

INFINITELY MANY LEFSCHETZ PENCILS ON RULED SURFACES

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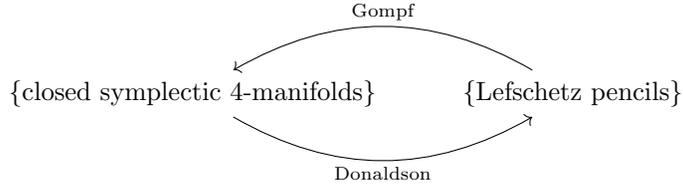
ABSTRACT. We show that any ruled surface X with $\chi(X) < 0$ admits infinitely many inequivalent Lefschetz pencils of fixed genus and number of base points. Our proof proceeds by building infinitely many inequivalent Lefschetz fibrations on a blow-up $X\#\overline{4\mathbb{C}\mathbb{P}^2}$ of X with constant fiber class, via a mechanism known as partial conjugation. Furthermore, there exists a symplectic form on X compatible with all such pencils, and similarly for the fibrations in $X\#\overline{4\mathbb{C}\mathbb{P}^2}$. This provides the first example of this phenomenon and makes progress on Problem 4.98 of the K3 list of problems in low-dimensional topology in the case of ruled surfaces.

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1. INTRODUCTION

Let X be a closed, symplectic 4-manifold. The work of Donaldson [Don99] and Gompf [Gom05] builds a correspondence between topological properties of X and properties of the *topological Lefschetz pencils* (X, f) : Any symplectic 4-manifold admits a pencil, and any 4-manifold admitting a pencil is symplectic.



This correspondence has proved remarkably fruitful in the study of symplectic manifolds (e.g. [DS03, Sti00, Ush06, FS04, BH16b, BK17, BH24, ABKP00, BHM23]). Nevertheless, there is not an a priori canonical Lefschetz pencil on a given symplectic manifold X . A fundamental open question is then the following.

Question 1.1 (K3 [BKRRar, Problem 4.98]; cf. Baykur–Hayano [BH16b, Question 6.4]). *Does every closed symplectic 4-manifold admit inequivalent Lefschetz pencils with the same fiber genus g , for sufficiently large g ? How about infinitely many?*

In this work we make progress on this problem in the case of *ruled surfaces*, or S^2 -bundles over Σ_g for some $g \geq 0$. In particular, we provide the first examples of infinitely many inequivalent pencils of fixed genus on the same symplectic 4-manifold.

Theorem 1.2. *Let X be a ruled surface with $\chi(X) = 4 - 4g < 0$. There exist infinitely many pairwise inequivalent Lefschetz pencils $\pi_n : X - B_n \rightarrow S^2$, $n \in \mathbb{Z}_{\geq 0}$, of genus $2g$ with fixed number of base points $|B_n| = 4$ and fixed homology class of regular fiber. There exists a symplectic form ω on X such that the smooth loci of all fibers of π_n for all $n \in \mathbb{Z}_{\geq 0}$ are symplectic submanifolds of (X, ω) .*

Blowing down disjoint (-1) -sections of inequivalent Lefschetz fibrations produce inequivalent Lefschetz pencils (see e.g. [BH16a, Corollary 3.10]). The following theorem forms an intermediate step in the proof of Theorem 1.2.

Theorem 1.3. *Let X be a ruled surface with $\chi(X) = 4 - 4g < 0$. There exist infinitely many pairwise inequivalent Lefschetz fibrations $\pi_n : X \# 4\overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$, $n \in \mathbb{Z}_{\geq 0}$ of genus $2g$ and fixed homology class of regular fiber. There exists a symplectic form ω on $X \# 4\overline{\mathbb{C}\mathbb{P}^2}$ such that the smooth loci of all fibers of π_n for all $n \in \mathbb{Z}_{\geq 0}$ are symplectic submanifolds of $(X \# 4\overline{\mathbb{C}\mathbb{P}^2}, \omega)$.*

Theorem 1.3 may be of independent interest, as it gives the first construction of infinitely many pairwise inequivalent Lefschetz fibrations of the same genus on a fixed smooth 4-manifold. See Section 1.3 for a summary of previously known constructions of finitely many such Lefschetz fibrations.

Remark 1.4 (On the fiber genus). An essential point of Question 1.1 is the existence of inequivalent Lefschetz pencils of the *same* fiber genus. Otherwise, infinitely many Lefschetz pencils can arise from increasingly large powers of a given line bundle.

The smallest genus of the regular fibers appearing in the fibrations of Theorem 1.3 is 4. Currently, the smallest known genus of inequivalent but diffeomorphic Lefschetz fibrations is 3, by work of Baykur–Hayano [BH16b, Theorem 6.2]. On the other hand, there are restrictions on the monodromy of potential genus-2 examples; see [BH16b, Remark 6.3]. Theorem 1.3 answers Question 6.4 of Baykur–Hayano [BH16b] positively for the case of (four-fold blowups of) ruled surfaces.

Remark 1.5 (The case of surface bundles over surfaces and mapping tori). Our result highlights a difference between Lefschetz fibrations and surface bundles. F.E.A. Johnson [Joh99] showed that if $\pi : M \rightarrow \Sigma_h$ is a Σ_g -bundle over Σ_h with $g, h \geq 2$ then there are only finitely many ways to realize M as the total space of a $\Sigma_{g'}$ -bundle $M \rightarrow \Sigma_{h'}$ with $g', h' \geq 2$, and Salter [Sal15] constructed 4-manifolds admitting arbitrarily many fiberings. In dimension 3, the set of surface bundles $Y \rightarrow S^1$ for a fixed 3-manifold Y is governed by the *Thurston norm* [Thu86]. Thurston showed, for example, that any irreducible and atoroidal 3-manifold fibers over S^1 with fixed genus in at most finitely many ways. Moreover, the isomorphism class of the fibering is determined by the homology class of the fiber [Tis70, Lemma 1].

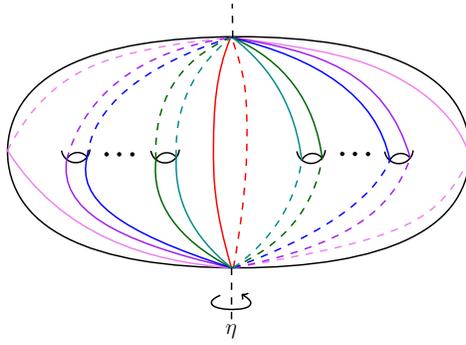


FIGURE 1. Vanishing cycles of the MCK Lefschetz fibration of genus $2g$ and the involution η .

1.1. A road map to the proof of Theorems 1.2 and 1.3. In this subsection we give a quick overview of each of the steps appearing in the proofs of Theorems 1.2 and 1.3. We emphasize that the proofs of Theorems 1.2 and 1.3 in the smooth category are essentially finished in Section 5; see Corollaries 5.3 and 5.8.

Let $g \geq 2$ and $X = \Sigma_g \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$. The starting point in our construction of infinitely many inequivalent Lefschetz fibrations on X is a quite peculiar Lefschetz fibration on $\pi : X \rightarrow S^2$ called the (even) Matsumoto–Cadavid–Korkmaz (MCK) Lefschetz fibration [Ham17].

The Matsumoto–Cadavid–Korkmaz Lefschetz fibration. The MCK fibration is induced by a factorization of an involution η on Σ_{2g} with two fixed points. Its vanishing cycles and the involution η are shown in Figure 1. This Lefschetz fibration was originally described by Matsumoto [Mat96, Proposition 4.2] for $g = 1$ and later generalized to higher genus by both Cadavid [Cad98] and Korkmaz [Kor01]. The MCK fibration has been the starting point of many interesting examples of Lefschetz fibrations (e.g. [Kor01, OS00a]). It is a hyperelliptic Lefschetz fibration, with the smallest number of singular fibers of a fibration of genus at least 6 for manifolds with $b_2^+ = 1$ [Sti02, Bay16b].

For our purposes, the most important properties of the MCK fibration are the following:

- (1) The total space is diffeomorphic to $\Sigma_g \times S^2 \# 4\overline{\mathbb{C}\mathbb{P}^2}$. Furthermore, the total space is invariant under *partial conjugations* by elements of the *Torelli group* $\mathcal{I}_{2g} \trianglelefteq \text{Mod}(\Sigma_{2g})$. (This relies heavily on results of Liu [Liu96]; see Section 5 for details).
- (2) It is a hyperelliptic Lefschetz fibration.

Our construction then proceeds as follows: Starting from the MCK fibration $\pi : X \rightarrow S^2$, we find a suitable mapping class $f \in \mathcal{I}_{2g}$ and perform partial conjugations by the powers f^n to obtain a family of fibrations $\{\pi_n : X_n \rightarrow S^2\}$. We now briefly describe the partial conjugation construction.

Navigating the set of Lefschetz fibrations. Let (X, π) be a Lefschetz fibration of genus h . A big part of the success in the study of Lefschetz fibrations is that they admit simple combinatorial descriptions via factorizations of the identity in $\text{Mod}(\Sigma_h)$ by right-handed Dehn twists [Mat96, Kas80]. From this perspective the following operation, called *partial conjugation* [Bay16a, Aur06], is quite natural. Let W_1, W_2 be products of positive Dehn twists such that

$$W_1 \cdot W_2 = 1 \in \text{Mod}(\Sigma_h).$$

Let $f \in \text{Mod}(\Sigma_h)$ be a mapping class that commutes with W_2 . Then because $fT_c f^{-1} = T_{f(c)}$ for any right-handed Dehn twist $T_c \in \text{Mod}(\Sigma_h)$, we obtain a new factorization

$$W_1 \cdot W_2^f = 1 \in \text{Mod}(\Sigma_h)$$

of the identity in $\text{Mod}(\Sigma_h)$ by right-handed Dehn twists, where W_2^f denotes the conjugation of the factors of W_2 by f . This new factorization defines a Lefschetz fibration (X_f, π_f) on a manifold X_f which has the same signature and Euler characteristic as the original manifold X .

Remark 1.6. Partial conjugation can also be understood from a topological point of view as a cutting and regluing operation; see Section 2 for details. In this way, partial conjugation is a generalization of the fiber-sum operation. However, the Lefschetz fibrations we construct in the proof of Theorem 1.3 are *not minimal*. In particular, the partial conjugations that we employ are *not* fiber-sums [Ush06].

Remark 1.7 (Effectiveness of partial conjugations). Let G_{W_2} be the group generated by the Dehn twists present in the word W_2 . If $f \in G_{W_2}$, then (X_f, π_f) is equivalent to (X, π) [Aur05, Lemma 6]. Thus, partial conjugation is more effective in changing the equivalence class of a fibration when G_{W_2} is “small.”

In order to distinguish the equivalence classes of the fibrations $\pi_n : X_n \rightarrow S^2$, we distinguish the monodromy groups of π_n up to conjugacy.

Monodromy group and the Johnson homomorphism. An invariant of a Lefschetz fibration is given by (the conjugacy class of) its *monodromy group*; see Section 2 for details. Let G be the monodromy group of (X, π) and G_n the monodromy group of (X_n, π_n) . Let $\mathcal{I}_{2g} \trianglelefteq \text{Mod}(\Sigma_{2g})$ be the *Torelli group*. If G and G_n are conjugate in $\text{Mod}(\Sigma_{2g})$ then $G \cap \mathcal{I}_{2g}$ and $G_n \cap \mathcal{I}_{2g}$ are also conjugate in $\text{Mod}(\Sigma_{2g})$ because \mathcal{I}_{2g} is normal in $\text{Mod}(\Sigma_{2g})$. Thus, it is enough to distinguish subgroups of \mathcal{I}_{2g} up to conjugation by $\text{Mod}(\Sigma_{2g})$. This observation allows us to *abelianize* the problem: Let $H := H_1(\Sigma_{2g}; \mathbb{Z})$ and

$$\tau : \mathcal{I}_{2g} \rightarrow (\wedge^3 H)/H$$

be the *Johnson homomorphism* [FM12, Section 6.6]. The Johnson homomorphism τ is $\text{Mod}(\Sigma_{2g})$ -equivariant, with respect to the conjugation action on \mathcal{I}_{2g} and the induced action $(\wedge^3 H)/H$ via the symplectic representation of $\text{Mod}(\Sigma_{2g})$ acting on H . Thus, it is enough to distinguish $\text{Mod}(\Sigma_{2g})$ -orbits in $(\wedge^3 H)/H$, and the hyperellipticity of (X, π) vastly simplifies this computation.

Remark 1.8. We employed the same idea in our previous work [LS24, Lemma 7.4]. The role of \mathcal{I}_{2g} was played by the point-pushing subgroup $\pi_1(\Sigma_g) \trianglelefteq \text{Mod}(\Sigma_{g,1})$, and the role of τ was played by the abelianization map $\pi_1(\Sigma_g) \rightarrow H_1(\Sigma; \mathbb{Z})$.

From fibrations to pencils. Hamada showed that the MCK fibration possesses four disjoint (-1) -sections by finding appropriate lifts of the corresponding factorization to $\text{Mod}(\Sigma_{2g}^4)$ [Ham17]. In order to relate Theorem 1.3 to Theorem 1.2, we find an appropriate lift of $f \in \mathcal{I}_{2g}$ and perform partial conjugations in $\text{Mod}(\Sigma_{2g}^4)$ in order to preserve this number of disjoint (-1) -sections. Furthermore, by using two distinct lifts given by Hamada [Ham17, Figure 21] the total space of the pencil can be either $\Sigma_g \times S^2$ or $\Sigma_g \tilde{\times} S^2$. The existence of four disjoint (-1) -sections also improves our understanding of the homology class of the fibers. In particular, we can show that all fibers of our pencils are representatives of a fixed homology class (Lemma 6.3).

Remark 1.9. In our previous work [LS24] we constructed examples, via partial conjugations, of infinitely many inequivalent factorizations in $\text{Mod}(\Sigma_{g,1})$ covering the *same* factorization of the identity in $\text{Mod}(\Sigma_g)$. Yet, these factorizations were *not* associated to pencils: the starting point for the construction was a Lefschetz fibration with a section of self-intersection at most -2 . In fact, our method in [LS24] fails for sections of self-intersection -1 (see Remark 5.9).

Symplectic structures on ruled surfaces. To finish the proof of Theorem 1.2 and Theorem 1.3, we leverage the classification of symplectic structures on ruled surfaces of Lalonde–McDuff [LM96]. In particular, it is enough to construct cohomologous symplectic forms compatible with the pencils. We achieve this via the Gompf–Thurston construction and by carefully choosing the gluing maps associated to the respective partial conjugation; see Section 6 for details.

Remark 1.10. Lin–Wu–Xie–Zhang [LWXZ25, Theorem 1.5] recently showed the existence of infinitely many isotopy classes of diffeomorphic symplectic forms on ruled surfaces with negative Euler characteristic. While we show that all the symplectic forms we construct on a minimal ruled surface are diffeomorphic, we are unable to tell if they are isotopic.

1.2. Lefschetz fibrations with homeomorphic total spaces. In Section 7 we show the broader applicability of our methods. Starting with a simply-connected hyperelliptic Lefschetz fibration, we easily construct infinitely many inequivalent but homeomorphic Lefschetz fibrations. We are unable to tell if these families of homeomorphic 4-manifolds are diffeomorphic. In particular, our construction gives a broader source of potential examples of the same phenomenon described by Theorem 1.3.

