\rho_n \star u)\|_p^p + \|T(\rho_n \star u) - Tu\|_p^p \\ &\leq 2\varepsilon + \int_0^1 \left| \int_x^{x+h} (\rho_n \star u)(t) dt \right|^p dx \\ &\leq 2\varepsilon + h^p \|\rho_n \star u\|_\infty^p \\ &\leq 2\varepsilon + h^p \|\nabla \rho_n\|_{p'} \|u\|_p \\ &\leq 2\varepsilon + h^p \|\nabla \rho_n\|_{p'}.\end{aligned}$$

It follows that $\|\tau_h Tu - Tu\|_p^p \leq 2\varepsilon + h^p \|\nabla \rho_n\|_{p'}$ for all $Tu \in \mathcal{F}$ for n fixed. Thus, $\lim_{h \to 0} \|\tau_h Tu - Tu\|_p^p \leq 2\varepsilon$ uniformly in h over $Tu \in \mathcal{F}$, and since $\varepsilon > 0$ was arbitrary, it follows that $\|\tau_h Tu - Tu\|_p \to 0$ uniformly in h over $Tu \in \mathcal{F}$. By the Fréchet-Kolmogorov theorem, it follows that $\mathcal{F} = T(B_E)$ is a compact subset of $L^p(0,1)$. ☐

2. Determine $EV(T)$ and $\sigma(T)$.

Solution

Suppose for some $u \in L^p(0,1)$ and $\lambda \in \mathbb{R} \setminus \{0\}$, we have $\int_0^x u(t) dt = \lambda u(x)$ for almost all x. Then $u \in C([0,1])$ and is differentiable with $u(x) = \lambda u'(x)$. Solving this differential equation we get that $u(x) = Ce^{\lambda x}$ for some constant C. But then we have that $C = u(0) = \int_0^0 u(t) dt = 0$, so that $u = 0$. It follows that $EV(T) \subset \{0\}$. And if $\lambda = 0$, then $\int_0^x u(t) dt = 0$ for almost all x implies that $u \equiv 0$ (since Tu is continuous so that $\int_0^x u(t) dt = 0$ for all $x \in [0,1]$). Thus, $EV(T) = \varnothing$. Since T is compact, $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\} = \varnothing$, so that $\sigma(T) \subset \{0\}$. Since $T(E) \subset C([0,1]) \subsetneq E$, it follows that $\sigma(T) = \{0\}$.

3. Give an explicit formula for $(T - \lambda I)^{-1}$ when $\lambda \in \rho(T)$.

Solution

Fix $\lambda \in \rho(T)$, so that $\lambda \neq 0$, and $f \in C([0,1])$. Set $u = (T - \lambda I)^{-1} f$. Then $f(x) = \int_0^x u(t) dt - \lambda u(x)$, or equivalently $f(x) = v(x) - \lambda v'(x)$, $v(0) = 0$, where $v(x) = \int_0^x u(x) dx$. Solving this initial value problem, we get that $u(x) = -\frac{1}{\lambda^2} e^{\frac{1}{\lambda} x} T(f \cdot \exp^{1/\lambda})(x) - \frac{1}{\lambda} f(x)$. By the density of $C([0,1])$ in $L^p(0,1)$ and the continuity of the expression with respect to f, it follows that $(T - \lambda I)^{-1} f(x) = -\frac{1}{\lambda^2} e^{\frac{1}{\lambda} x} T(f \cdot \exp^{1/\lambda})(x) - \frac{1}{\lambda} f(x)$ for all $f \in L^p(0,1)$.

4. Determine $T^\star$.

Solution

Fix $u \in L^p(0,1)$ and $v \in L^{p'}(0,1)$. We have that

$$\begin{aligned}\int_0^1 Tu(x)v(x)dx &= \int_0^1 \int_0^x u(t)dt\, v(x)dx \\ &= \iint_{[0,1]^2} u(t)v(x)\chi_{[0,x]}(t,x)dt\,dx \\ &= \iint_{[0,1]^2} u(t)v(x)\chi_{[t,1]}(t,x)dx\,dt \\ &= \int_0^1 u(t)\int_t^1 v(x)dx\,dt.\end{aligned}$$

Thus, $T^\star v(x) = \int_x^1 v(t)dt$.

6.22

Let E be a Banach space, and let $T \in \mathcal{L}(E)$. Given a polynomial $Q(t) = \sum_{k=0}^p a_k t^k$ with $a_k \in \mathbb{R}$, let $Q(T) = \sum_{k=0}^p a_k T^k$.

1. Prove that $Q(EV(T)) \subset EV(Q(T))$.

   Proof. Fix $\lambda \in EV(T)$. By definition, there exists some nonzero $u \in E$ with $Tu = \lambda u$. Observe that $Q(T)u = \sum_{k=0}^p a_k\lambda^k u = Q(\lambda)u$, so that $u \in N(Q(T) - Q(\lambda)I)$. Thus, $Q(\lambda) \in EV(Q(T))$ and it follows that $Q(EV(T)) \subset EV(Q(T))$. □

2. Prove that $Q(\sigma(T)) \subset \sigma(Q(T))$.

   Proof. Suppose that $\lambda \in Q(\sigma(T))$. By part 1, we may assume WLOG that $T - \lambda I$ is injective, so that $T - \lambda I$ must not be surjective. Observe that since the polynomial $P(t) = Q(t) - Q(\lambda)$ has λ as a root, there exists some polynomials $\tilde{Q}$ such that $(t-\lambda)\tilde{Q}(t) = P(t) = Q(t) - Q(\lambda)$. Thus, we have that $(T - \lambda I)\circ\tilde{Q}(T) = Q(T) - Q(\lambda)I$, and it is immediate that $Q(T) - Q(\lambda)I$ is not surjective since $T - \lambda I$ is not surjective. It follows that $Q(\lambda) \in \sigma(Q(T))$, proving that $Q(\sigma(T)) \subset \sigma(Q(T))$. □