1.3. Previous work. Despite the great deal of interest in Lefschetz pencils and fibrations, Question 1.1 remains wide open in its full generality. The first (published) examples of manifolds admitting at least two inequivalent Lefschetz fibrations were given by Park–Yun [PY09], constructed using knot surgery operations of Fintushel–Stern [FS04] and distinguished by their monodromy groups. (Park–Yun also mentioned that Ivan Smith had an earlier example in his thesis of two inequivalent fibrations on $T^2 \times \Sigma_2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$.) See also further work of Park–Yun [PY17] generalizing this construction to find arbitrarily large numbers of inequivalent but diffeomorphic fibrations. Later, Baykur–Hayano used monodromy substitutions to construct pairs of inequivalent Lefschetz pencils on blow-ups of Calabi–Yau K3 surfaces [BH16b, Theorem 6.2] whose fibers are ambiently homeomorphic. Using completely different methods, Baykur showed that any symplectic manifold X admits at least two distinct pencils (up to equivalence and fibered Luttinger surgery) on a blowup of X , and arbitrarily many such pencils if X is not rational or ruled [Bay16a, Bay19]. Despite this progress, the methods used to distinguish the fibrations were inherently finite: Park–Yun [PY17] relied on a finite graph to distinguish the monodromy groups, and Baykur bounds the number of inequivalent pencils by a number of a priori chosen points to blow-up [Bay16a]. Further examples of pairs of inequivalent pencils were also given by Hamada [Ham17, Theorem 7] and Baykur–Hayano–Monden [BHM23, Theorem 6.4].

1.4. Questions. We end this introduction with a set of questions prompted by our results.

In [Bay16a, Remark 3.5] Baykur wonders if the MCK fibration of genus $2g$ on $\Sigma_g \times \overline{S^2 \# 4\mathbb{C}\mathbb{P}^2}$ is unique up to equivalence and fibered Luttinger surgery. Although we are unable to tell if the partial conjugations we employ can be obtained by a sequence of fibered Luttinger surgeries, our results show that Luttinger surgery is necessary in a potential uniqueness statement. More generally, we ask the following refinement to Question 1.1:

Question 1.11. *Does there exist a 4-manifold X admitting infinitely many Lefschetz fibrations that are not related via sequences of partial conjugations?*

A plausible starting point to finding more families of infinitely many inequivalent but diffeomorphic Lefschetz fibrations may be in determining the diffeomorphism types of the example described in Section 7.

Question 1.12. *Are the 4-manifolds Z_n and Z_m diffeomorphic for all $n, m \in \mathbb{Z}_{\geq 0}$, where $\pi_n : Z_n \rightarrow S^2$ is the Lefschetz fibration constructed in Theorem 7.3?*

Finally, we point out that we construct infinite families of *non-hyperelliptic* Lefschetz fibrations of even genus (starting with genus 4) in this paper, leaving open the following questions.

Question 1.13. *Do there exist infinitely many inequivalent Lefschetz fibrations of odd genus with diffeomorphic total spaces? Of genus 2?*

Question 1.14. *Do there exist infinitely many inequivalent hyperelliptic Lefschetz fibrations of fixed genus and diffeomorphic total spaces?*

1.5. Organization of the paper. In Section 2, we review the necessary background material about Lefschetz fibrations and mapping class groups of surfaces. In Section 3, we describe in detail the Matsumoto–Cadavid–Korkmaz (MCK) Lefschetz fibration $\pi : X \rightarrow S^2$ and define its twisted versions $\{\pi_n : X_n \rightarrow S^2\}$. In Section 4 we show how to use the Johnson homomorphism as a tool to detect inequivalent Lefschetz fibrations. In particular, we show that the family of fibrations $\{\pi_n : X_n \rightarrow S^2\}$ is pairwise inequivalent. Section 5 contains the proof of Theorems 1.2 and 1.3 in the smooth category. Section 6 describes the construction of the symplectic forms ω_n compatible with $\pi_n : X_n \rightarrow S^2$ and

finishes the proof of Theorems 1.2 and 1.3. In Section 7, we describe a new family of examples of pairwise inequivalent Lefschetz fibrations with homeomorphic total space. Appendix A contains some routine computations in $\text{Mod}(\Sigma_{2g}^4)$ and associated mapping class groups. In particular, it contains the proof of Theorem 3.3. Appendix B contains the proof of Proposition 6.8, which allows us to find nice representatives for the gluing maps employed in our partial conjugation construction.

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2. LEFSCHETZ FIBRATIONS AND MAPPING CLASS GROUPS

In this section we recall the relationship between Lefschetz fibrations and mapping class groups and some tools for studying them.

2.1. Lefschetz pencils, fibrations, and monodromy. Let M^4 be a closed, oriented, smooth 4-manifold. A *Lefschetz pencil* is a smooth map $\pi : M - B \rightarrow S^2$ for some nonempty, finite set $B \subseteq M$ with finitely many critical points $p_1, \dots, p_r \in M - B$ such that

- (a) for each *base point* $p \in B$, there are a smooth, orientation-compatible chart $U \cong \mathbb{C}^2$ around $p \in M$ and a diffeomorphism $S^2 \cong \mathbb{C}\mathbb{P}^1$ with respect to which π takes the form

$$\pi(z, w) = [z : w] \in \mathbb{C}\mathbb{P}^1,$$

and

- (b) for each critical point p_i , there are smooth, orientation-compatible charts $U_i \cong \mathbb{C}^2$ around $p_i \in M$ and $V_i \cong \mathbb{C}$ around $\pi(p_i) \in S^2$ with respect to which π takes the form

$$\pi(z, w) = z^2 + w^2.$$

A *Lefschetz fibration* is a smooth, surjective map $\pi : M \rightarrow S^2$ with finitely many critical points $p_1, \dots, p_r \in M$, each satisfying (b) above. We assume that π is injective on the set $\{p_1, \dots, p_r\}$ of its critical points and that Lefschetz fibrations are *relatively minimal*, i.e. no fiber of π contains an embedded (-1) -sphere. The *genus* of a Lefschetz fibration $\pi : M \rightarrow S^2$ is the genus of any regular fiber $\pi^{-1}(b)$, $b \in S^2$. Two Lefschetz fibrations $\pi_1 : M_1 \rightarrow S^2$ and $\pi_2 : M_2 \rightarrow S^2$ are *equivalent* if there exist diffeomorphisms $\Phi : M_1 \rightarrow M_2$ and $\varphi : S^2 \rightarrow S^2$ so that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ S^2 & \xrightarrow{\varphi} & S^2 \end{array}$$

A choice of a regular value $b \in S^2$ and a choice of a diffeomorphism $\Phi_b : \pi^{-1}(b) \xrightarrow{\sim} \Sigma_g$ determine the *monodromy representation* of a Lefschetz fibration $\pi : M \rightarrow S^2$, which is an antihomomorphism of groups

$$\rho : \pi_1(S^2 - \{q_1, \dots, q_r\}, b) \rightarrow \text{Mod}(\Sigma_g),$$

where $q_1, \dots, q_r \in S^2$ denote the singular values of π (cf. [GS99, p. 291]). The monodromy representation ρ is characterized by the property that for any loop $\gamma \in \pi_1(S^2 - \{q_1, \dots, q_r\}, b)$ and any $\varphi \in \text{Diff}^+(\Sigma_g)$ that represents the mapping class $\rho(\gamma)$, there is an isomorphism of Σ_g -bundles over S^1

$$\Phi : \gamma^* M \rightarrow M_\varphi := \Sigma_g \times [0, 1] / ((\varphi(x), 0) \sim (x, 1))$$

between the pullback of π along γ and the *mapping torus* M_φ . Viewing γ as a map $[0, 1] \rightarrow S^2 - \{q_1, \dots, q_r\}$ with $\gamma(0) = \gamma(1) = b$, the restriction of Φ to $(\gamma^* \pi)^{-1}(0) = \pi^{-1}(b) \rightarrow \Sigma_g \times \{0\}$ agrees with the identification $\Phi_b : \pi^{-1}(b) \rightarrow \Sigma_g$ chosen in the definition of the monodromy representation.

If $\gamma_i \in \pi_1(S^2 - \{q_1, \dots, q_r\}, b)$ is a loop obtained from a small, counterclockwise loop around q_i connected to b by a path in $S^2 - \{q_1, \dots, q_r\}$, the monodromy $\rho(\gamma_i)$ is a right-handed Dehn twist $T_{\ell_i} \in \text{Mod}(\Sigma_g)$ about a *vanishing cycle* ℓ_i , an isotopy class of some essential simple closed curve in Σ_g . Given an additional choice of generators $\gamma_1, \dots, \gamma_r \in \pi_1(S^2 - \{q_1, \dots, q_r\}, b)$ such that $\gamma_1 \gamma_2 \dots \gamma_r = 1$, the monodromy representation determines a *monodromy factorization*, a relation in $\text{Mod}(\Sigma_g)$ of the form

$$T_{\ell_r} \dots T_{\ell_1} = 1 \in \text{Mod}(\Sigma_g).$$

More generally, one can consider a *positive factorization* of any mapping class $h \in \text{Mod}(\Sigma_{g,n}^m)$, i.e. a factorization of h consisting only of right-handed Dehn twists $T_\ell \in \text{Mod}(\Sigma_{g,n}^m)$. Two positive factorizations of h in $\text{Mod}(\Sigma_{g,n}^m)$ are said to be *Hurwitz equivalent* if they are related by a sequence of the following two types of moves:

- (a) (*Elementary transformation*) For any $1 \leq i \leq r - 1$,

$$T_{\ell_r} \dots T_{\ell_{i+1}} T_{\ell_i} \dots T_{\ell_1} \leftrightarrow T_{\ell_r} \dots (T_{\ell_{i+1}} T_{\ell_i} T_{\ell_{i+1}}^{-1}) T_{\ell_{i+1}} \dots T_{\ell_1}.$$

- (b) (*Global conjugation*) For any $f \in \text{Mod}(\Sigma_{g,n}^m)$ that commutes with $h \in \text{Mod}(\Sigma_{g,n}^m)$

$$T_{\ell_r} \dots T_{\ell_1} \leftrightarrow (f T_{\ell_r} f^{-1}) \dots (f T_{\ell_1} f^{-1}).$$

The following theorem characterizes equivalence classes of Lefschetz fibrations by the Hurwitz equivalence classes of their monodromy factorizations.

Theorem 2.1 (Kas [Kas80], Matsumoto [Mat96]). *Let $g \geq 2$. There is a bijection*

$$\left\{ \begin{array}{l} \text{genus-}g \text{ Lefschetz fibrations, up} \\ \text{to equivalence} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{positive factorizations of the identity in} \\ \text{Mod}(\Sigma_g), \text{ up to Hurwitz equivalence} \end{array} \right\}$$

given by monodromy factorizations.

This bijection immediately implies the following criterion for two Lefschetz fibrations to be inequivalent.

Corollary 2.2. *Let $g \geq 2$ and let $\pi_1 : M_1 \rightarrow S^2$ and $\pi_2 : M_2 \rightarrow S^2$ be genus- g Lefschetz fibrations with monodromy representations ρ_1 and ρ_2 respectively. If the images $\text{im}(\rho_1)$ and $\text{im}(\rho_2)$ are not conjugate as subgroups of $\text{Mod}(\Sigma_g)$ then the Lefschetz fibrations π_1 and π_2 are inequivalent.*

Adding the data of disjoint sections of Lefschetz fibrations yields more refined information on the monodromy side. Fix n -many disjoint sections $\sigma_1, \dots, \sigma_n : S^2 \rightarrow M$ of a Lefschetz fibration $\pi : M \rightarrow S^2$. By considering the monodromy factorization of π with respect to the sections $\sigma_1, \dots, \sigma_n$ in $\text{Mod}(\Sigma_g^n)$ instead (where the i th boundary component δ_i corresponds to the boundary of a neighborhood of the i th section $\sigma_i(b)$ in $\pi^{-1}(b)$), we obtain a lift

$$T_{\tilde{\ell}_r} \dots T_{\tilde{\ell}_1} = T_{\delta_1}^{a_1} \dots T_{\delta_n}^{a_n} \in \text{Mod}(\Sigma_g^n)$$

of the monodromy factorization $T_{\ell_r} \dots T_{\ell_1} = 1 \in \text{Mod}(\Sigma_g)$ of π under the capping and forgetful homomorphism $\text{Mod}(\Sigma_g^n) \rightarrow \text{Mod}(\Sigma_g)$. For all $1 \leq i \leq r$, the essential simple closed curve $\tilde{\ell}_i \subseteq \Sigma_g^n$ is isotopic to ℓ_i in Σ_g (after capping off the boundary components) and $a_i \in \mathbb{N}$ satisfies $-a_i = [\sigma_i(S^2)]^2$ where $[\sigma_i(S^2)]^2$ denotes the self-intersection of $\sigma_i(S^2)$ in M [Smi01, Lemma 2.3]. By considering only the sections $\sigma_1, \dots, \sigma_n$ themselves and not their normal neighborhoods in M , we obtain a similar lift

$$T_{\tilde{\ell}_r} \dots T_{\tilde{\ell}_1} = 1 \in \text{Mod}(\Sigma_{g,n})$$

of the monodromy factorization $T_{\ell_r} \dots T_{\ell_1} = 1 \in \text{Mod}(\Sigma_g)$ under the forgetful map $\text{Mod}(\Sigma_{g,n}) \rightarrow \text{Mod}(\Sigma_g)$. Conversely, any lifts of the monodromy factorization of π in $\text{Mod}(\Sigma_g)$ to $\text{Mod}(\Sigma_g^n)$ or $\text{Mod}(\Sigma_{g,n})$ as above give rise to n -many disjoint sections $\sigma_1, \dots, \sigma_n : S^2 \rightarrow M$ of $\pi : M \rightarrow S^2$ of self-intersections $-a_1, \dots, -a_n$ respectively.

By blowing up the set B (with $n = |B|$) of base points of any Lefschetz pencil $\pi : M - B \rightarrow S^2$, we obtain an associated Lefschetz fibration $\pi : M \# n \overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$ with n -many disjoint sections $\sigma_1, \dots, \sigma_n$ of self-intersection -1 . Conversely, taking a Lefschetz fibration $\pi : M \rightarrow S^2$ and blowing down any set of n -many (-1) -sections $\sigma_1, \dots, \sigma_n$ yields a Lefschetz pencil structure on N (where $M \rightarrow N$ is this blow down map so that $M \cong N \# n \overline{\mathbb{C}\mathbb{P}^2}$). Two Lefschetz pencils $\pi_1 : M_1 - B_1 \rightarrow S^2$ and $\pi_2 : M_2 - B_2 \rightarrow S^2$ are *equivalent* if there exist diffeomorphisms $\Phi : M_1 \rightarrow M_2$ and $\varphi : S^2 \rightarrow S^2$ so that $\Phi(B_1) = B_2$ and the following diagram commutes:

$$\begin{array}{ccc} M_1 - B_1 & \xrightarrow{\Phi} & M_2 - B_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ S^2 & \xrightarrow{\varphi} & S^2 \end{array}$$

One can detect when two Lefschetz pencils are inequivalent by considering their associated Lefschetz fibrations:

Proposition 2.3 (cf. Baykur–Hayano [BH16a, Corollary 3.10]). *Let $\pi_1 : M_1 - B_1 \rightarrow S^2$ and $\pi_2 : M_2 - B_2 \rightarrow S^2$ be Lefschetz pencils with $n = |B_1| = |B_2|$. If the Lefschetz fibrations $M_1 \# n \overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$ and $M_2 \# n \overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$ associated to the Lefschetz pencils π_1 and π_2 respectively are inequivalent then the Lefschetz pencils π_1 and π_2 are inequivalent.*

2.2. Partial conjugations. One way to obtain new positive factorizations of the identity in $\text{Mod}(\Sigma_{g,n})$ from old ones is via *partial conjugation*. Given some positive factorization

$$(1) \quad T_{\tilde{\ell}_r} \dots T_{\tilde{\ell}_{i+1}} T_{\tilde{\ell}_i} \dots T_{\tilde{\ell}_1} = 1 \in \text{Mod}(\Sigma_{g,n})$$

and any $\tilde{f} \in \text{Mod}(\Sigma_{g,n})$ that commutes with the mapping class $T_{\tilde{\ell}_i} \dots T_{\tilde{\ell}_1}$ in $\text{Mod}(\Sigma_{g,n})$, we form a new positive factorization

$$(2) \quad T_{\tilde{\ell}_r} \dots T_{\tilde{\ell}_{i+1}} (\tilde{f} T_{\tilde{\ell}_i} \tilde{f}^{-1}) \dots (\tilde{f} T_{\tilde{\ell}_1} \tilde{f}^{-1}) = 1 \in \text{Mod}(\Sigma_{g,n}).$$

We now describe this operation in terms of Lefschetz fibrations and sections. Let $\pi : M \rightarrow S^2$ denote the Lefschetz fibration with sections $\sigma_1, \dots, \sigma_n : S^2 \rightarrow M$ corresponding to monodromy factorization (1). Let $b \in S^2$ denote the chosen regular value of π and let $q_1, \dots, q_r \in S^2$ denote the singular values of π so that q_i corresponds to the vanishing cycle $\tilde{\ell}_i$ for all $1 \leq i \leq r$. We may then write the base S^2 as a union of two disks $S^2 = D_1 \cup_{\partial} D_2$ with $D_i \cong D^2$, where

- (a) the base point $b \in S^2$ is contained in $\partial D_1 = \partial D_2$,
- (b) the chosen generators $\gamma_1, \dots, \gamma_i$ and singular values q_1, \dots, q_i are contained in D_1 and the chosen generators $\gamma_{i+1}, \dots, \gamma_r$ and singular values q_{i+1}, \dots, q_r are contained in D_2 , and
- (c) the based loop $(\partial D_1, b)$ can be oriented so that as an element of $\pi_1(S^2 - \{q_1, \dots, q_r\}, b)$,

$$[\partial D_1] = \gamma_1 \dots \gamma_i.$$

Let

$$\rho : \pi_1(S^2 - \{q_1, \dots, q_r\}, b) \rightarrow \text{Mod}(\Sigma_{g,n})$$

denote the monodromy representation corresponding to π and $\sigma_1, \dots, \sigma_n$. Then (c) implies that

$$\rho([\partial D_1]) = T_{\tilde{\ell}_i} \dots T_{\tilde{\ell}_1} \in \text{Mod}(\Sigma_{g,n}).$$

In other words, if $\tilde{\eta} \in \text{Diff}^+(\Sigma_{g,n})$ is any representative of $T_{\tilde{\ell}_i} \dots T_{\tilde{\ell}_1} \in \text{Mod}(\Sigma_{g,n})$ then there is an isomorphism of Σ_g -bundles over S^1

$$\pi^{-1}(\partial D_1) \rightarrow M_{\tilde{\eta}} = (\Sigma_g \times [0, 1]) / ((\tilde{\eta}(x), 0) \sim (x, 1))$$

that sends the section σ_i to the section $s_i : S^1 \rightarrow M_{\tilde{\eta}}$ defined by the marked point p_i of $\Sigma_{g,n}$ fixed by $\tilde{\eta}$.

Let $\varphi \in \text{Diff}^+(\Sigma_{g,n})$ be any representative of $\tilde{f} \in \text{Mod}(\Sigma_{g,n})$. Because \tilde{f} and $T_{\tilde{\ell}_i} \dots T_{\tilde{\ell}_1}$ commute in $\text{Mod}(\Sigma_{g,n})$, there is an isotopy $\varphi_t : \Sigma_{g,n} \times [0, 1] \rightarrow \Sigma_{g,n}$ with $\varphi_0 = \varphi$ and $\varphi_1 = \tilde{\eta}^{-1} \circ \varphi \circ \tilde{\eta}$. This isotopy induces an isomorphism $\mathcal{F} : M_{\tilde{\eta}} \rightarrow M_{\tilde{\eta}}$ of Σ_g -bundles

$$\mathcal{F} : (x, t) \mapsto (\varphi_t(x), t)$$

that fixes each section $s_i : S^1 \rightarrow M_{\tilde{\eta}}$ for all $1 \leq i \leq n$. Let \mathcal{F} also denote the corresponding map $\pi^{-1}(\partial D_1) \rightarrow \pi^{-1}(\partial D_2)$ under the identifications $\pi^{-1}(\partial D_i) \rightarrow M_{\tilde{\eta}}$. Then π induces a new Lefschetz fibration

$$(3) \quad \pi^{-1}(D_1) \cup_{\mathcal{F}, \pi^{-1}(\partial D_1) \rightarrow \pi^{-1}(\partial D_2)} \pi^{-1}(D_2) \rightarrow S^2$$

with n -many sections induced by the sections $\sigma_1, \dots, \sigma_n$ of π . This resulting Lefschetz fibration and sections have monodromy factorization (2) with respect to the original identification

$$\Phi_b : (\pi^{-1}(b), \sigma_1(b), \dots, \sigma_n(b)) \rightarrow \Sigma_{g,n},$$

viewing b as a point in ∂D_2 .

2.3. The Torelli group and the Johnson homomorphism. Let $g \geq 2$. The *Torelli group* $\mathcal{I}_g \leq \text{Mod}(\Sigma_g)$ is the kernel of the symplectic representation

$$\text{Mod}(\Sigma_g) \rightarrow \text{Aut}(H_1(\Sigma_g; \mathbb{Z})) \cong \text{Sp}(2g, \mathbb{Z}).$$

We will study the Torelli group via a surjective group homomorphism

$$\tau : \mathcal{I}_g \rightarrow (\wedge^3 H_1(\Sigma_g)) / H_1(\Sigma_g)$$

called the *Johnson homomorphism* [Joh80] (also see [FM12, Section 6.6]). Here, the inclusion $H_1(\Sigma_g) \hookrightarrow \wedge^3 H_1(\Sigma_g)$ is given by

$$c \mapsto \left(\sum_{i=1}^g a_i \wedge b_i \right) \wedge c$$

where $a_1, b_1, \dots, a_g, b_g \in H_1(\Sigma_g)$ is any standard symplectic basis.

One useful property of τ is its *naturality property*: for any $h \in \text{Mod}(\Sigma_g)$ and any $f \in \mathcal{I}_g$,

$$(4) \quad \tau(hfh^{-1}) = h_*(\tau(f))$$

where on the right hand side, $\text{Mod}(\Sigma_g)$ acts on $(\wedge^3 H_1(\Sigma_g)) / H_1(\Sigma_g)$ via the symplectic representation [FM12, Equation (6.1)]. We record one easy but important consequence of naturality below:

Corollary 2.4. *Fix a hyperelliptic involution $\iota \in \text{Mod}(\Sigma_g)$ and let $\text{SMod}(\Sigma_g)$ denote the hyperelliptic mapping class group of Σ_g , i.e. the centralizer of ι in $\text{Mod}(\Sigma_g)$. There is an inclusion*

$$\text{SMod}(\Sigma_g) \cap \mathcal{I}_g \leq \ker(\tau).$$

This corollary follows from naturality and the fact that ι induces the negation map on a torsion-free abelian group $\wedge^3 H_1(\Sigma_g) / H_1(\Sigma_g)$.

3. PARTIAL CONJUGATIONS OF THE MATSUMOTO–CADAVID–KORKMAZ LEFSCHETZ FIBRATION

In this section we construct a genus- $2g$ Lefschetz fibration $\pi_n : X_n \rightarrow S^2$ for each $n \in \mathbb{Z}_{\geq 0}$ and for all $g \geq 2$ by considering partial conjugations of the Matsumoto–Cadavid–Korkmaz (MCK) Lefschetz fibration. We will prove later that the fibrations π_n are pairwise inequivalent (Section 4), that the 4-manifolds X_n are pairwise diffeomorphic (Section 5), and pairwise symplectomorphic (Section 6).

To define the MCK Lefschetz fibration of genus $2g$, first consider the isotopy classes of curves shown in Figure 2. Dehn twists about these curves form a factorization of an important involution $[\eta] \in \text{Mod}(\Sigma_{2g})$, where $\eta \in \text{Diff}^+(\Sigma_{2g})$ denotes the order-2 diffeomorphism depicted in Figure 2.

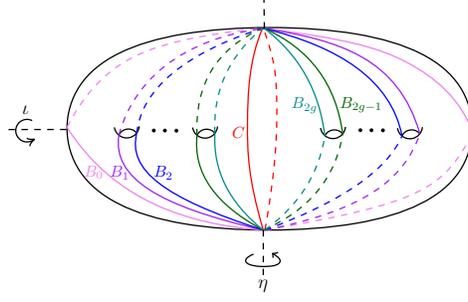


FIGURE 2. Vanishing cycles of the MCK Lefschetz fibration of genus $2g$, the involution η , and a hyperelliptic involution ι .

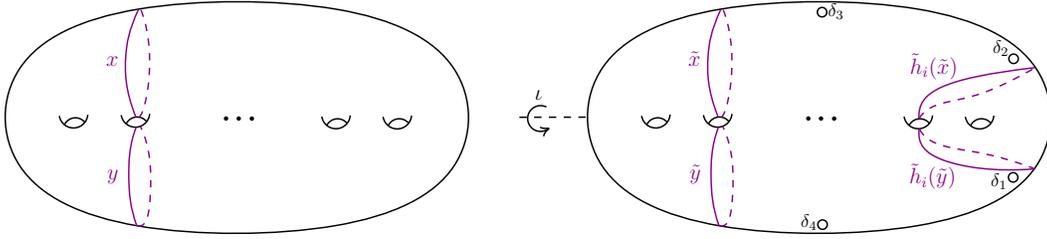


FIGURE 3. Left: Curves used to define $f \in \mathcal{I}_{2g}$; Right: Lifts \tilde{x}, \tilde{y} of x, y to Σ_{2g}^4 and their images under $\tilde{h}_i \in \text{Mod}(\Sigma_{2g}^4)$ defined in Lemma 3.4 for both $i = 1, 2$ (cf. Lemma A.6). A hyperelliptic involution ι acting on Σ_{2g}^4 permuting the boundary components.

Theorem 3.1 (Cadavid [Cad98, Theorem 5.1.1], Korkmaz [Kor01, Theorem 3.4]). *Let*

$$h := T_{B_0} T_{B_1} \dots T_{B_{2g}} T_C \in \text{Mod}(\Sigma_{2g}).$$

Then $h = [\eta]$. In particular, $h^2 = 1 \in \text{Mod}(\Sigma_{2g})$.

With Lemma 3.1 in hand, we are ready to define the MCK Lefschetz fibration of genus $2g$.

Definition 3.2. The *Matsumoto–Cadavid–Korkmaz Lefschetz fibration* (or *MCK Lefschetz fibration*) of genus $2g$ is the Lefschetz fibration $\pi_0 : X_0 \rightarrow S^2$ with monodromy factorization

$$T_{B_0} T_{B_1} \dots T_{B_{2g}} T_C T_{B_0} T_{B_1} \dots T_{B_{2g}} T_C = 1 \in \text{Mod}(\Sigma_{2g}).$$

Hamada [Ham17] constructed multiple sets of four disjoint (-1) -sections of the MCK Lefschetz fibration. In this paper, we will consider two sets of sections given in [Ham17].

Theorem 3.3 (Hamada [Ham17, Sections 3.4.1 – 3.4.2]). *For $i = 1, 2$, let $B_{0,1}^i, B_{1,1}^i, \dots, B_{2g,1}^i, C_1^i$ and $B_{0,2}^i, B_{1,2}^i, \dots, B_{2g,2}^i, C_2^i$ be the isotopy classes of curves in Σ_{2g}^4 as shown in Figure 4. Then*

$$T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i} T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i} = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4} \in \text{Mod}(\Sigma_{2g}^4).$$

For both completeness and convenience of the reader, we prove Theorem 3.3 in Appendix A and deduce Theorem 3.1 from it. The following lemma constructs a positive factorization of $T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4} \in \text{Mod}(\Sigma_{2g}^4)$ for each $n \in \mathbb{Z}$ via partial conjugation and Theorem 3.3.

Lemma 3.4. *Let $B_{0,1}^i, B_{1,1}^i, \dots, B_{2g,1}^i, C_1^i$ and $B_{0,2}^i, B_{1,2}^i, \dots, B_{2g,2}^i, C_2^i$ for $i = 1, 2$ be the isotopy classes of curves in Σ_{2g}^4 as shown in Figure 4 and let \tilde{x}, \tilde{y} be the isotopy classes of curves in Σ_{2g}^4 as shown in the right side of Figure 3. Let*

$$\tilde{h}_i := T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i} \in \text{Mod}(\Sigma_{2g}^4).$$

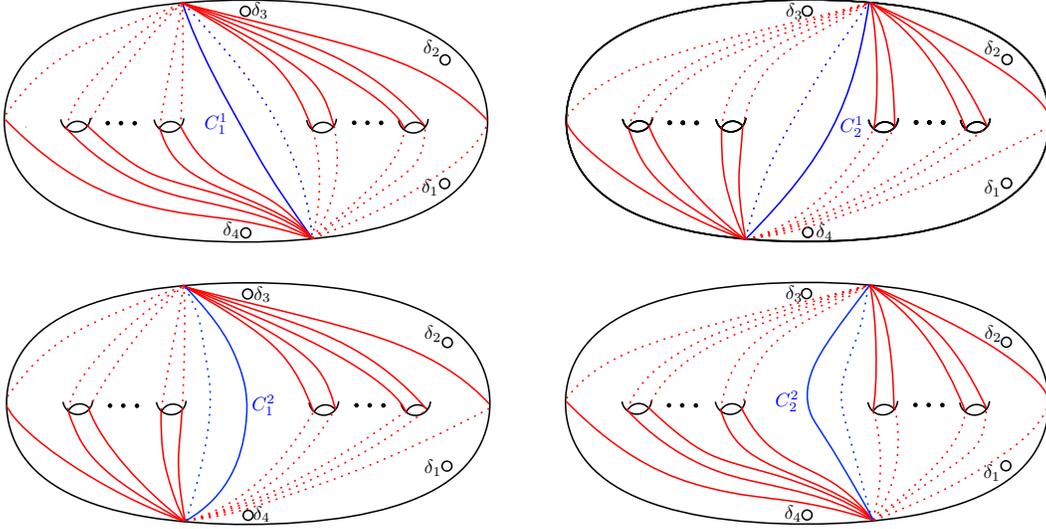


FIGURE 4. Lifts of the vanishing cycles of the MCK Lefschetz fibration of genus $2g$ to Σ_{2g}^4 found by Hamada [Ham17]. The four boundary components of Σ_{2g}^4 are denoted by $\delta_1, \delta_2, \delta_3, \delta_4$. Top left: $B_{0,1}^1, \dots, B_{2g,1}^1, C_1^1$; Top right: $B_{0,2}^1, \dots, B_{2g,2}^1, C_2^1$; Bottom left: $B_{0,1}^2, \dots, B_{2g,1}^2, C_1^2$; Bottom right: $B_{0,2}^2, \dots, B_{2g,2}^2, C_2^2$

Then

$$\tilde{f} := T_{\tilde{x}} T_{\tilde{y}}^{-1} T_{\tilde{h}_1(\tilde{x})} T_{\tilde{h}_1(\tilde{y})}^{-1} = T_{\tilde{x}} T_{\tilde{y}}^{-1} T_{\tilde{h}_2(\tilde{x})} T_{\tilde{h}_2(\tilde{y})}^{-1} \in \text{Mod}(\Sigma_{2g}^4)$$

and for any $n \in \mathbb{Z}$ and for $i = 1, 2$,

$$T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i} T_{\tilde{f}^n(B_{0,2}^i)} T_{\tilde{f}^n(B_{1,2}^i)} \dots T_{\tilde{f}^n(B_{2g,2}^i)} T_{\tilde{f}^n(C_2^i)} = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4} \in \text{Mod}(\Sigma_{2g}^4).$$