3. Construct an example in $E = \mathbb{R}^2$ for which the above inclusions are strict.

   Solution

   Pick A to be rotation by $\pi/2$ and observe that A has no eigenvalues but $A^2 = -I$ has the eigenvalue -1. Since $\sigma(T) = EV(T)$ whenever $\dim E < \infty$, this example works for both the spectrum and the set of eigenvalues.

   In what follows we assume that E is a Hilbert space (identified with its dual space $H^\star$) and that $T = T^\star$.

4. Assume here that the polynomial Q has no real root, i.e., $Q(t) \neq 0 \quad \forall t \in \mathbb{R}$. Prove that $Q(T)$ is bijective.

   Proof. Fix $\lambda > 0$ and let $t^2 + bt + c$ be a polynomial in $\mathbb{R}$ with no real roots. Define the bilinear form $a : E \times E \to \mathbb{R}$; $(u,v) \mapsto (T^2u + bTu + cu, T^2v + bTv + cv)$. Clearly a is a bounded bilinear form by the continuity of $T^2 + bT + cI$ and for any $u \in E$, we have that

$$\begin{aligned}a(u,u) &= ((T + \frac{b}{2}I)^2u + (c - \frac{b^2}{4})u, (T + \frac{b}{2}I)^2u + (c - \frac{b^2}{4})u) \\ &= |(T + \frac{b}{2}I)^2u|^2 + 2(c - \frac{b^2}{4})|(T + \frac{b}{2}I)u|^2 + (c - \frac{b^2}{4})^2|u|^2 \\ &\geq (c - \frac{b^2}{4})^2|u|^2\end{aligned}$$

   , so that a is also coercive. Thus, by the Lax-Milgram Theorem, for each $f \in E$, there exists a unique element $u \in E$ such that $(T^2u + bTu + cu, T^2v + bTv + cv) = a(u,v) = (f, T^2v + bTv + cv)$ for all $v \in E$, proving that $T^2 + bT + cI$ is bijective. Since every polynomial in $\mathbb{R}$ with no real roots can be decomposed as $c(t^2 + b_1 t + c_1)^{n_1} \cdots (t^2 + b_k t + c_k)^{n_k}$ for some $b_1, \ldots, b_k, c_1, \ldots, c_k \in \mathbb{R}$ such that $c_i - \frac{b_i^2}{4} > 0$ for each $i \in \{1, \ldots, k\}$ and $c \neq 0$, it follows that $Q(T) = c(T^2 + b_1 T + c_1 I)^{n_1} \circ \cdots \circ (T^2 + b_k T + c_k I)^{n_k}$ which, from our above analysis, is the composition of bijections and is therefore bijective. ☐

5. Deduce that for *every* polynomial Q, we have
   (i) $Q(EV(T)) = EV(Q(T))$,
   (ii) $Q(\sigma(T)) = \sigma(Q(T))$.

   Proof. Fix $\lambda \in EV(Q(T))$. Then since $Q(T) - \lambda I$ is not injective, it follows from part 4 that the polynomial $Q(t) - \lambda$ must have a root $\alpha \in \mathbb{R}$. Thus, we can decompose the polynomial $Q(t) - \lambda$ as $(t - \alpha_1)^{n_1} \ldots (t - \alpha_k)^{n_k} \tilde{Q}(t) = Q(t) - \lambda$ for some polynomial $\tilde{Q}(t)$ with no real roots, and such that $n_1 \geq 1$. Then $\tilde{Q} \circ (T - \alpha_1 I)^{n_1} \circ \cdots \circ (T - \alpha_k I)^{n_k}(T) = Q(T) - \lambda I$. By part 4, we know that $\tilde{Q}(T)$ is bijective and since $Q(T) - \lambda I$ is not injective, it must be the case that $(T - \alpha_i I)^{n_i}$ is not injective for some i, so that $T - \alpha_i I$ is not injective. Thus, $\alpha_i \in EV(T)$. Since α_i is a root of $Q(t) - \lambda$, it follows that $Q(\alpha_i) = \lambda$, so that $\lambda \in Q(EV(T))$. It follows that $Q(EV(T)) = EV(Q(T))$. Notice that replacing "injective" above with "bijective" proves part (ii). ☐

6.23 Spectral radius.

Let E be a Banach space and let $T \in \mathcal{L}(E)$. Set

$$a_n = \log \|T^n\|, \quad n \geq 1.$$

1. Check that
$$a_{i+j} \leq a_i + a_j \quad \forall i, j \geq 1.$$

   Proof. Fix $u \in E$ and $i, j \geq 1$, and observe that
$$\begin{aligned} \|T^{i+j} u\| &= \|T^i (T^j u)\| \\ &\leq \|T^i\| \|T^j\| \|u\|, \end{aligned}$$
   so that $a_{i+j} = \log \|T^{i+j}\| \leq \log(\|T^i\| \|T^j\|) = a_i + a_j$. ☐