Proof. First, note that $T_{\tilde{h}_1(\tilde{x})} = T_{\tilde{h}_2(\tilde{x})}$ and $T_{\tilde{h}_1(\tilde{y})} = T_{\tilde{h}_2(\tilde{y})}$ in $\text{Mod}(\Sigma_{2g}^4)$ by Lemma A.6(a), showing the first desired equality. Furthermore,

$$\tilde{f} = T_{\tilde{x}} T_{\tilde{y}}^{-1} T_{\tilde{h}_i(\tilde{x})} T_{\tilde{h}_i(\tilde{y})}^{-1} = T_{\tilde{h}_i(\tilde{x})} T_{\tilde{h}_i(\tilde{y})}^{-1} T_{\tilde{x}} T_{\tilde{y}}^{-1}$$

where the second equality follows by Lemma A.6(a). Now compute that

$$\tilde{f} \tilde{h}_i = T_{\tilde{h}_i(\tilde{x})} T_{\tilde{h}_i(\tilde{y})}^{-1} T_{\tilde{x}} T_{\tilde{y}}^{-1} \tilde{h}_i = \tilde{h}_i T_{\tilde{x}} T_{\tilde{y}}^{-1} \tilde{h}_i^{-1} T_{\tilde{x}} T_{\tilde{y}}^{-1} \tilde{h}_i = \tilde{h}_i T_{\tilde{x}} T_{\tilde{y}}^{-1} T_{\tilde{h}_i^{-1}(\tilde{x})} T_{\tilde{h}_i^{-1}(\tilde{y})}^{-1} = \tilde{h}_i \tilde{f},$$

where the last equality follows from Lemma A.6(b). Finally, the lemma follows from Theorem 3.3 and the fact that \tilde{h}_j and \tilde{f} commute. \square

By applying the capping and forgetful homomorphisms $\text{Mod}(\Sigma_{2g}^4) \rightarrow \text{Mod}(\Sigma_{2g})$, we now rephrase Lemma 3.4 in terms of $\text{Mod}(\Sigma_{2g})$. Consider the curves $x, y \subseteq \Sigma_{2g}$ shown in Figure 3 and define

$$f := T_x T_y^{-1} T_{h(x)} T_{h(y)}^{-1} \in \mathcal{I}_{2g},$$

where we note that f is contained in the Torelli group \mathcal{I}_{2g} because f is a composition of two bounding pair maps $T_x T_y^{-1}$ and $T_{h(x)} T_{h(y)}^{-1}$. The following corollary is then an immediate consequence of Lemma 3.4.

Corollary 3.5. *Let $g \geq 2$. For any $n \in \mathbb{Z}$,*

$$T_{B_0} T_{B_1} \dots T_{B_{2g}} T_C T_{f^n(B_0)} T_{f^n(B_1)} \dots T_{f^n(B_{2g})} T_{f^n(C)} = 1 \in \text{Mod}(\Sigma_{2g}).$$

Finally, we define the Lefschetz fibrations of interest using Corollary 3.5.

Definition 3.6. Let $g \geq 2$. For any $n \in \mathbb{Z}_{\geq 0}$, let $\pi_n : X_n \rightarrow S^2$ denote the Lefschetz fibration of genus $2g$ with monodromy factorization

$$T_{B_0} T_{B_1} \cdots T_{B_{2g}} T_C T_{f^n(B_0)} T_{f^n(B_1)} \cdots T_{f^n(B_{2g})} T_{f^n(C)} = 1 \in \text{Mod}(\Sigma_{2g}).$$

4. DISTINGUISHING LEFSCHETZ FIBRATIONS VIA THE JOHNSON HOMOMORPHISM

The goal of this section is to prove the following theorem.

Theorem 4.1. *For any $n \neq m \in \mathbb{Z}_{\geq 0}$, the Lefschetz fibrations $\pi_n : X_n \rightarrow S^2$ and $\pi_m : X_m \rightarrow S^2$ are inequivalent.*

The main idea of the proof is to apply Corollary 2.2 by studying the images of the monodromy representations of π_n , intersected with the Torelli group \mathcal{I}_{2g} . As such, consider the following groups for any $n \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} G_n &:= \langle T_{B_0}, T_{B_1}, \dots, T_{B_{2g}}, T_C, T_{f^n(B_0)}, T_{f^n(B_1)}, \dots, T_{f^n(B_{2g})}, T_{f^n(C)} \rangle, \\ G_n^{\mathcal{I}} &:= G_n \cap \mathcal{I}_{2g}, \\ A_n &:= \langle [T_{B_i}^{-1}, f^n], [T_C^{-1}, f^n] : 0 \leq i \leq 2g \rangle. \end{aligned}$$

By construction, G_n is the image of the monodromy representation of $\pi_n : X_n \rightarrow S^2$. We also point out that A_n is a subgroup of $G_n^{\mathcal{I}}$. To see this, write for any curve $c = C$ or B_i for some $0 \leq i \leq 2g$ that

$$T_c^{-1} T_{f^n(c)} = [T_c^{-1}, f^n] = (T_c^{-1} f^n T_c) f^{-n}.$$

The first equality shows that each generator $[T_c^{-1}, f^n]$ of A_n is an element of G_n . The second equality shows that each $[T_c^{-1}, f^n]$ is an element of \mathcal{I}_{2g} because both f^{-n} and $T_c^{-1} f^n T_c$ are elements of \mathcal{I}_{2g} .

In what follows, we study the image $\tau(G_n^{\mathcal{I}})$ of $G_n^{\mathcal{I}}$ under the Johnson homomorphism (cf. Section 2.3)

$$\tau : \mathcal{I}_{2g} \rightarrow (\wedge^3 H) / H, \quad H := H_1(\Sigma_{2g}).$$

To do so, we first describe $G_n^{\mathcal{I}}$ in terms of $G_0^{\mathcal{I}}$ and A_n .

Lemma 4.2. *For any $n \in \mathbb{Z}_{\geq 0}$, there is an equality of subgroups of \mathcal{I}_{2g}*

$$G_n^{\mathcal{I}} = \langle G_0^{\mathcal{I}}, k A_n k^{-1} : k \in G_0 \rangle.$$

Proof. Recall that $A_n \leq G_n^{\mathcal{I}}$ and that $G_0 \leq G_n$. Therefore, it suffices to show the inclusion

$$G_n^{\mathcal{I}} \leq \langle G_0^{\mathcal{I}}, k A_n k^{-1} : k \in G_0 \rangle.$$

By construction, there is an equality of subgroups of $\text{Mod}(\Sigma_{2g})$

$$G_n = \langle G_0, A_n \rangle.$$

Take any $k \in G_n^{\mathcal{I}} \leq G_n$ and write

$$k = \ell_1 m_1 \ell_2 m_2 \cdots \ell_s m_s$$

for some $\ell_i \in G_0, m_i \in A_n$ for $1 \leq i \leq s$. Applying the identity $ab = (aba^{-1})a$ repeatedly to the given factorization of k , write

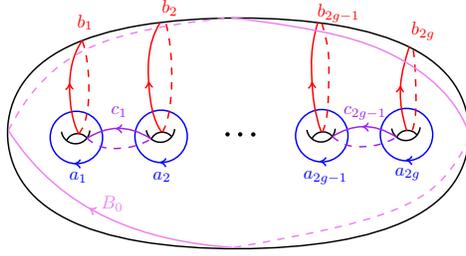
$$k = \left(\prod_{i=1}^s r_i m_i r_i^{-1} \right) r$$

for some $r \in G_0$ and $r_1, \dots, r_s \in G_0$. Because each $m_i \in A_n$ is contained in \mathcal{I}_{2g} and because k is contained in \mathcal{I}_{2g} , the mapping class r is contained in \mathcal{I}_{2g} , and hence in $G_0^{\mathcal{I}}$. \square

The following lemma gives a first description of the image $\tau(G_n^{\mathcal{I}})$ using Lemma 4.2. Another key tool is the naturality (4) of the Johnson homomorphism in conjunction with fact that $G_0^{\mathcal{I}}$ is hyperelliptic.

Lemma 4.3. *For any $n \in \mathbb{Z}_{\geq 0}$, there is an inclusion of subgroups*

$$\tau(G_n^{\mathcal{I}}) \leq n (\wedge^3 H) / H.$$

FIGURE 5. Some homology classes in H

Proof. For any $\ell \in G_n^{\mathcal{I}}$, write using Lemma 4.2

$$\ell = \ell_1(k_1 h_1 k_1^{-1}) \ell_2(k_2 h_2 k_2^{-1}) \dots \ell_s(k_s h_s k_s^{-1})$$

for some $\ell_i \in G_0^{\mathcal{I}}$, $k_i \in G_0$, and $h_i \in A_n$ for $1 \leq i \leq s$. Applying τ to both sides (and applying naturality (4)) yields

$$\tau(\ell) = \left(\sum_{i=1}^s \tau(\ell_i) \right) + \left(\sum_{i=1}^s k_i \cdot \tau(h_i) \right).$$

On the other hand, recall that each generator of G_0 is hyperelliptic (see Figure 2), and hence $G_0^{\mathcal{I}}$ is contained in the hyperelliptic mapping class group $\text{SMod}(\Sigma_{2g})$. By Corollary 2.4, $\tau(\ell_i) = 0$ for all $1 \leq i \leq s$.

Because h_i is an element of A_n for all $1 \leq i \leq s$ and because each k_i preserves the subgroup $n(\wedge^3 H)/H$, it now suffices to show that $\tau(A_n)$ is contained in $n(\wedge^3 H)/H$. To see this, note that for any $k \in \text{Mod}(\Sigma_{2g})$

$$\tau([k^{-1}, f^n]) = \tau(k^{-1} f^n k) - \tau(f^n) = n(\tau(k^{-1} f k) - \tau(f)) \in n(\wedge^3 H)/H. \quad \square$$

In order to further analyze the image $\tau(G_n^{\mathcal{I}})$, we need to verify that certain elements of $(\wedge^3 H)/H$ are nonzero and primitive. We record the following algebraic lemma for this purpose.

Lemma 4.4. *Let $k \geq 2$ and let $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k$ be a symplectic \mathbb{Z} -basis of $H_1(\Sigma_k)$. Let $\gamma_{2i-1} := \alpha_i$ and $\gamma_{2i} := \beta_i$ for all $1 \leq i \leq k$. The set*

$$\{\gamma_i \wedge \gamma_j \wedge \gamma_\ell : 1 \leq i < j < \ell \leq 2k, (i, j, \ell) \neq (i, 2k-1, 2k), (1, 2, 2k-1), (1, 2, 2k)\}$$

forms a \mathbb{Z} -basis of the free abelian group $(\wedge^3 H_1(\Sigma_k))/H_1(\Sigma_k)$.

Proof. Let

$$\omega := \sum_{i=1}^k \alpha_i \wedge \beta_i = \sum_{i=1}^k \gamma_{2i-1} \wedge \gamma_{2i} \in \wedge^2 H_1(\Sigma_k).$$

The wedge ω is independent of the choice of symplectic basis and so the inclusion $H_1(\Sigma_k) \hookrightarrow \wedge^3 H_1(\Sigma_k)$ is given by

$$c \mapsto \omega \wedge c.$$

The group $\wedge^3 H_1(\Sigma_k)$ is torsion-free and has a \mathbb{Z} -basis

$$\{\gamma_i \wedge \gamma_j \wedge \gamma_\ell : 1 \leq i < j < \ell \leq 2k\}.$$

The set

$$\{\gamma_i \wedge \gamma_j \wedge \gamma_\ell : 1 \leq i < j < \ell \leq 2k, (i, j, \ell) \neq (i, 2k-1, 2k), (1, 2, 2k-1), (1, 2, 2k)\} \cup \{\omega \wedge \gamma_i : 1 \leq i \leq 2k\}$$

is another \mathbb{Z} -basis of $\wedge^3 H_1(\Sigma_k)$. To see this, note that the size of the new basis agrees with that of the old and that each element of the old basis can be expressed in terms of this new basis because

$$\begin{aligned}\gamma_i \wedge \gamma_{2k-1} \wedge \gamma_{2k} &= \omega \wedge \gamma_i - \sum_{j=1}^{k-1} \gamma_{2j-1} \wedge \gamma_{2j} \wedge \gamma_i \quad \text{for } 1 \leq i \leq 2k-2, \\ \gamma_1 \wedge \gamma_2 \wedge \gamma_{2k-1} &= \omega \wedge \gamma_{2k-1} - \sum_{j=2}^{k-1} \gamma_{2j-1} \wedge \gamma_{2j} \wedge \gamma_{2k-1}, \\ \gamma_1 \wedge \gamma_2 \wedge \gamma_{2k} &= \omega \wedge \gamma_{2k} - \sum_{j=2}^{k-1} \gamma_{2j-1} \wedge \gamma_{2j} \wedge \gamma_{2k}.\end{aligned}$$

Because $\{\omega \wedge \gamma_i : 1 \leq i \leq 2k\}$ is a \mathbb{Z} -basis of $H_1(\Sigma_k) \leq \wedge^3 H_1(\Sigma_k)$, the quotient $(\wedge^3 H_1(\Sigma_k)) / H_1(\Sigma_k)$ is torsion-free. Furthermore, the \mathbb{Z} -span of

$$\{\gamma_i \wedge \gamma_j \wedge \gamma_\ell : 1 \leq i < j < \ell \leq 2k, (i, j, \ell) \neq (i, 2k-1, 2k), (1, 2, 2k-1), (1, 2, 2k)\}$$

maps isomorphically onto the quotient $\wedge^3 H_1(\Sigma_k) \rightarrow (\wedge^3 H_1(\Sigma_k)) / H_1(\Sigma_k)$ and hence forms a \mathbb{Z} -basis of the quotient. \square

The following lemma refines our description of $\tau(G_n^{\mathcal{I}})$ through a more detailed calculation of the Johnson homomorphism. In the proof, we use the notation of Figure 5 to compute in $(\wedge^3 H) / H$.

Lemma 4.5. *There exists a primitive $v \in (\wedge^3 H) / H$ so that nv is contained in $\tau(G_n^{\mathcal{I}})$ for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Let

$$v := \tau([T_{B_0}^{-1}, f]).$$

To see that nv is contained in $\tau(G_n^{\mathcal{I}})$ for all $n \in \mathbb{Z}_{\geq 0}$, recall that $A_n \leq G_n^{\mathcal{I}}$ and compute that

$$\tau([T_{B_0}^{-1}, f^n]) = \tau(T_{B_0}^{-1} f^n T_{B_0}) - \tau(f^n) = n(\tau(T_{B_0}^{-1} f T_{B_0}) - \tau(f)) = nv.$$

It remains to show that v is primitive. First, note that

$$\tau(T_x T_y^{-1}) = a_1 \wedge b_1 \wedge b_2 = a_1 \wedge b_1 \wedge c_1,$$

where the first equality follows from [FM12, p. 195-196] and the second equality follows because $b_2 = b_1 + c_1$ as elements of H . Therefore,

$$\begin{aligned}\tau(f) &= \tau(T_x T_y^{-1}) + \tau(T_{h(x)} T_{h(y)}^{-1}) \\ &= \tau(T_x T_y^{-1}) + h \cdot \tau(T_x T_y^{-1}) \quad \text{by naturality (4) of } \tau \\ &= a_1 \wedge b_1 \wedge c_1 + (-a_{2g}) \wedge (-b_{2g}) \wedge c_{2g-1}.\end{aligned}$$

Now compute in $(\wedge^3 H) / H$

$$\begin{aligned}v &= \tau(T_{B_0}^{-1} f T_{B_0}) - \tau(f) \\ &= T_{B_0}^{-1} (a_1 \wedge b_1 \wedge c_1 + a_{2g} \wedge b_{2g} \wedge c_{2g-1}) - (a_1 \wedge b_1 \wedge c_1 + a_{2g} \wedge b_{2g} \wedge c_{2g-1}) \quad \text{by naturality (4) of } \tau \\ &= (a_1 \wedge (b_1 - B_0) \wedge c_1 + a_{2g} \wedge (b_{2g} - B_0) \wedge c_{2g-1}) - (a_1 \wedge b_1 \wedge c_1 + a_{2g} \wedge b_{2g} \wedge c_{2g-1}) \\ &= (a_1 \wedge c_1 + a_{2g} \wedge c_{2g-1}) \wedge B_0.\end{aligned}$$

Here and in the rest of this proof, we also denote by B_0 the homology class of B_0 , oriented as in Figure 5.

There exists a symplectic basis $\alpha_1, \beta_1, \dots, \alpha_{2g}, \beta_{2g}$ of $H_1(\Sigma_{2g})$ with

$$\alpha_1 = -a_1, \beta_1 = c_1, \alpha_2 = -a_{2g}, \beta_2 = c_{2g-1}, \alpha_3 = B_0.$$

By Lemma 4.4, the set $\{a_1 \wedge c_1 \wedge B_0, a_{2g} \wedge c_{2g-1} \wedge B_0\}$ can be completed to a \mathbb{Z} -basis of $(\wedge^3 H_1(\Sigma_{2g})) / H_1(\Sigma_{2g})$. Therefore, the sum $a_1 \wedge c_1 \wedge B_0 + a_{2g} \wedge c_{2g-1} \wedge B_0$ is primitive. \square

The following proposition forms the main computational tool in the proof of Theorem 4.1.

Proposition 4.6. *If $n \neq m \in \mathbb{Z}_{\geq 0}$ then G_n and G_m are not conjugate as subgroups of $\text{Mod}(\Sigma_{2g})$.*

Proof. Suppose that there exists $k \in \text{Mod}(\Sigma_{2g})$ so that

$$kG_nk^{-1} = G_m \leq \text{Mod}(\Sigma_{2g}).$$

Because \mathcal{I}_{2g} is normal in $\text{Mod}(\Sigma_{2g})$, this implies that

$$kG_n^{\mathcal{I}}k^{-1} = G_m^{\mathcal{I}} \leq \mathcal{I}_{2g}.$$

By naturality (4) of τ and Lemma 4.3,

$$\tau(G_m^{\mathcal{I}}) = k \cdot \tau(G_n^{\mathcal{I}}) \leq n(\wedge^3 H) / H.$$

By Lemma 4.5, there exists a primitive class $v \in (\wedge^3 H) / H$ so that mv is contained in $n(\wedge^3 H) / H$; in other words, n divides m . By symmetry, m also divides n , i.e. $n = m$. \square

The main theorem of this section now follows as an immediate consequence.

Proof of Theorem 4.1. If $n \neq m \in \mathbb{Z}_{\geq 0}$ then the G_n and G_m are not conjugate as subgroups of $\text{Mod}(\Sigma_{2g})$. By construction, G_n and G_m are the images $\text{im}(\rho_n)$ and $\text{im}(\rho_m)$ of the monodromy representations ρ_n and ρ_m of π_n and π_m respectively. By Corollary 2.2, π_n and π_m are inequivalent Lefschetz fibrations. \square

5. RULED SURFACES AND THEIR BLOWUPS

The goal of this section is to determine the diffeomorphism type of the Lefschetz fibrations $\pi_n : X_n \rightarrow S^2$ of Section 3 (Proposition 5.2). Along the way, we will also determine the diffeomorphism type of the blowdown of X_n of the four (-1) -sections found in Section 3 (Proposition 5.6).

5.1. Diffeomorphism type of X_n . We first compute the algebraic topology invariants of X_n .

Lemma 5.1. *For any $n \in \mathbb{Z}_{\geq 0}$,*

$$\sigma(X_n) = -4, \quad \chi(X_n) = 8 - 4g, \quad b_1(X_n) = 2g, \quad b_2^+(X_n) = 1, \quad b_2(X_n) = 6$$

Proof. By Korkmaz's computations [Kor01, Section 5], the lemma holds in the case of $n = 0$, i.e. the MCK Lefschetz fibration $\pi_0 : X_0 \rightarrow S^2$. (In fact, Korkmaz determines the diffeomorphism type of X_0 for $g \geq 3$; also see Proposition 5.2.)

To compute $\sigma(X_n)$ for $n \geq 1$, consider the Lefschetz fibration $Z \rightarrow D^2$ with monodromy factorization

$$T_{B_0} \dots T_{B_{2g}} T_C \in \text{Mod}(\Sigma_{2g}).$$

Then X_n is formed by gluing two copies of Z together along their boundaries by some diffeomorphism $\partial Z \rightarrow \partial Z$ (which varies with n) for any $n \geq 0$ (cf. Section 2.2). By Novikov additivity,

$$\sigma(X_n) = 2\sigma(Z) = \sigma(X_0) = -4$$

for all $n \geq 0$.

To compute $\chi(X_n)$ for all $n \in \mathbb{Z}_{\geq 0}$, note that $\pi_n : X_n \rightarrow S^2$ has $(4g + 4)$ -many vanishing cycles and that

$$\chi(X_n) = 4 - 8g + (4g + 4) = 8 - 4g.$$

To compute $b_1(X_n)$ for $n \geq 1$, recall first that the positive factorizations of $T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4} \in \text{Mod}(\Sigma_{2g}^4)$ given in Lemma 3.4 are lifts of the positive factorization of the identity in $\text{Mod}(\Sigma_{2g})$ given in Corollary 3.5 that defines the Lefschetz fibration π_n , and hence $\pi_n : X_n \rightarrow S^2$ admits sections. Therefore,

$$H_1(X_n) \cong H_1(\Sigma_{2g}) / \mathbb{Z}\{[B_0], \dots, [B_{2g}], [C], [f^n(B_0)], \dots, [f^n(B_{2g})], [f^n(C)]\}.$$

Because f^n acts trivially on $H_1(\Sigma_{2g})$, this implies that $b_1(X_n) = b_1(X_0) = 2g$ for all $n \geq 1$.

Finally, compute $b_2^+(X_n)$ and $b_2(X_n)$ by solving the system of equations

$$\begin{aligned} 8 - 4g &= \chi(X_n) = 1 - 2g + (b_2^+(X_n) + b_2^-(X_n)) - 2g + 1, \\ -4 &= \sigma(X_n) = b_2^+(X_n) - b_2^-(X_n). \end{aligned} \quad \square$$

The algebraic topology invariants of X_n determine its diffeomorphism type by a theorem of Liu [Liu96].

Proposition 5.2. *For any $n \in \mathbb{Z}_{\geq 0}$, the 4-manifold X_n is diffeomorphic to $(\Sigma_g \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$ and to $(\Sigma_g \tilde{\times} S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$.*

Proof. By the Gompf–Thurston construction, the 4-manifold X_n admits a symplectic form. Let M_n be a minimal, symplectic 4-manifold so that $M_n \# k\overline{\mathbb{C}\mathbb{P}^2}$ is diffeomorphic to X_n . Because $b_2(X_n) = 6$ and $b_2^+(X_n) = 1$ by Lemma 5.1, there is a bound $0 \leq k \leq 5$. By Lemma 5.1,

$$\sigma(M_n) = -4 + k, \quad \chi(M_n) = 8 - 4g - k, \quad b_1(M_n) = 2g, \quad b_2^+(M_n) = 1.$$

Then because $g \geq 2$,

$$c_1^2(M_n) = 2\chi(M_n) + 3\sigma(M_n) = 4 - 8g + k \leq 9 - 8g < 0.$$

Liu’s theorem [Liu96, Theorem A] shows that M_n is an irrational ruled surface. Hence M_n is an S^2 -bundle over Σ_g because $b_1(M_n) = 2g$, i.e. M_n is diffeomorphic to $\Sigma_g \times S^2$ or to $\Sigma_g \tilde{\times} S^2$, and $k = 4$ by Euler characteristic considerations. Since $(\Sigma_g \times S^2) \# \overline{\mathbb{C}\mathbb{P}^2}$ and $(\Sigma_g \tilde{\times} S^2) \# \overline{\mathbb{C}\mathbb{P}^2}$ are diffeomorphic, the proposition follows. \square

An immediate corollary is the smooth portion of Theorem 1.3.

Corollary 5.3. *Let X be a ruled surface with $\chi(X) = 4 - 4g < 0$. There exist infinitely many inequivalent Lefschetz fibrations $X \# 4\overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$ of genus $2g$.*

Proof. By assumption, $X = \Sigma_g \times S^2$ or $\Sigma_g \tilde{\times} S^2$ with $g \geq 2$. By Proposition 5.2, there are diffeomorphisms $X_n \cong X \# 4\overline{\mathbb{C}\mathbb{P}^2}$ for all $n \in \mathbb{Z}_{\geq 0}$. By Theorem 4.1, the Lefschetz fibrations $\pi_n : X \# 4\overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$ of genus $2g$ are pairwise inequivalent for all $n \in \mathbb{Z}_{\geq 0}$. \square

5.2. Lefschetz pencils on ruled surfaces. In this section we prove Theorem 1.2 by studying the topology of X_n after blowing down certain sets of disjoint (-1) -sections.

Definition 5.4. Fix any $n \in \mathbb{Z}_{\geq 0}$. For $i = 1$ or 2 , let $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n, \sigma_{4,i}^n$ denote the four disjoint (-1) -sections of $\pi_n : X_n \rightarrow S^2$ defined by the positive factorization (Lemma 3.4)

$$(5) \quad T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i} T_{\tilde{f}^n(B_{0,2}^i)} T_{\tilde{f}^n(B_{1,2}^i)} \cdots T_{\tilde{f}^n(B_{2g,2}^i)} T_{\tilde{f}^n(C_2^i)} = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4} \in \text{Mod}(\Sigma_{2g}^4)$$

where the curves $B_{0,1}^i, \dots, B_{2g,1}^i, C_1^i, B_{1,2}^i, \dots, B_{2g,2}^i, C_2^i$ in Σ_{2g}^4 are as shown in Figure 4.

Similarly as in the proof of Proposition 5.2, let M_n^i denote the 4-manifold obtained by blowing down the (-1) -sections $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n$, and $\sigma_{4,i}^n$ in X_n .

Lemma 5.5. *Let*

$$Q_{M_n^i} : H_2(M_n^i; \mathbb{Z}) \times H_2(M_n^i; \mathbb{Z}) \rightarrow \mathbb{Z}$$

denote the intersection form of M_n^i . The lattice $(H_2(M_n^i), Q_{M_n^i})$ is even if $i = 1$ and is odd if $i = 2$.

Proof. First, we determine the intersection form of M_n^i . The blowdown $X_n \rightarrow M_n^i$ decomposes the lattice $(H_2(X_n), Q_{X_n})$ as an orthogonal direct sum

$$(H_2(X_n), Q_{X_n}) \cong (H_2(M_n^i), Q_{M_n^i}) \oplus (\mathbb{Z}\{\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n, \sigma_{4,i}^n\}).$$

Consider the reducible singular fiber $F \subseteq X_n$ of π_n corresponding to the vanishing cycle C_1^i in the monodromy factorization of π_n as in (5). Then F is a union of two genus- g surfaces F_1 and F_2 intersecting transversely once, and

$$Q_{X_n}([F_1], [F_1]) = Q_{X_n}([F_2], [F_2]) = -1, \quad Q_{X_n}([F_1], [F_2]) = 1.$$

Below, we determine the parity of the lattice $(H_2(M_n^i), Q_{M_n^i})$ depending on the index i . Note that $H_2(M_n^i)$ is torsion-free for both $i = 1, 2$ because $H_2(X_n)$ is, by Proposition 5.2.

- (1) If $i = 1$ then for some $\alpha_1, \alpha_2 \in H_2(M_n^1)$

$$[F_1] = \alpha_1 - [\sigma_{4,1}^n], \quad [F_2] = \alpha_2 - [\sigma_{1,1}^n] - [\sigma_{2,1}^n] - [\sigma_{3,1}^n]$$

as homology classes in $H_2(X_n)$ with respect to the orthogonal sum decomposition given above, up to possibly permuting the indices of F_1, F_2 . Then because the sections $\sigma_{j,1}^n$ have self-intersection -1 ,

$$Q_{M_n^1}(\alpha_1, \alpha_1) = 0, \quad Q_{M_n^1}(\alpha_2, \alpha_2) = 2, \quad Q_{M_n^1}(\alpha_1, \alpha_2) = 1.$$

The restriction of $Q_{M_n^1}$ to the \mathbb{Z} -span $\mathbb{Z}\{\alpha_1, \alpha_2\}$ is unimodular. Moreover, $H_2(M_n^1) \cong \mathbb{Z}^2$ as a group by Lemma 5.1, and so $\mathbb{Z}\{\alpha_1, \alpha_2\}$ has finite index in $H_2(M_n^1)$. These two facts together imply that $\mathbb{Z}\{\alpha_1, \alpha_2\} = H_2(M_n^1)$. Therefore, the lattice $(H_2(M_n^1), Q_{M_n^1})$ is even because $Q_{M_n^1}(\alpha_j, \alpha_j)$ is even for both $j = 1, 2$.

- (2) If $i = 2$ then $[F_1] \in H_2(X_n)$ is orthogonal to the sections $\sigma_{1,2}^n, \sigma_{2,2}^n, \sigma_{3,2}^n, \sigma_{4,2}^n$ and so $[F_1]$ is contained in $H_2(M_n^2)$ under the orthogonal sum decomposition given above, and

$$Q_{M_n^2}([F_1], [F_1]) = -1.$$

Because there exists an element of $H_2(M_n^2)$ with odd self-intersection, the lattice $(H_2(M_n^2), Q_{M_n^2})$ is odd. \square

The following proposition uses Liu's theorem [Liu96, Theorem A] and the parity of the intersection form of M_n^i to determine its diffeomorphism type.

Proposition 5.6. *There are diffeomorphisms*

$$M_n^i \cong \begin{cases} \Sigma_g \times S^2 & \text{if } i = 1, \\ \Sigma_g \tilde{\times} S^2 & \text{if } i = 2. \end{cases}$$

Proof. For both $i = 1, 2$, the (-1) -sections $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n, \sigma_{4,i}^n$ are disjoint. By the Gompf–Thurston construction [GS99, Theorem 10.2.18], there exists a symplectic form on X_n turning the sections $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n, \sigma_{4,i}^n$ into symplectic submanifolds. Blowing down these sections then yields a symplectic form on M_n^i . Using Lemma 5.1, compute that

$$\sigma(M_n^i) = 0, \quad \chi(M_n^i) = 4 - 4g, \quad b_2^+(M_n^i) = 1, \quad b_1(M_n^i) = 2g, \quad b_2(M_n^i) = 2, \quad c_1^2(M_n^i) = 8 - 8g.$$

Therefore, $c_1^2(M_n^i) < 0$ because $g \geq 2$.

Let N_n^i be a minimal symplectic manifold so that M_n^i is a symplectic blowup of N_n^i . If M_n^i is not minimal then because $b_2^-(M_n^i) = 1$, we can compute that

$$b_2^+(N_n^i) = b_2 = 1, \quad c_1^2(N_n^i) = 9 - 8g < 0.$$

By Liu's theorem [Liu96, Theorem A], N_n^i is an irrational ruled surface, which is a contradiction because $\sigma(N_n^i) \neq 0$. Therefore, M_n^i is a minimal symplectic 4-manifold to which Liu's theorem [Liu96, Theorem A] applies, i.e. M_n^i is an irrational ruled surface with $b_1(M_n^i) = 2g$. This means that M_n^i is diffeomorphic to either $\Sigma_g \times S^2$ or $\Sigma_g \tilde{\times} S^2$. Because $\Sigma_g \times S^2$ has even intersection form and $\Sigma_g \tilde{\times} S^2$ has odd intersection form, Lemma 5.5 gives the desired conclusion. \square

With Proposition 5.6, we are able to define the Lefschetz pencils of interest on ruled surfaces.