2. Deduce that
$$\lim_{n \to +\infty} (a_n / n) \text{ exists and coincides with } \inf_{m \geq 1} (a_m / m).$$

   Proof. Fix $m \geq 1$ and for any n, let $n = mq + r$ where r is the remainder when dividing n by m, so that $0 \leq m < r$. Thus, we have that $a_n \leq a_{mq} + a_r \leq q a_m + a_r \leq \frac{n}{m} a_m + a_r$, so that $\frac{a_n}{n} \leq \frac{a_m}{m} + \frac{a_r}{n} \leq \frac{a_m}{m} + \frac{1}{n} \max_{i \in \{0, \ldots, m-1\}} |a_i|$. It follows that $\limsup_{n \to +\infty} \frac{a_n}{n} \leq \frac{a_m}{m}$. Since this inequality holds for all $m \geq 1$, we have that $\limsup_{n \to +\infty} \frac{a_n}{n} \leq \inf_{m \geq 1} \frac{a_m}{m} \leq \liminf_{m \to +\infty} \frac{a_m}{m}$. Thus, $\lim_{n \to +\infty} (a_n / n)$ exists and is equal to $\inf_{m \geq 1} (a_m / m)$. ☐

3. Conclude that $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}$ exists and that $r(T) \leq \|T\|$. Construct an example in $E = \mathbb{R}^2$ such that $r(T) = 0$ and $\|T\| = 1$. The number $r(T)$ is called the *spectral radius* of T.

   Proof. Since $\lim_{n \to \infty} \frac{1}{n} \log \|T^n\| = \lim_{n \to \infty} \log \|T^n\|^{1/n}$ exists from part 2, by the continuity of exp, it follows that $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = \lim_{n \to \infty} \exp(\log \|T^n\|^{1/n})$ exists. Moreover, observe that $r(T) = \exp(\lim_{n \to \infty} a_n / n) \leq \exp(\inf_{m \geq 1} a_m / m) \leq \exp(a_1) = \|T\|$. Take
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
   and observe that $Ae_2 = e_1$ and $A^2 = 0$, so that $A^n = 0$ for all $n \geq 2$. It follows that $\|A\| = 1$ and $r(A) = 0$. ☐

4. Prove that $\sigma(T) \subset [-r(T), +r(T)]$. Deduce that if $\sigma(T) \neq \varnothing$, then
$$\max\{|\lambda| : \lambda \in \sigma(T)\} \leq r(T).$$

Proof. By Exercise 6.22, we have that $\sigma(T)^n \subset \sigma(T^n) \subset [-\|T^n\|, +\|T^n\|]$, so that $\sigma(T) \subset [-\|T^n\|^{1/n}, +\|T^n\|^{1/n}]$ for all $n \geq 1$. It follows that $\sigma(T) \subset \bigcap_{n\geq 1}[-\|T^n\|^{1/n}, +\|T^n\|^{1/n}] \subset [-r(T), +r(T)]$. Thus, if there exists some $\lambda \in \sigma(T)$, then $\lambda \in [-r(T), +r(T)]$, so that $|\lambda| \leq r(T)$, and it follows that $\max\{|\lambda| : \lambda \in \sigma(T)\} \leq r(T)$. □

5. Construct an example in $E = \mathbb{R}^3$ such that $\sigma(T) = \{0\}$, while $r(T) = 1$.

Solution

Fix the standard basis for $\mathbb{R}^3$ and take

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then since A is a projection onto the x, y plane, followed by rotation in the x, y plane by $\pi/2$, it's clear that A has no eigenvalues and that $\|A^n\| = 1$ for all n, so that $r(A) = 1$.

In what follows we take $E = \mathcal{L}^p(0,1)$ with $1 \leq p \leq \infty$. Consider the operator $T \in \mathcal{L}(E)$ defined by

$$Tu(t) = \int_0^t u(s)ds.$$

6. Prove by induction that for $n \geq 2$,

$$(T^n u)(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1}u(\tau)d\tau.$$

Proof. When $n = 1$, this is exactly the definition of Tu. Suppose that the relation holds for some $n \geq 1$. Then we have that

$$\begin{aligned}
(T^{n+1}u)(t) &= (T(T^n u))(t) \\
&= \int_0^t \frac{1}{(n-1)!}\int_0^s (s-\tau)^{n-1}u(\tau)d\tau\, ds \\
&= \int_0^t \frac{1}{(n-1)!}u(\tau)\int_\tau^t (s-\tau)^{n-1}ds\, d\tau \\
&= \int_0^t \frac{1}{n!}(t-\tau)^n u(\tau)d\tau,
\end{aligned}$$

where the third equality is just an application of Fubini. Thus, we conclude that the relation holds for all n by induction. □

7. Deduce that $\|T^n\| \leq \frac{1}{n!}$

Proof. If $p = \infty$, then we have that for any $u \in L^\infty(0,1)$ such that $\|u\|_\infty \leq 1$ and for all $t \in [0,1]$

$$|(T^n u)(t)| \leq \frac{1}{(1-n)!}\int_0^1 (1-\tau)^{n-1}d\tau = \frac{1}{n!}.$$

Thus, $\|T^n\| \leq \frac{1}{n!}$. Moreover, if $1 \leq p < \infty$ then applying Young's inequality, we have that for all $u \in L^p(0,1)$ with $\|u\|_p \leq 1$,

$$\begin{aligned}
\|T^n u\|_p^p &\leq \frac{1}{(n-1)!^p}\int_0^1 \left(\int_0^1 (t-\tau)^{n-1}|u(\tau)|d\tau\right)^p dt \\
&= \frac{1}{(n-1)!^p}\|t^{n-1} \star |u|\|_p^p \\
&\leq \frac{1}{(n-1)!^p}\|t^{n-1}\|_1^p\|u\|_p^p \\
&= \frac{1}{n!^p}.
\end{aligned}$$