Definition 5.7. For any $n \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2$, let $B_n^i \subseteq M_n^i$ denote the image of the four sections $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n$, and $\sigma_{4,i}^n$ under the blowdown $X_n \rightarrow M_n^i$. Let

$$\pi_{n,i} : M_n^i - B_n^i \rightarrow S^2$$

denote the the Lefschetz pencil induced by the Lefschetz fibration $\pi_n : X_n \rightarrow S^2$.

The following corollary proves the smooth portion of Theorem 1.2.

Corollary 5.8. *Let X be a ruled surface with $\chi(X) = 4 - 4g < 0$. There are infinitely many pairwise inequivalent Lefschetz pencils $X - B \rightarrow S^2$ of genus $2g$ and $\#B = 4$.*

Proof. By assumption, $X = \Sigma_g \times S^2$ or $\Sigma_g \tilde{\times} S^2$ with $g \geq 2$. By Proposition 5.6, there exists $i \in \{1, 2\}$ such that X is diffeomorphic to M_n^i for all $n \in \mathbb{Z}_{\geq 0}$. We may assume that $B_n^i = B$ under these diffeomorphisms for all $n \in \mathbb{Z}_{\geq 0}$.

Because the Lefschetz fibrations $\pi_n : X_n \rightarrow S^2$ are pairwise inequivalent by Theorem 4.1, Proposition 2.3 implies that the pencils $\pi_{n,i} : X - B \rightarrow S^2$ and $\pi_{m,i} : X - B \rightarrow S^2$ are inequivalent Lefschetz pencils if $n \neq m$. \square

Remark 5.9 (Comparison with [LS24]). In [LS24, Corollary 1.5, Remark 3.4] we constructed a fiber-sum indecomposable Lefschetz fibration $M_g \rightarrow S^2$ for every genus $g \geq 2$ that admits infinitely many homologically distinct sections of equal self-intersection. One may hope to blow down these sections to obtain infinitely many pairwise inequivalent Lefschetz pencils. However, this requires the sections to have self-intersection -1 . Li [Li99, Corollary 3] showed that any closed, symplectic 4-manifold admitting infinitely many homology classes represented by smoothly embedded spheres of self-intersection -1 is rational or ruled; on the other hand, the symplectic manifold M_g constructed is neither rational nor ruled because $b_2^+(M_g) \geq 2$ [LS24, Proposition 6.6].

Even aside from the specific examples $\pi : M_g \rightarrow S^2$ above, the constructions of [LS24] always yield sections σ of self-intersection $[\sigma]^2 \leq -2$. Below, we prove this upper bound using an idea of Smith [Smi01] to study the action of $\text{Mod}(\Sigma_{g,1})$ on $\partial\mathbb{H}^2 \cong S^1$.

Let $p \in \Sigma_{g,1}$ denote the marked point and fix some lift $\tilde{p} \in \mathbb{H}^2$ of $p \in \Sigma_{g,1}$ under the covering $\mathbb{H}^2 \rightarrow \Sigma_g$. There is a well-defined homomorphism [FM12, Section 5.5.4]

$$\partial : \text{Mod}(\Sigma_{g,1}) \hookrightarrow \text{Homeo}^+(\partial\mathbb{H}^2)$$

where $\partial h \in \text{Homeo}^+(\partial\mathbb{H}^2)$ is the homeomorphism induced by the lift $\tilde{\varphi} \in \text{Diff}^+(\mathbb{H}^2, \tilde{p})$ of any representative $\varphi \in \text{Diff}^+(\Sigma_{g,1})$ of h . For any Dehn twist $T_\ell \in \text{Mod}(\Sigma_{g,1})$, the homeomorphism ∂T_ℓ fixes countably many points of $\partial\mathbb{H}^2$ and moves all other points of $\partial\mathbb{H}^2$ clockwise [Smi01, Proposition 2.1].

Consider any sequence $T_{\ell_1}, \dots, T_{\ell_r} \in \text{Mod}(\Sigma_{g,1})$ of Dehn twists such that

$$T_{\ell_r} \dots T_{\ell_1} = 1 \in \text{Mod}(\Sigma_{g,1})$$

and let $\pi : M \rightarrow S^2$ and $\sigma : S^2 \rightarrow M$ denote the Lefschetz fibration and section corresponding to this positive factorization. Because each $\partial T_{\ell_i} \in \text{Homeo}^+(\partial\mathbb{H}^2)$ fixes points, the sequence $\partial T_{\ell_1}, \dots, \partial T_{\ell_r}$ of homeomorphisms admits a well-defined rotation number $c \in \mathbb{N}$. According to [Smi01, Lemma 2.3], the rotation number c is equal to $-[\sigma]^2$, where $[\sigma]^2$ denotes the self-intersection of σ .

Consider a Lefschetz fibration $\pi : M \rightarrow S^2$ and sections $\sigma_k : S^2 \rightarrow M$ for any $k \in \mathbb{Z}$ constructed in [LS24]. The monodromy factorization of σ_k is of the form [LS24, Theorem 2.1(b)]

$$T_{\ell_{r_1+r_2}} \dots T_{\ell_{r_1+1}} T_{P_\gamma^k(\ell_{r_1})} \dots T_{P_\gamma^k(\ell_1)} = 1 \in \text{Mod}(\Sigma_{g,1})$$

for some point-push mapping class $P_\gamma \in \pi_1(\Sigma_g, p) \trianglelefteq \text{Mod}(\Sigma_{g,1})$ commuting with the product $T_{\ell_{r_1}} \dots T_{\ell_1} \in \text{Mod}(\Sigma_{g,1})$ such that for some $1 \leq i \leq r_1$ and for some $r_1 + 1 \leq j \leq r_1 + r_2$,

$$\hat{i}([\gamma], [\ell_i]) \neq 0 \quad \text{and} \quad \hat{i}([\gamma], [\ell_j]) \neq 0$$

where \hat{i} denotes the (algebraic) intersection form of Σ_g . In particular, no power of $T_{\ell_i}(\gamma)$ or $T_{\ell_j}(\gamma)$ is contained in $\langle \gamma \rangle \leq \pi_1(\Sigma_g, p)$.

The homeomorphism ∂P_γ of $\partial\mathbb{H}^2$ coincides with the action of a deck transformation of $\mathbb{H}^2 \rightarrow \Sigma_g$ [FM12, p. 150-151] and hence has exactly two fixed points, one attracting point p_1 and one repelling point p_2 . Because P_γ and $T_{\ell_{r_1}} \dots T_{\ell_1}$ commute, the homeomorphisms $\partial(T_{\ell_{r_1}} \dots T_{\ell_1})$ and $\partial(T_{\ell_{r_1+r_2}} \dots T_{\ell_{r_1+1}})$ must also fix each point $p_1, p_2 \in \partial\mathbb{H}^2$. Because no power of $T_{\ell_i} P_\gamma T_{\ell_i}^{-1}$ is contained in $\langle P_\gamma \rangle$, the homeomorphism $\partial(T_{\ell_i} P_\gamma T_{\ell_i}^{-1})$ moves at least one point p_1 or p_2 . In other words, ∂T_{ℓ_i} moves at least one point p_1 or p_2 clockwise, and similarly for ∂T_{ℓ_j} . Hence each sequence $\partial T_{\ell_1}, \dots, \partial T_{\ell_{r_1}}$ and $\partial T_{\ell_{r_1+1}}, \dots, \partial T_{\ell_{r_1+r_2}}$ moves the point p_1 around $\partial\mathbb{H}^2$ clockwise at least once. Therefore, the rotation number $c \in \mathbb{N}$ of the sequence $\partial T_{\ell_1}, \dots, \partial T_{\ell_{r_1+r_2}}$ is at least 2, and $[\sigma_k]^2 = -c \leq -2$.

6. SYMPLECTIC FORMS

Throughout this section, fix $g \geq 2$ and let M^1 and M^2 denote the 4-manifolds

$$M^1 := \Sigma_g \times S^2, \quad M^2 := \Sigma_g \tilde{\times} S^2.$$

The goal of this section is to prove that the Lefschetz pencils $\pi_{n,i} : M_n^i - B_n^i \rightarrow S^2$ defined in Definition 5.7 are symplectic for a common symplectic form on M^i for all $n \in \mathbb{Z}_{\geq 0}$ with respect to some diffeomorphism $M_n^i \cong M^i$.

Theorem 6.1. *Let $i = 1$ or $i = 2$. There exist a symplectic form ω on M^i and diffeomorphisms $\Psi_n^i : M_n^i \rightarrow M^i$ for all $n \in \mathbb{Z}_{\geq 0}$ such that the smooth locus of the irreducible components of all fibers of $\pi_{n,i} : M_n^i - B_n^i \rightarrow S^2$ are all symplectic submanifolds of $(M_n^i, (\Psi_n^i)^*\omega)$. Moreover, the regular fibers F_n of $\pi_{n,i}$ are all homologous in $H_2(M^i; \mathbb{Z})$ under Ψ_n^i , i.e.*

$$(\Psi_n^i)_*([F_n]) = (\Psi_m^i)_*([F_m]) \in H_2(M^i; \mathbb{Z})$$

for all $n, m \in \mathbb{Z}_{\geq 0}$.

The following theorem about the Lefschetz fibrations $\pi_n : X \rightarrow S^2$ will also follow from our proof of Theorem 6.1.

Theorem 6.2. *There exist a symplectic form ω on $(\Sigma_g \times S^2) \# \overline{4\mathbb{C}\mathbb{P}^2}$ and diffeomorphisms $\Phi_n : X_n \rightarrow (\Sigma_g \times S^2) \# \overline{4\mathbb{C}\mathbb{P}^2}$ for each $n \in \mathbb{Z}_{\geq 0}$ such that the smooth locus of the irreducible components of all fibers of $\pi_n : X_n \rightarrow S^2$ are all symplectic submanifolds of $(X_n, \Phi_n^*\omega)$. Moreover, the regular fibers F_n of π_n are all homologous in $H_2((\Sigma_g \times S^2) \# \overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$ under Φ_n , i.e.*

$$(\Phi_n)_*([F_n]) = (\Phi_m)_*([F_m]) \in H_2((\Sigma_g \times S^2) \# \overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$$

for all $n, m \in \mathbb{Z}_{\geq 0}$.

We begin with an overview of the proofs of Theorems 6.1 and 6.2.

- (1) The existence of the diffeomorphism $\Phi_n : X_n \rightarrow M^i \# \overline{4\mathbb{C}\mathbb{P}^2}$ sending regular fibers (and in fact the irreducible components of reducible singular fibers) to the same homology class follows from a more careful analysis of the diffeomorphism found in Proposition 5.6. This is done in Lemma 6.3, relying on work of Liu [Liu96] and minimal genus computations in ruled surfaces due to Li–Li [LL97]. Blowing down produces the diffeomorphisms Ψ_n^i .
- (2) Since X_n is the blow-up of a ruled surface, it suffices to construct appropriate symplectic forms ω_n of X_n such that $[(\Phi_n^{-1})^*(\omega_n)] \in H^2(M^i \# \overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$ is independent of $n \in \mathbb{Z}_{\geq 0}$ by work of Lalonde–McDuff [LM96] and McDuff [McD96].
- (3) The first step is to build a suitable symplectic form ω_0 in X_0 by the Gompf–Thurston construction. Realize X_0 (with four sections $\sigma_{1,i}^0, \sigma_{2,i}^0, \sigma_{3,i}^0, \sigma_{4,i}^0$) as the fiber sum of two fibrations over the two hemispheres on S^2 glued over the equator E , and let A be an annulus neighborhood of E . An understanding of the (co)homology of $\pi_0^{-1}(A)$ allows us to control ω_0 on A . Furthermore, we construct ω_0 to be a standard symplectic form near the four (-1) -sections $\sigma_{1,i}^0, \sigma_{2,i}^0, \sigma_{3,i}^0, \sigma_{4,i}^0$ to ensure that the fibers remain symplectic in the blow-down M_0^i . This is all achieved in Lemma 6.9 and Proposition 6.10.
- (4) Finally, to construct the forms ω_n we use the description of X_n via partial conjugation: It is enough to find a representative φ of the twisting mapping class $f \in \mathcal{I}_{2g}$ that induces a bundle symplectomorphism of $(\pi_0^{-1}(A), \omega_0|_A)$ acting by the identity near the sections $\sigma_{1,i}^0, \sigma_{2,i}^0, \sigma_{3,i}^0, \sigma_{4,i}^0$. In fact, we construct both the “fiber part” of $\omega_0|_A$ and φ simultaneously in Proposition 6.8 (and discussion thereafter). The key observation is that the mapping class $\tilde{h}_i \in \text{Mod}(\Sigma_{2g}^4)$ described in Theorem 3.3 and Lemma 3.4 is *reducible*.

6.1. (Co)homology computations. The following lemma determines the homology classes of various fiber classes and section classes of $\pi_n : X_n \rightarrow S^2$ in $H_2(X_n; \mathbb{Z})$. We will also specify a cohomology class $[\nu_i] \in H^2(X_0; \mathbb{R})$ that will be a key input to the Gompf–Thurston construction of a symplectic form ω_0 on X_0 . Below, recall that $\sigma_{j,i}^n : S^2 \rightarrow X_n$ for $1 \leq j \leq 4$ and for $i = 1, 2$ is the (-1) -section of π_n specified in Definition 5.4.

Lemma 6.3. *For any $n \in \mathbb{Z}_{\geq 0}$, let $F_1^n \cup F_2^n$ and $F_3^n \cup F_4^n$ denote the two reducible singular fibers of $\pi_n : X_n \rightarrow S^2$ corresponding to the vanishing cycles C_1^i and $\tilde{f}^n(C_2^i)$ respectively. Let F_1^n, F_3^n and F_2^n, F_4^n denote the irreducible components depicted on the left and right side respectively of the corresponding vanishing cycle in Figure 4. Let F^n denote any regular fiber of π_n .*

(a) *For each $i = 1, 2$, there exists a diffeomorphism*

$$\Phi_n^i : X_n \rightarrow M^i \# 4\overline{\mathbb{C}\mathbb{P}^2}$$

such that

$$(\Phi_n^i)_*([F^n]) = (\Phi_0^i)_*([F^0]), \quad (\Phi_n^i)_*([F_j^n]) = (\Phi_0^i)_*([F_j^0]), \quad (\Phi_n^i)_*([\sigma_{j,i}^n]) = E_j$$

for all $1 \leq j \leq 4$, where $E_1, \dots, E_4 \in H_2(M^i \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$ denote the exceptional divisors coming from each summand $\overline{\mathbb{C}\mathbb{P}^2}$.

(b) *For each $i = 1, 2$, the homology classes $[F_1^n], [F_2^n], [\sigma_{1,i}^n], [\sigma_{2,i}^n], [\sigma_{3,i}^n]$, and $[\sigma_{4,i}^n]$ span $H_2(X_n; \mathbb{R})$, i.e.*

$$H_2(X_n; \mathbb{R}) = \mathbb{R}\{[F_1^n], [F_2^n], [\sigma_{1,i}^n], [\sigma_{2,i}^n], [\sigma_{3,i}^n], [\sigma_{4,i}^n]\} \cong \mathbb{R}^6.$$

(c) *For each $i = 1, 2$, there exists a cohomology class $[\nu_i] \in H^2(X_0; \mathbb{R})$ such that*

$$\langle [\nu_i], [F^0] \rangle = 1, \quad \langle [\nu_i], [F_j^0] \rangle > 0, \quad \langle [\nu_i], [\sigma_{j,i}^0] \rangle = 0$$

for all $1 \leq j \leq 4$.