Taking pth roots, we get that $\|T^n u\|_p \leq \frac{1}{n!}$ so that $\|T^n\| \leq \frac{1}{n!}$. □

8. Prove that the spectral radius of T is 0.

Proof. By Stirling's formula we have that

$$\begin{aligned} r(T) &= \lim_{n\to\infty} \|T^n\|^{1/n} \\ &\leq \limsup_{n\to\infty} \frac{1}{n!^{1/n}} \\ &\leq \limsup_{n\to\infty} \frac{e}{\sqrt{2\pi n}^{1/n} n} = 0. \end{aligned}$$

□

9. Show that $\sigma(T) = \{0\}$. Compare with Exercise 6.20.

Proof. The cases $1 \leq p < \infty$ were already proven in Exercise 6.20. Thus, we may assume WLOG that $p = \infty$. From part 4, we have that $\sigma(T) \subset [-r(T), +r(T)] = \{0\}$. Since $T(L^\infty(0,1)) \subset C([0,1]) \subsetneq L^\infty(0,1)$, it follows that $\sigma(T) = \{0\}$. □

6.24

Assume that $T \in \mathcal{L}(H)$ is self-adjoint.

1. Prove that the following are equivalent:

   (i) $(Tu, u) \geq 0 \quad \forall u \in H$,

   (ii) $\sigma(T) \subset [0, \infty)$.

   Proof. Suppose (i). Then by Proposition 6.9, $\sigma(T) \subset [\inf_{\substack{u\in H\\|u|=1}}(Tu,u), \sup_{\substack{u\in H\\|u|=1}}(Tu,u)] \subset [0, \|T\|] \subset [0,\infty)$. Thus, (i) $\Longrightarrow$ (ii). Now suppose (ii). Fix $u \in H$ and suppose for a contradiction that $(Tu, u) < 0$. Then clearly $\|u\| \neq 0$ and $(T\frac{u}{\|u\|}, \frac{u}{\|u\|}) < 0$, so that $\inf_{\substack{u\in H\\|u|=1}}(Tu,u) < 0$. But then by Proposition 6.9, we have that $\inf_{\substack{u\in H\\|u|=1}}(Tu,u) \in \sigma(T)$, contradicting the fact that $\sigma(T) \subset [0,\infty)$. Thus, by contradiction (ii) $\Longrightarrow$ (i). □

2. Prove that the following properties are equivalent:

   (iii) $\|T\| \leq 1$ and $(Tu, u) \geq 0 \quad \forall u \in H$,

   (iv) $0 \leq (Tu,u) \leq |u|^2 \quad \forall u \in H$,

   (v) $\sigma(T) \subset [0,1]$,

   (vi) $(Tu,u) \geq |Tu|^2 \quad \forall u \in H$.

   Proof. Suppose (iii). Fix $u \in H$ and observe that $0 \leq (Tu,u) \leq |Tu||u| \leq \|T\||u|^2 \leq |u|^2$. Thus, (iii) $\Longrightarrow$ (iv). Suppose (iv). Then we have that for all $u \in H$ such that $|u| = 1$, $0 \leq (Tu,u) \leq 1$. Thus, by Proposition 6.9, it follows that $\sigma(T) \subset [0,1]$ so that (iv) $\Longrightarrow$ (v). Suppose (v). Then for all $\varepsilon > 0$, we have that $-\varepsilon \in \rho(T)$ so that $T + \varepsilon I$ is invertible. Moreover, since $\sigma(T) \subset [0,1]$, it follows that $\sigma(T+\varepsilon I) \subset [\varepsilon, 1+\varepsilon]$, and so by Exercise 6.15, $\sigma((T+\varepsilon I)^{-1}) = \sigma(T+\varepsilon I)^{-1} \subset [\frac{1}{1+\varepsilon}, \varepsilon^{-1}]$. Thus, since $(T+\varepsilon I)^{-1}$ is self-adjoint, we can apply Proposition 6.9 to conclude that $((T+\varepsilon I)^{-1}u, u) \geq \frac{|u|^2}{1+\varepsilon}$ for all $u \in H$. Fix $u \in H$ and set $f = Tu + \varepsilon u$. Then we have that $\frac{|Tu+\varepsilon u|^2}{1+\varepsilon} = \frac{|f|^2}{1+\varepsilon} \leq ((T+\varepsilon I)^{-1}f, f) = (u, Tu+\varepsilon u)$. Since this inequality holds for all $\varepsilon > 0$, we can take the limit as $\varepsilon \to 0$ to conclude that $|Tu|^2 \leq (Tu,u)$, and so (v) $\Longrightarrow$ (vi). Finally, suppose (vi). Then clearly $(Tu,u) \geq |Tu|^2 \geq 0$ and $|Tu|^2 \leq (Tu,u) \leq |Tu||u|$, so that $|Tu| \leq |u|$, proving that $\|T\| \leq 1$. Thus, (vi) $\Longrightarrow$ (iii). □

3. Prove that the following properties are equivalent:

   (vii) $(Tu,u) \leq |Tu|^2 \quad \forall u \in H$,

   (viii) $(0,1) \subset \rho(T)$.

Proof. Suppose (vii). Set $U = 2T - I$ and observe that for all $u \in H$, $|Uu|^2 = 4|Tu|^2 - 4(Tu,u) + |u|^2 \geq |u|^2$, so that $|Uu| \geq |u|$ for all u. Thus, for any $\lambda \in (-1,+1)$, we have that $|Uu - \lambda u|^2 = |Uu|^2 - 2\lambda(Uu,u) + \lambda^2|u|^2 \geq |u|^2 - 2\lambda|u|^2 + \lambda^2|u|^2 = (\lambda-1)^2|u|^2$. Since $U - \lambda I$ is self-adjoint, we can apply Theorem 2.20 to conclude that $U - \lambda I$ is surjective. Clearly we also have that $U - \lambda I$ is also injective and so $\lambda \in \rho(U)$ for all $\lambda \in (-1,+1)$. Applying Exercise 6.22 part 5, it follows that $(-1,+1) \subset \rho(U) = 2\rho(T) - 1$, and so $\rho(T) \supset (0,1)$, which verifies that (vii) $\Longrightarrow$ (viii). Suppose (viii). Then $\rho(U) = 2\rho(T) - 1 \supset (-1,+1)$ and so U is invertible, and applying Exercise 6.22, we have that $\sigma(U^{-1}) = \sigma(U)^{-1} \subset [-1,+1]$. By Proposition 6.9, it follows that $\|U^{-1}\| \leq 1$ and so for all $u \in H$, $|u|^2 = |U^{-1}Uu|^2 \leq |Uu|^2 = 4|Tu|^2 - 4(Tu,u) + |u|^2$. The statement follows after rearranging. □

Evan's Sobolev Spaces Solutions

In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary δU. As usual, all given functions are assumed smooth, unless otherwise stated.

1.

Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

Proof. Fix $u \in C^{0,\beta}(U)$. Then observe that since $|u(x) - u(y)| \leq C|x-y|^\beta \leq C|x-y|^\gamma \leq C|x-y|$ for all $x, y \in U$, it follows that $u \in C^{0,\gamma}(U) \cap C^{0,1}(U)$. Moreover, for any $x \neq y \in U$, we have that

$$\begin{aligned}\frac{|u(x)-u(y)|}{|x-y|^\gamma} &= \left(\frac{|u(x)-u(y)|}{|x-y|^\beta}\right)^{\frac{1-\gamma}{1-\beta}} \left(\frac{|u(x)-u(y)|}{|x-y|}\right)^{\frac{\gamma-\beta}{1-\beta}} \\ &\leq [u]_\beta^{\frac{1-\gamma}{1-\beta}} [u]_1^{\frac{\gamma-\beta}{1-\beta}},\end{aligned}$$

so that $[u]_\gamma \leq [u]_\beta^{\frac{1-\gamma}{1-\beta}} [u]_1^{\frac{\gamma-\beta}{1-\beta}}$. It follows that

$$\begin{aligned}\|u\|_{C^{0,\gamma}(U)} &\leq \|u\|_\infty^{\frac{1-\gamma}{1-\beta}} \|u\|_\infty^{\frac{\gamma-\beta}{1-\beta}} + [u]_\beta^{\frac{1-\gamma}{1-\beta}} [u]_1^{\frac{\gamma-\beta}{1-\beta}} \\ &= \left(\|u\|_\infty + [u]_\beta\right)^{\frac{1-\gamma}{1-\beta}} \left(\frac{\|u\|_\infty}{\|u\|_\infty + [u]_\beta}\left(\frac{\|u\|_\infty(\|u\|_\infty + [u]_\beta)}{\|u\|_\infty}\right)^{\frac{\gamma-\beta}{1-\beta}} + \frac{[u]_\beta}{\|u\| + [u]_\beta}\left(\frac{[u]_1(\|u\|_\infty + [u]_\beta)}{[u]_\beta}\right)^{\frac{\gamma-\beta}{1-\beta}}\right) \\ &\leq \left(\|u\|_\infty + [u]_\beta\right)^{\frac{1-\gamma}{1-\beta}} \left(\|u\|_\infty + [u]_1\right)^{\frac{\gamma-\beta}{1-\beta}},\end{aligned}$$

where I have applied the convexity of the map $t \mapsto t^{\frac{\gamma-\beta}{1-\beta}}$ in the final inequality. The desired inequality follows. □

3.

Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$.

Proof. By possibly adding a constant, we may assume WLOG that Since $u \in W^{1,p}(0,1)$, there exists $v \in L^p(0,1)$ such that $\int_0^1 u\varphi' = -\int_0^1 v\varphi$ for all test functions $\varphi \in C_c^\infty(0,1)$. Set $U(x) := \int_0^x v(t)dt$. Observe that for any test function $\varphi \in C_c^\infty(0,1)$,

$$\begin{aligned}\int_0^1 (u(x) - U(x))\varphi'(x)dx &= \int_0^1 (u(x) - \int_0^x v(t)dt)\varphi'(x)dx \\ &= \int_0^1 u(x)\varphi'(x)dx - \int_0^1 v(t)\int_t^1 \varphi'(x)dxdt \\ &= -\int_0^1 v(x)\varphi(x)dx + \int_0^1 v(t)\varphi(t)dt = 0.\end{aligned}$$

It follows that $u = U$ a.e. on $(0,1)$. From Analysis 1, we know that $U(x) = \int_0^x v(t)dt$ is absolutely continuous, and the statement follows. ☐

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p dt \right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

Proof. Since we may assume WLOG that u is absolutely continuous by part (a), and because the fundamental theorem of calculus part II applies to absolutely continuous functions, it follow that $u(y) - u(x) = \int_x^y u'(t)dt$ for a.e. $x, y \in [0,1]$. Thus, by applying Hölder's inequality, we have that for a.e. $x, y \in [0,1]$

$$\begin{aligned} |u(x) - u(y)| &\leq \int_x^y |u'(t)|dt \\ &\leq \left(\int_0^1 \chi_{[x,y]}^{p'} \right)^{1/p'} \left(\int_0^1 |u'(t)|^p dt \right)^{1/p} \\ &= |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p dt \right)^{1/p}. \end{aligned}$$

☐

4.

Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U.

Proof. Since we are working in $\mathbb{R}^n$ and $\overline{V} \subset U$, there exists open W such that $\overline{V} \subset W \subset \overline{W} \subset U$, and since $\overline{V}$ is compact, we can further take $\overline{W}$ to be compact. That is, we have $V \subset\subset W \subset\subset U$. Now let ρ_n be a sequence of mollifiers and observe that for each n, $\zeta_n = \chi_W \star \rho_n$ is smooth and for $\frac{1}{n} < \frac{1}{2}\min(\text{dist}(\overline{V}, \partial W), \text{dist}(\overline{W}, \partial U))$, we have that

$$\begin{aligned} \zeta_n(x) &= \int_{\mathbb{R}^n} \rho_n(x-y)\chi_W(y)dy \\ &= \int_W \rho_n(x-y)dy \\ &= \int_{W \cap B_{1/n}(x)} \rho_n(x-y)dy. \end{aligned}$$

Thus, for all $x \in V$, we have that $B_{1/n}(x) \subset W$, so that $\zeta_n|_V \equiv 1$, and for all $x \in \partial U$ and $z \in B_{1/n}(x)$, $\zeta_n(z) = 0$ (since $B_{1/n}(x) \cap W = \varnothing$). The statement follows. ☐

5.

Let U be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\overline{U}) \cap L^p(U)$.

Proof. Fix p and suppose for a contradiction that there exists a bounded linear operator $T : L^p(U) \to L^p(\partial U)$ such that $Tu = u|_{\partial U}$ for all $u \in C(\overline{U}) \cap L^p(U)$. Consider the sequence of functions

$$u_n(x) = \frac{1}{1 + n\text{dist}(x, \partial U)}.$$

Observe that each u_n is continuous and $0 \leq u_n \leq 1$, so that $(u_n) \subset C(\overline{U}) \cap L^p(U)$. Moreover, we clearly have that $u_n \to 0$ pointwise, so that by the dominated convergence theorem, $\int_U |u_n|^p \to 0$ as $n \to \infty$. Since $u_n|_{\partial U} \equiv 1$ for all n, we have that $\text{Area}(\partial U) = \|Tu_n\|^p_{L^p(\partial U)} \leq \|T\|^p \|u_n\|^p_{L^p(U)} \to 0$. That is, we find that ∂U has zero area, which is impossible. The statement follows by contradiction. ☐

6.

Prove that for all $u \in C_c^\infty(U)$

$$\|Du\|_{L^2} \le C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}.$$

Assume U is bounded, ∂U is smooth, and prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

Proof. Fix $u \in C_c^\infty(U)$. Integrating by parts (and using the fact that u has compact support), we see that

$$\begin{aligned}\|Du\|_{L^2}^2 &= \int_U Du \cdot Du \\ &\le C\int_U |u||D^2u| \\ &\le C\|u\|_{L^2}\|D^2u\|_{L^2}.\end{aligned}$$

The first part of the problem follows.

For the second part, fix $u \in H^2(U) \cap H_0^1(U)$ and pick sequences $(v_n) \subset C_c^\infty(U)$ converging to u in $H_0^1(U)$ and $(w_n) \subset C^\infty(\overline{U})$ converging to u in $H^2(U)$ (the first sequence exists by the definition of $H_0^1(U)$, and the second exists since ∂U is smooth, so the we can extend U to $U \subset\subset V$, and $C_c^\infty(V)$ is dense in $H^2(\overline{U})$). Integrating by parts (and using the fact that the terms have compact support), we have that for all n

$$\begin{aligned}\int_U Dv_n \cdot Dw_n &\le C\int_U |v_n||D^2w_n| \\ &\le C\|v_k\|_{L^2}\|D^2w_k\|_{L^2}.\end{aligned}$$

Clearly the RHS goes to $C\|u\|_{L^2}\|D^2u\|_{L^2}$ as $n \to \infty$. Moreover, we have that for all n,

$$\int_U \left(Dv_n \cdot Dw_n - (Du)^2\right) \le \|Dv_n\|_{L^2}\|Dw_n - Du\|_{L^2} + \|Du\|_{L^2}\|Dv_n - Du\|_{L^2} \to 0,$$

so that $\int_U Dv_n \cdot Dw_n \to \|Du\|_{L^2}^2$. It follows that $\|Du\|_{L^2} \le C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2}$, as required. □

7.

Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U.$$

Prove u is constant a.e. in U.

Proof. Let (ρ_n) be a sequence of mollifiers, fix $\varepsilon > 0$ and pick $n > 1/\varepsilon$. Set $U_n = \{x \in U : \operatorname{dist}(x, \partial U) > \frac{1}{n}\}$. Observe that $u_n = \rho_n \star u \in C^\infty(U_n)$ and $Du_n(x) = \rho_n \star Du(x) = 0$ for all $x \in U_n$. Since U_n is connected and u_n is smooth, it follows that u_n is equal to a constant c_n on U_n. Thus, since $u_n = c_n \to u$ in $L^p(U_m)$ for any fixed m, it follows that there exists a subsequence (c_{n_k}) such that $c_{n_k} \to u$ a.e. on U_m. Thus, u is constant a.e. on each U_m, and taking $m \to \infty$, we have that u is constant a.e. on U. □

8.

Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,p}(U)$, such that u is not Lipschitz continuous on U.

Solution

Pick $U := B^0(0,2) \setminus \{(x,0) : x \le 0\}$. Using polar coordinates, define $u(r,\theta) = r\sin(\theta/2)$. Observe that u is smooth and bounded in U so that $u \in W^{1,p}(U)$. Fix $\varepsilon > 0$ and, using polar coordinates, set $K_\varepsilon = \{(r,\theta) : r \in [\frac{1}{2}, \frac{3}{2}] \quad \& \quad \theta \in$

$[-\pi+\varepsilon, \pi-\varepsilon]\} \subset U$. If we pick $x, y \in K_\varepsilon$ such that the polar coordinates of x are $(1, -\pi+\varepsilon)$ and the polar coordinates of y are $(1.\pi-\varepsilon)$, then we see that

$$\begin{aligned}
\sup_{x\neq y\in U} \frac{|u(x)-u(y)|}{|x-y|} &\geq \sup_{x\neq y\in K_\varepsilon} \frac{|u(x)-u(y)|}{|x-y|} \\
&\geq \frac{|u(x)-u(y)|}{|x-y|} \\
&= \frac{|\sin(\frac{-\pi+\varepsilon}{2}) - \sin(\frac{\pi-\varepsilon}{2})|}{|\cos(-\pi+\varepsilon) - \sin(\pi-\varepsilon)|} \\
&= \frac{2\cos(\varepsilon/2)}{|\cos(-\pi+\varepsilon) - \sin(\pi-\varepsilon)|} \to \infty \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus, u is not Lipschitz continuous on U.

9.

Verify that if $n > 1$, the unbounded function $u = \log\log\left(1+\frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U = B^0(0,1)$.

Proof. Observe that u is differentiable at all points away from 0. To see that u has a weak derivative that coincides with its strong derivatives away from 0, observe that for any $\varepsilon > 0$ and test function $\varphi \in C_c^\infty(U)$

$$\begin{aligned}
\int_{U\setminus B(0,\varepsilon)} u\varphi_{x_i} &= -\int_{U\setminus B(0,\varepsilon)} u_{x_i}\varphi + \int_{\partial B(0,\varepsilon)} u\varphi\nu^i dS \\
&\leq -\int_{U\setminus B(0,\varepsilon)} u_{x_i}\varphi + C\|\varphi\|_\infty \log\left(\frac{\log(1+1/\varepsilon)}{\varepsilon^{1/n-1}}\varepsilon^{(n-1/n)^2}\right).
\end{aligned}$$

Thus, if knew that u and each u_{x_i} belong to $L^n(U)$, and therefore also to $L^1(U)$ by the boundedness of U, it would follow that

$$\begin{aligned}
\int_U u\varphi_{x_i} &= \lim_{\varepsilon\to 0}\int_{U\setminus B(0,\varepsilon)} u\varphi_{x_i} \\
&= \lim_{\varepsilon\to 0}\left(-\int_{U\setminus B(0,\varepsilon)} u_{x_i}\varphi + C\|\varphi\|_\infty \log\left(\frac{\log(1+1/\varepsilon)}{\varepsilon^{1/n-1}}\varepsilon^{(n-1/n)^2}\right)\right) \\
&= -\lim_{\varepsilon\to 0}\int_{U\setminus B(0,\varepsilon)} u_{x_i}\varphi \\
&= -\int_U u_{x_i}\varphi.
\end{aligned}$$

Hence, to finish the proof, it suffices to verify that u and each u_{x_i} belong to $L^n(U)$. To this end, observe that since u depends only only the radius of $x \in U$,

$$\begin{aligned}
\int_U |u|^n &= \int_0^1 \int_{\partial B(0,r)} |u(s)|^n dS(s)\, dr \\
&= \text{Area}(\partial U)\int_0^1 r^{n-1}|u(r)|^n dr.
\end{aligned}$$

And since $\lim_{r\to 0} r^{n-1}|u(r)|^n = 0$, it follows that $u \in L^n(U)$ (to see why this limit is so, we apply L'Hôpital to get that

$$\begin{aligned}
\lim_{r\to 0} r^{n-1}\left|\log\log\left(1+\frac{1}{r}\right)\right|^n &= \left|\lim_{r\to 0}\frac{-1}{\log(1+1/r)(1+1/r)r^2(1/n-1)r^{1/n-2}}\right|^n \\
&= \left(\frac{1}{1-\frac{1}{n}}\right)^n \left|\lim_{r\to 0}\frac{r^{1/n}}{\log(1+1/r)}\right|^n \\
&= \lim_{r\to 0} r^{2-1/n}(1+1/r) = 0.)
\end{aligned}$$

Finally, observe that $|u_{x_i}(x)| \leq \frac{1}{\log(1+|x|)(|x|^2+|x|)}$. And using the same method of estimation as above, we get that

$$\begin{aligned}\int_U |u_{x_i}|^n &= \text{Area}(\partial U)\int_0^1 r^{n-1}|u_{x_i}(r)|^n dr\\ &\leq C\int_0^1 \frac{1}{\log(1+1/r)^n(r^2+r)}\\ &= -\int_\infty^{\log(2)} \frac{dy}{y^n} < \infty.\end{aligned}$$

The statement follows. □

10.

Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show that there exists a constant C, depending only on n and α, such that

$$\int_U u^2 dx \leq C\int_U |Du|^2 dx,$$

provided

$$|\{x \in U : u(x) = 0\}| \geq \alpha, \quad u \in H^1(U).$$

Proof. Set $E = \{x \in U : u(x) = 0\}$. By assumption, we have that $|E| > 0$ and $u_E := ⨍_E u(x)dx = 0$. Thus, it clearly suffices to prove that there exists a constant C such that $\|v - v_E\|_{L^2(U)} \leq C\|Dv\|_{L^2(U)}$ for all $v \in H^1(U)$, C depending only on n and U. For a contradiction, suppose not. Then for each $k \geq 1$, there exists some $u_k \in H^1(U)$ such that $\|u_k - (u_k)_E\|_{L^2(U)} > k\|Du_k\|_{L^2(U)}$. Now set

$$v_k = \frac{u_k - (u_k)_E}{\|u_k - (u_k)_E\|_{L^2(U)}}.$$

Then we have that $(v_k)_E = 0$ and $\|v_k\|_{L^2(U)} = 1$, and so $\|Dv_k\| < \frac{1}{k}$ for each k. By the Rellich-Kondrachov Compactness theorem, there exists a subsequence (v_{k_j}) and $v \in L^2(U)$ such that $v_{k_j} \to v$ in $L^2(U)$. By continuity, we have that $v_E = 0$ and $\|v\|_{L^2(U)} = 1$. Moreover, we have that for all test functions $\varphi \in C_c^\infty(U)$,

$$\int_U v\varphi_{x_i} = \lim_{j\to\infty}\int_U v_{k_j}\varphi_{x_i} = -\lim_{j\to\infty}\int_U v_{k_j,x_i}\varphi = 0.$$

Thus, $v \in H^1(U)$ and $Dv = 0$ a.e. Since U is connected, v is constant. But since $v_E = 0$, v is constant on E and $|E| > 0$, it follows that $v = 0$, contradicting that $\|v\|_{L^2(U)} = 1$. The statement follows by contradiction. □

11.

Show that for each $n \geq 3$ there exosys a constant C so that

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq C\int_{\mathbb{R}^n} |Du|^2 dx$$

for all $u \in H^1(\mathbb{R}^n)$

Proof. Suppose first that $u \in C_c^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Set $F(x) = \frac{x}{|x|}$, and observe that since our domain is $\mathbb{R}^n$, integration by parts gives that

$$\int u^2 \text{div}(F) = -\int D(u^2)\cdot F = -2\int Du \cdot uF.$$

Thus, applying Cauchy Schwarz, we have

$$\left|\int u^2 \text{div}(F)\right| \leq 2\|Du\|_2\|uF\|_2,$$

or equivalently,

$$\left(\int \frac{u^2}{|x|^2}\right)^2 \leq \frac{4}{(n-2)^2}\int |Du|^2 dx \int \frac{u^2}{|x|^2}dx.$$

Thus, $\int \frac{u^2}{|x|^2} \leq \frac{4}{(n-2)^2}\int |Du|^2$ for all $u \in C_c^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Now fix $u \in H^1(\mathbb{R}^n)$ and observe that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$, so that there exists $(u_n) \subset C_c^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ such that $u_n \to u$ in $H^1(\mathbb{R}^n)$. By possibly picking a subsequence and relabeling, we may assume WLOG that $u_n \to u$ pointwise a.e. Now by Fatou, we have that

$$\int \frac{u^2}{|x|^2} \leq \liminf_n \int \frac{u_n}{|x|^2} \leq \frac{4}{(n-2)^2}\liminf_n \int |Du_n|^2 = \frac{4}{(n-2)^2}\int |Du|^2.$$

□

12.

Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u)u_{x_i} \quad (i = 1,\ldots,n).$$

Proof. By density, pick $(u_k) \subset C^\infty(U)\cap W^{1,p}(U)$ such that $u_k \to u$ in $W^{1,p}(U)$. Then using the fact that F' is bounded, there exists some C such that

$$|F(u_k(x)) - F(u_j(x))| \leq C|u_k(x) - u_j(x)|$$

for all $x \in U$. Thus, for any test function $\varphi \in C_c^\infty(U)$

$$\int F(u)D\varphi = \lim_{k\to\infty}\int F(u_k)D\varphi = \lim_{k\to\infty} -\int F'(u_k)Du_k\varphi.$$

By possibly picking a subsequence of (u_k) and relabeling, we may assume WLOG that $u_k \to u$ a.e. and there $F'(u_k) \to F(u)$ a.e. It follows that

$$\left|\int (F'(u_k)Du_k - F'(u)Du)\varphi\right| \leq C\int \|Du_k\|_\infty |F'(u_k) - F'(u)||\varphi| + \|F'(u)\|_\infty \|u_k - u\|_{W^{1,p}(U)}|\varphi| \to 0$$

as $k \to \infty$. Thus, $\int F(u)D\varphi = \lim_{k\to\infty} -\int F'(u_k)Du_k\varphi = \int F'(u)Du\varphi$. Finally, since U is bounded $F \in C^1$ and $u \in L^p(U)$, $F(u) \in L^p(U)$, and since $F'(u)$ is bounded, $F'(u)u_{x_i} \in L^p(U)$ $(i = 1,\ldots,n)$. Thus, $F(u) \in W^{1,p}(U)$, as required. □