Proof. Fix $n \in \mathbb{Z}_{\geq 0}$ and $i = 1, 2$. Recall from Proposition 5.6 that blowing down the sections $\sigma_{1,i}^n, \sigma_{2,i}^n, \sigma_{3,i}^n$, and $\sigma_{4,i}^n$ in X_n yields a diffeomorphism

$$\Phi_n^i : X_n \rightarrow M^i \# 4\overline{\mathbb{C}\mathbb{P}^2}$$

with $(\Phi_n^i)_*([\sigma_{j,i}^n]) = E_j$ for all $1 \leq j \leq 4$. The connected sum structure determines an orthogonal decomposition

$$H_2(M^i \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{Z}) \cong H_2(M^i; \mathbb{Z}) \oplus \mathbb{Z}\{E_1, E_2, E_3, E_4\}.$$

There exists a diffeomorphism $M^i \rightarrow M^i$ acting by negation on $H_2(M^i; \mathbb{Z})$ by [LL97, Theorem 3]. Therefore, there exists a diffeomorphism $\Psi^i \in \text{Diff}^+(M^i \# 4\overline{\mathbb{C}\mathbb{P}^2})$ such that

$$\Psi_*^i = -\text{Id} \oplus \text{Id} : H_2(M^i; \mathbb{Z}) \oplus \mathbb{Z}\{E_1, E_2, E_3, E_4\} \rightarrow H_2(M^i; \mathbb{Z}) \oplus \mathbb{Z}\{E_1, E_2, E_3, E_4\}$$

by [Wal64, Lemma 2].

Irreducible components of reducible fibers of Lefschetz fibrations have self-intersection -1 . Because F_1^n and F_2^n intersect once transversely (similarly, F_3^n and F_4^n),

$$(6) \quad Q_{X_n}([F_j^n], [F_j^n]) = -1, \quad Q_{X_n}([F_1^n], [F_2^n]) = Q_{X_n}([F_3^n], [F_4^n]) = 1,$$

for all $1 \leq j \leq 4$, and $Q_{X_n}([F_j^n], [F_k^n]) = 0$ for all other pairs $1 \leq j, k \leq 4$. The intersection numbers $Q_{X_n}([F_j^n], [\sigma_{k,i}^n])$ for $1 \leq j, k \leq 4$ are specified by Figure 4. By computing with these intersection numbers, we prove (a), (b), (c) separately in each case $i = 1$ and $i = 2$.

Case 1: $i = 1$. Suppose $i = 1$ and consider $M^1 = S_g \times S^2$. There exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H_2(M^1; \mathbb{Z})$ such that

$$\begin{aligned} (\Phi_n^1)_*[F_1^n] &= \alpha_1 - E_4, & (\Phi_n^1)_*[F_2^n] &= \alpha_2 - E_1 - E_2 - E_3, \\ (\Phi_n^1)_*[F_3^n] &= \alpha_3 - E_3, & (\Phi_n^1)_*[F_4^n] &= \alpha_4 - E_1 - E_2 - E_4. \end{aligned}$$

Then (6) implies that

$$Q_{M^1}(\alpha_1, \alpha_1) = 0, \quad Q_{M^1}(\alpha_2, \alpha_2) = 2, \quad Q_{M^1}(\alpha_1, \alpha_2) = 1.$$

Let $[\Sigma_g], [S^2] \in H_2(M^1; \mathbb{Z})$ denote the classes of each factor. The only pairs of classes $\alpha_1, \alpha_2 \in H_2(M^1; \mathbb{Z})$ satisfying the prescribed intersection pattern are

$$(\alpha_1, \alpha_2) = \pm([\Sigma_g], [\Sigma_g] + [S^2]), \pm([S^2], [\Sigma_g] + [S^2]).$$

By blowing down the sections $\sigma_{j,1}^n$ for $1 \leq j \leq 4$, the diffeomorphism Φ_n^1 induces a diffeomorphism $M_n^1 \rightarrow M^1 = \Sigma_g \times S^2$ sending $[F^n] \in H_2(M_n^1; \mathbb{Z})$ to $\alpha_1 + \alpha_2 \in H_2(M^1; \mathbb{Z})$, where we also denote by $F^n \subseteq M_n^1$ the image of a regular fiber $F^n \subseteq X_n$ under the blowdown $X_n \rightarrow M_n^1$. Because $F^n \subseteq M_n^1$ is a fiber of a genus- $2g$ Lefschetz pencil structure on M_n^1 , there is a symplectic form on M_n^1 turning F^n into a symplectic submanifold.¹ By the resolution of the symplectic Thom conjecture [OS00b, Theorem 1.1], the smooth minimal genus of $[F^n] \in H_2(M_n^1; \mathbb{Z})$ (and hence of $\alpha_1 + \alpha_2 \in H_2(M^1; \mathbb{Z})$) is $2g$. However, the smooth minimal genus of $\pm(2[S^2] + [\Sigma_g]) \in H_2(M^1; \mathbb{Z})$ is g by work of Li-Li [LL97, Theorem 1]. Therefore, (α_1, α_2) is not equal to $\pm([S^2], [\Sigma_g] + [S^2])$. By the same argument applied to (α_3, α_4) and by noting that $[F^n] = [F_1^n] + [F_2^n] = [F_3^n] + [F_4^n]$, we conclude that

$$(\alpha_1, \alpha_2) = (\alpha_3, \alpha_4) = \pm([\Sigma_g], [\Sigma_g] + [S^2]).$$

After possibly replacing Φ_n^1 with $\Psi^1 \circ \Phi_n^1$,

$$(\Phi_n^1)_*([F_j^n]) = (\Phi_0^1)_*([F_j^0])$$

for all $1 \leq j \leq 4$. This proves (a) for $i = 1$.

The classes α_1 and α_2 form an \mathbb{R} -basis of $H_2(M^1; \mathbb{R}) \cong \mathbb{R}^2$, and so

$$\{(\Phi_n^1)_*([F_1^n]), (\Phi_n^1)_*([F_2^n]), E_1, E_2, E_3, E_4\}$$

forms an \mathbb{R} -basis of $H_2(M^1 \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R})$. Noting that $(\Phi_n^1)_*([\sigma_{j,1}^n]) = E_j$ for all $1 \leq j \leq 4$ concludes the proof of (b) for $i = 1$.

Finally, let $\beta \in H^2(M^1 \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R})$ be any class such that

$$\langle \beta, \alpha_1 \rangle = \langle \beta, \alpha_3 \rangle = \frac{1}{2}, \quad \langle \beta, \alpha_2 \rangle = \langle \beta, \alpha_4 \rangle = \frac{1}{2}, \quad \langle \beta, E_j \rangle = 0 \text{ for all } 1 \leq j \leq 4$$

which exists because $\alpha_1, \alpha_2, E_1, \dots, E_4$ are linearly independent in $H_2(M^1 \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R})$. Letting $[\nu_1] = (\Phi_0^1)^*(\beta)$, compute for all $1 \leq j \leq 4$ that

$$\langle [\nu_1], [F_j^0] \rangle = \langle [\nu_1], [F_j^0] \rangle = \frac{1}{2}, \quad \langle [\nu_1], [F^0] \rangle = \langle [\nu_1], [F_1^0] + [F_2^0] \rangle = 1, \quad \langle [\nu_1], [\sigma_{j,1}^0] \rangle = 0,$$

which proves (c) for $i = 1$.

Case 2: $i = 2$. Suppose $i = 2$ and consider $M^2 = \Sigma_g \tilde{\times} S^2$. There exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H_2(M^2; \mathbb{Z})$ such that

$$\begin{aligned} (\Phi_n^2)_*[F_1^n] &= \alpha_1, & (\Phi_n^2)_*[F_2^n] &= \alpha_2 - E_1 - E_2 - E_3 - E_4, \\ (\Phi_n^2)_*[F_3^n] &= \alpha_3 - E_3 - E_4, & (\Phi_n^2)_*[F_4^n] &= \alpha_4 - E_1 - E_2, \end{aligned}$$

Then (6) implies that

$$\begin{aligned} Q_{M^2}(\alpha_1, \alpha_1) &= -1, & Q_{M^2}(\alpha_2, \alpha_2) &= 3, & Q_{M^2}(\alpha_1, \alpha_2) &= 1, \\ Q_{M^2}(\alpha_3, \alpha_3) &= 1, & Q_{M^2}(\alpha_4, \alpha_4) &= 1, & Q_{M^2}(\alpha_3, \alpha_4) &= 1. \end{aligned}$$

¹One can make a compatible symplectic form ω_n on X_n standard near the sections, in a way that is compatible with the fibration (see Proposition 6.10(1)). The form on M_n^1 is the blow-down of ω_n [MP94, Section 5].

Let $[\Sigma_g] \in H_2(M^2; \mathbb{Z})$ denote the class of a section of the S^2 -bundle $M^2 \rightarrow \Sigma_g$ of self-intersection 1 and let $[S^2] \in H_2(M^2; \mathbb{Z})$ denote the class of the fiber. The only pairs of classes $\alpha_1, \alpha_2 \in H_2(M^2; \mathbb{Z})$ and $\alpha_3, \alpha_4 \in H_2(M^2; \mathbb{Z})$ satisfying the prescribed intersection pattern are

$$\begin{aligned} (\alpha_1, \alpha_2) &= \pm([\Sigma_g] - [S^2], -3[\Sigma_g] + [S^2]), \pm([\Sigma_g] - [S^2], [\Sigma_g] + [S^2]) \\ (\alpha_3, \alpha_4) &= \pm([\Sigma_g], [\Sigma_g]) \end{aligned}$$

By blowing down the sections $\sigma_{j,2}^n$ for $1 \leq j \leq 4$, the diffeomorphism Φ_n^2 induces a diffeomorphism $M_n^2 \rightarrow M^2 = \Sigma_g \tilde{\times} S^2$ sending $[F_2^n]$ to α_2 , where we also denote by $F_2^n \subseteq M_n^2$ the image of the genus- g surface $F_2^n \subseteq X_n$ under the blowdown $X_n \rightarrow M_n^2$. By work of Li-Li [LL97, Theorem 2], the smooth minimal genus of $\pm(-3[\Sigma_g] + [S^2]) \in H_2(M^2; \mathbb{Z})$ is $3g - 1$. Therefore, $\alpha_2 \neq \pm(-3[\Sigma_g] + [S^2])$ because α_2 is represented by a smooth genus- g surface and $g < 3g - 1$ for all $g \geq 2$. Combining with the fact that $[F^n] = [F_1^n] + [F_2^n] = [F_3^n] + [F_4^n]$, we conclude that

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \pm([\Sigma_g] - [S^2], [\Sigma_g] + [S^2], [\Sigma_g], [\Sigma_g]).$$

Therefore, after possibly replacing Φ_n^2 with $\Psi^2 \circ \Phi_n^2$,

$$(\Phi_n^2)_*[F_j^n] = (\Phi_0^2)_*[F_j^0]$$

for all $1 \leq j \leq 4$. This proves (a) for $i = 2$.

The classes α_1 and α_2 form an \mathbb{R} -basis of $H_2(M^2; \mathbb{R}) \cong \mathbb{R}^2$, and so

$$\{(\Phi_n^2)_*[F_1^n], (\Phi_n^2)_*[F_2^n], E_1, E_2, E_3, E_4\}$$

forms an \mathbb{R} -basis of $H_2(M^2 \# 4\overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{R})$. Noting that $(\Phi_n^2)_*[\sigma_{j,2}^n] = E_j$ for all $1 \leq j \leq 4$ concludes the proof of (b) for $i = 2$.

Let $\beta \in H^2(M^2 \# 4\overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{R})$ be any class such that

$$\langle \beta, \alpha_1 \rangle = \frac{1}{4}, \quad \langle \beta, \alpha_3 \rangle = \langle \beta, \alpha_4 \rangle = \frac{1}{2}, \quad \langle \beta, E_j \rangle = 0 \text{ for all } 1 \leq j \leq 4$$

which exists because $\alpha_1, \alpha_3, E_1, \dots, E_4$ are linearly independent in $H_2(M^2 \# 4\overline{4\mathbb{C}\mathbb{P}^2}; \mathbb{R})$. Letting $[\nu_2] = (\Phi_0^2)^*(\beta)$, compute for all $1 \leq j \leq 4$ that

$$\begin{aligned} \langle [\nu_2], [F_1^0] \rangle &= \frac{1}{4}, & \langle [\nu_2], [F_2^0] \rangle &= \frac{3}{4}, & \langle [\nu_2], [F_3^0] \rangle &= \langle [\nu_2], [F_4^0] \rangle = \frac{1}{2}, \\ \langle [\nu_2], [F^0] \rangle &= \langle [\nu_2], [F_1^0] + [F_2^0] \rangle = 1, & \langle [\nu_2], [\sigma_{j,2}^0] \rangle &= 0, \end{aligned}$$

which proves (c) for $i = 2$. □

Remark 6.4 (Capping and forgetting points). Recall the mapping classes in $\text{Mod}(\Sigma_{2g}^4)$

$$\tilde{h}_i = T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}, \quad \tilde{f} = T_{\tilde{x}} T_{\tilde{y}}^{-1} T_{\tilde{h}_i(\tilde{x})} T_{\tilde{h}_i(\tilde{y})}^{-1}$$

defined in Lemma 3.4. We also denote by $\tilde{h}_i, \tilde{f} \in \text{Mod}(\Sigma_{2g,4})$ the images under the homomorphism $\text{Mod}(\Sigma_{2g}^4) \rightarrow \text{Mod}(\Sigma_{2g,4})$ that caps the four boundary components of Σ_{2g}^4 with disks with a marked point. Throughout the rest of this section we fix $i = 1$ or $i = 2$ and drop the index from our notation, i.e. let

$$M := M^i, \quad [\nu] := [\nu_i] \in H^2(X_0; \mathbb{R}), \quad \sigma_j^n := \sigma_{j,i}^n \text{ for } 1 \leq j \leq 4, \quad \tilde{h} := \tilde{h}_i \in \text{Mod}(\Sigma_{2g,4}).$$

It is sometimes more convenient to consider the image

$$(7) \quad \hat{h} \in \text{Mod}(\Sigma_{2g,2})$$

of $\tilde{h} \in \text{Mod}(\Sigma_{2g,4})$ under the forgetful map $\text{Mod}(\Sigma_{2g,4}) \rightarrow \text{Mod}(\Sigma_{2g,2})$ that forgets the marked points p_3, p_4 . We observe that $\hat{h} \in \text{Mod}(\Sigma_{2g,2})$ is independent of i and has order 2; see Corollary A.5 in Appendix A.

Recall that the Lefschetz fibration π_0 was defined by the monodromy factorization

$$T_{B_0}T_{B_1}\dots T_{B_{2g}}T_C T_{B_0}T_{B_1}\dots T_{B_{2g}}T_C = 1 \in \text{Mod}(\Sigma_{2g})$$

in Definition 3.2. Let $q_{4g+4}, \dots, q_1 \in S^2$ denote the singular values of $\pi_0 : X_0 \rightarrow S^2$ corresponding to the vanishing cycles $B_0, B_1, \dots, B_{2g}, C, B_0, B_1, \dots, B_{2g}, C$ respectively. Let $\gamma_1, \dots, \gamma_{4g+4} \in \pi_1(S^2 - \{q_1, \dots, q_{4g+4}\}, b)$ be the chosen generators, each γ_i a loop around q_i , that yields the above monodromy factorization for π_0 .

As in Section 2.2, write $S^2 = D_1 \cup_{\partial} D_2$ such that the basepoint $b \in S^2$ is contained in $E := \partial D_1$ and such that the loops $\gamma_1, \dots, \gamma_{2g+2}$ are contained in D_1 , the loops $\gamma_{2g+3}, \dots, \gamma_{4g+4}$ are contained in D_2 , and

$$[E] = \gamma_1 \dots \gamma_{2g+2} \in \pi_1(S^2 - \{q_1, \dots, q_{4g+4}\}, b).$$

In other words, there is an isomorphism of Σ_{2g} -bundles over S^1

$$\pi_0^{-1}(E) \xrightarrow{\sim} M_\eta$$

for any representative $\eta \in \text{Diff}^+(\Sigma_{2g})$ of $h = T_{B_0}T_{B_1}\dots T_{B_{2g}}T_C \in \text{Mod}(\Sigma_{2g})$ (cf. Theorem 3.1).

We first record an elementary computation of the homology of $\pi_0^{-1}(E)$.

Lemma 6.5. *Let $\eta \in \text{Diff}^+(\Sigma_{2g})$ be any diffeomorphism of order 2 representing $h \in \text{Mod}(\Sigma_{2g})$. For any set of simple closed curves $\ell_1, \dots, \ell_{2g} \subseteq \Sigma_{2g}$ which together form an \mathbb{R} -basis of $H_1(\Sigma_{2g}; \mathbb{R})^{(h)}$, the homology classes*

$$[\Sigma_{2g}], [T_{\ell_1}], \dots, [T_{\ell_{2g}}]$$

form an \mathbb{R} -basis of $H_2(M_\eta; \mathbb{R})$, where $[\Sigma_{2g}]$ denotes the fiber class of $M_\eta \rightarrow S^1$ and T_{ℓ_i} is the torus

$$T_{\ell_i} = \ell_i \times [0, 1] \cup \eta(\ell_i) \times [0, 1]$$

for each $i = 1, \dots, 2g$. In particular,

$$H_2(M_\eta; \mathbb{R}) \cong \mathbb{R}^{2g+1}.$$

Proof. We first show that $H_1(\Sigma_{2g}; \mathbb{R})^{(h)}$ is $2g$ -dimensional. Consider the description of $h \in \text{Mod}(\Sigma_{2g})$ in Theorem 3.1 as the mapping class of the involution shown in Figure 2. This description shows for all $1 \leq i \leq g$ that

$$\begin{aligned} h(a_i - a_{2g+1-i}) &= a_i - a_{2g+1-i}, & h(a_i + a_{2g+1-i}) &= -(a_i + a_{2g+1-i}), \\ h(b_i - b_{2g+1-i}) &= b_i - b_{2g+1-i}, & h(b_i + b_{2g+1-i}) &= -(b_i + b_{2g+1-i}), \end{aligned}$$

where the homology classes $a_i, b_i \in H_1(\Sigma_{2g}; \mathbb{Z})$ are as shown in Figure 5. The set

$$\{a_i - a_{2g+1-i}, a_i + a_{2g+1-i}, b_i - b_{2g+1-i}, b_i + b_{2g+1-i} : 1 \leq i \leq g\}$$

is an \mathbb{R} -basis of $H_1(\Sigma_{2g}; \mathbb{R})$, and hence

$$H_1(\Sigma_{2g}; \mathbb{R})^{(h)} = \mathbb{R}\{a_i - a_{2g+1-i}, b_i - b_{2g+1-i} : 1 \leq i \leq g\} \cong \mathbb{R}^{2g}.$$

Next, we compute $H_2(M_\eta; \mathbb{R})$. Consider the double cover

$$p : \Sigma_{2g} \times \mathbb{R}/\mathbb{Z} \rightarrow M_\eta$$

which is the quotient of the order-2 action on $\Sigma_{2g} \times \mathbb{R}/\mathbb{Z}$ given by

$$\eta_0 : (x, t) \mapsto \left(\eta(x), t + \frac{1}{2} \right) \in M_\eta = (\Sigma_{2g} \times [0, 1]) / ((\eta(x), 0) \sim (x, 1)).$$

By transfer, the restriction

$$p_* : H_2(\Sigma_{2g} \times \mathbb{R}/\mathbb{Z}; \mathbb{R})^{(\eta_0)} \rightarrow H_2(M_\eta; \mathbb{R})$$

is an isomorphism. The fixed subspace $H_2(\Sigma_{2g} \times \mathbb{R}/\mathbb{Z}; \mathbb{R})^{(\eta_0)}$ has an \mathbb{R} -basis

$$\{[\Sigma_{2g}], [\ell_i \times \mathbb{R}/\mathbb{Z}] : 1 \leq i \leq 2g\}.$$

Compute that $p_*([\Sigma_{2g}]) = [\Sigma_{2g}] \in H_2(M_\eta; \mathbb{R})$ and

$$p_*([\ell_i \times \mathbb{R}/\mathbb{Z}]) = [T_{\ell_i}] \in H_2(M_\eta; \mathbb{R}). \quad \square$$

Using Lemma 6.5, the following lemma computes the map on cohomology induced by the inclusion $\pi_0^{-1}(E) \hookrightarrow X_0$.

Lemma 6.6. *Let $\iota : \pi_0^{-1}(E) \hookrightarrow X_0$ denote the inclusion. The image of $\iota^* : H^2(X_0; \mathbb{R}) \rightarrow H^2(\pi_0^{-1}(E); \mathbb{R})$ is $\mathbb{R}\{\text{PD}([\sigma(E)])\}$, where $\sigma : S^2 \rightarrow X_0$ is any section of $\pi_0 : X_0 \rightarrow S^2$.*

Proof. Write $X_0 = Y_1 \cup_{\pi_0^{-1}(E)} Y_2$ with

$$Y_1 = \pi_0^{-1}(D_1), \quad Y_2 = \pi_0^{-1}(D_2).$$

Because Y_i is a genus- $2g$ Lefschetz fibration over D^2 for each $i = 1, 2$, it is a finite union of 2-handles $D^2 \times D^2$ attached to $\Sigma_{2g} \times D^2$, each with attaching regions $D^2 \times S^1$ [GS99, Section 8.2]. There is a Mayer–Vietoris sequence of the form

$$H^2(\sqcup_{j=1}^m D^2 \times S^1) \rightarrow H^3(Y_i) \rightarrow H^3(\Sigma_{2g} \times D^2) \oplus H^3(\sqcup_{j=1}^m (D^2 \times D^2))$$

for some $m \geq 0$, which implies that $H^3(Y_i; \mathbb{R}) = 0$ for each $i = 1, 2$. Moreover, $H^3(X_0; \mathbb{R}) \cong \mathbb{R}^{2g}$ because X_0 is diffeomorphic to $(\Sigma_g \times S^2) \# 4\mathbb{C}\mathbb{P}^2$ by Proposition 5.2, and $H^2(\pi_0^{-1}(E); \mathbb{R}) \cong \mathbb{R}^{2g+1}$ by Lemma 6.5. Combining these three computations with the Mayer–Vietoris sequence for $X_0 = Y_1 \cup_{\pi_0^{-1}(E)} Y_2$ yields

$$H^2(X_0; \mathbb{R}) \xrightarrow{(j_1)^* \oplus (j_2)^*} H^2(Y_1; \mathbb{R}) \oplus H^2(Y_2; \mathbb{R}) \xrightarrow{(k_1)^* - (k_2)^*} \underbrace{H^2(\pi_0^{-1}(E); \mathbb{R})}_{\cong \mathbb{R}^{2g+1}} \rightarrow \underbrace{H^3(X_0; \mathbb{R})}_{\cong \mathbb{R}^{2g}} \rightarrow 0$$

where $j_i : Y_i \hookrightarrow X_0$ and $k_i : \pi_0^{-1}(E) \hookrightarrow Y_i$ are inclusion maps for $i = 1, 2$ such that $\iota = j_i \circ k_i$ for $i = 1, 2$. Therefore, the image of $(k_1)^*$ (and hence the image of ι^*) is at most 1-dimensional.

Observe that

$$\iota^*(\text{PD}([\sigma(S^2)])) = \text{PD}([\sigma(E)]),$$

where $\text{PD}([\sigma(S^2)])$ and $\text{PD}([\sigma(E)])$ denote the Poincaré duals of $[\sigma(S^2)]$ and $[\sigma(E)]$ in X_0 and $\pi_0^{-1}(E)$ respectively. The class $\text{PD}([\sigma(E)])$ is nonzero (e.g. because $\sigma(E)$ intersects a fiber F in $\pi_0^{-1}(E)$ once transversely), and hence $\text{PD}([\sigma(E)])$ spans the image of ι^* , i.e.

$$\text{im}(\iota^*) = \mathbb{R}\{\text{PD}([\sigma(E)])\} \subseteq H^2(\pi_0^{-1}(E); \mathbb{R}). \quad \square$$

6.2. Gompf–Thurston construction for (X_n, ω_n) . The goal of this subsection is to prove the following proposition which builds a suitable symplectic form ω_n on X_n . A key point of the construction below is the good control over the cohomology classes $[\omega_n] \in H^2(X_n; \mathbb{R})$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proposition 6.7. *For any $n \in \mathbb{Z}_{\geq 0}$, let $F_1^n \cup F_2^n$ and $F_3^n \cup F_4^n$ denote the two reducible singular fibers of $\pi_n : X_n \rightarrow S^2$ with $F_1^n, F_2^n, F_3^n, F_4^n$ denoting the irreducible components as defined in Lemma 6.3. For each $n \in \mathbb{Z}_{\geq 0}$, there exists a symplectic form ω_n on X_n satisfying the following properties:*

- (a) *The smooth loci of the irreducible components of the fibers of $\pi_n : X_n \rightarrow S^2$ are all symplectic submanifolds of (X_n, ω_n) .*
- (b) *For all $n \in \mathbb{Z}_{\geq 0}$ and for all $1 \leq j \leq 4$,*

$$\langle [\omega_n], [F_j^n] \rangle = \langle [\omega_0], [F_j^0] \rangle.$$

In particular, $\langle [\omega_n], [F_j^n] \rangle$ is independent of n .

- (c) *(Standard near the sections) For each $1 \leq j \leq 4$, there exists an open tubular neighborhood N_j^n of $\sigma_j^n(S^2)$ such that $\pi_n|_{N_j^n} : N_j^n \rightarrow S^2$ is isomorphic to the tautological line bundle $\pi_L : L \rightarrow \mathbb{C}\mathbb{P}^1$ via an orientation-preserving diffeomorphism $\phi_j^n : N_j^n \rightarrow L$. Furthermore, for a small enough neighborhood $\sigma_j^n(S^2) \subseteq V_j^n \subseteq N_j^n$,*

$$\omega_n|_{V_j^n} = (\phi_j^n)^*(a_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2} + \pi_L^* \omega_{S^2})$$

where $\pi_{\mathbb{C}^2} : L \rightarrow \mathbb{C}^2$ is the standard blow-down map, $\omega_{\mathbb{C}^2}$ is the standard symplectic form on \mathbb{C}^2 , ω_{S^2} is the Fubini–Study form on S^2 , and a_j is some positive constant independent of n . In particular, all sections σ_j^n are symplectic and

$$\langle [\omega_n], [\sigma_j^n(S^2)] \rangle = \pi$$

is independent of n .

In order to explicitly build a suitable form around $\pi_0^{-1}(E) \subseteq X_0$ and around each section σ_j^n , we will need to find nice representative diffeomorphisms of the monodromy $\hat{h} \in \text{Mod}(\Sigma_{2g,2})$ and $\tilde{h} \in \text{Mod}(\Sigma_{2g,4})$, as well as compatible representatives of the conjugating mapping class $\tilde{f} \in \text{Mod}(\Sigma_{2g,4})$. This is achieved in the following proposition, whose proof is given in Appendix B.

Proposition 6.8. *Let α, β, γ , and δ be disjoint curves in $\Sigma_{2g,4}$ representing the isotopy classes $\tilde{x}, \tilde{y}, \tilde{h}(\tilde{x}), \tilde{h}(\tilde{y})$ respectively. There exist*

- a unit-area symplectic form θ on Σ_{2g} ,
- symplectomorphisms $\varphi \in \text{Diff}^+(\Sigma_{2g,4})$ and $\hat{\eta} \in \text{Diff}^+(\Sigma_{2g,2})$ of (Σ_{2g}, θ) with

$$[\varphi] = \tilde{f} \in \text{Mod}(\Sigma_{2g,4}), \quad [\hat{\eta}] = \hat{h} \in \text{Mod}(\Sigma_{2g,2}), \quad \text{and} \quad \hat{\eta}^2 = \text{Id}_{\Sigma_{2g,2}},$$
- an isotopy $\kappa_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ with $\kappa_0 = \text{Id}_{\Sigma_{2g,2}}$,
- a disjoint union $W = W_\alpha \sqcup W_\beta \sqcup W_\gamma \sqcup W_\delta$ of tubular neighborhoods of $\alpha, \beta, \gamma, \delta$ in $\Sigma_{2g,4}$, and
- a disjoint union $O = O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$ of neighborhoods of p_1, p_2, p_3 and p_4

satisfying the following properties:

- (a) The symplectomorphism φ is supported on W , i.e.

$$\varphi|_{\Sigma_{2g,4}-W} = \text{Id}|_{\Sigma_{2g,4}-W}.$$

- (b) The symplectomorphisms φ and $\hat{\eta}$ commute, i.e.

$$\varphi \circ \hat{\eta} = \hat{\eta} \circ \varphi.$$

- (c) The diffeomorphism $\tilde{\eta} := \hat{\eta} \circ \kappa_1$ fixes O pointwise and satisfies

$$[\tilde{\eta}] = \tilde{h} \in \text{Mod}(\Sigma_{2g,4}).$$

- (d) For all $t \in [0, 1]$

$$\kappa_t(O) \cap W = \emptyset.$$

In particular, $W \cap O = \emptyset$.

Isomorphisms near the equator E . For the remainder of this section, fix the notation of Proposition 6.8. Let $N \cong E \times (-2, 2)$ be a tubular neighborhood of E in S^2 disjoint from the set of singular values of π_0 . Shrinking N yields another tubular neighborhood $A \cong E \times (-1, 1)$ of E in S^2 .

The four marked points p_1, p_2, p_3, p_4 fixed by $\tilde{\eta} \in \text{Diff}^+(\Sigma_{2g,4})$ define sections \tilde{s}_i of $M_{\tilde{\eta}} \times (-2, 2) \rightarrow S^1 \times (-2, 2)$ for $1 \leq i \leq 4$ by

$$(8) \quad \tilde{s}_i : (t, s) \mapsto ((p_i, t), s) \in M_{\tilde{\eta}} \times (-2, 2),$$

where we recall that the mapping torus $M_{\tilde{\eta}}$ is identified with

$$M_{\tilde{\eta}} = (\Sigma_{2g,4} \times [0, 1]) / ((\tilde{\eta}(x), 0) \sim (x, 1)).$$

There exists an isomorphism of Σ_{2g} -bundles over $S^1 \times (-2, 2)$

$$(9) \quad \tilde{\mathcal{G}} : \pi_0^{-1}(N) \rightarrow M_{\tilde{\eta}} \times (-2, 2)$$

that sends each section σ_i , $1 \leq i \leq 4$ of π_0 to the section \tilde{s}_i of $M_{\tilde{\eta}} \times (-2, 2) \rightarrow S^1 \times (-2, 2)$.

The isotopy $\kappa_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ defines a map of Σ_{2g} -bundles

$$\begin{aligned} \mathcal{K} : M_{\tilde{\eta}} \times (-2, 2) &\rightarrow M_{\tilde{\eta}}, \\ ((x, t), s) &\mapsto (\kappa_t(x), t). \end{aligned}$$

Finally, define a map of Σ_{2g} -bundles $\hat{\mathcal{G}}$ to be the composition

$$(10) \quad \hat{\mathcal{G}} := \mathcal{K} \circ \tilde{\mathcal{G}} : \pi_0^{-1}(N) \rightarrow M_{\hat{\eta}}$$

Because κ_t fixes the points p_1, p_2 for all $t \in [0, 1]$, the map $\hat{\mathcal{G}}$ sends each section σ_1 and σ_2 of π_0 to the section \hat{s}_1 and \hat{s}_2 of $M_{\hat{\eta}} \rightarrow S^1$ respectively, where \hat{s}_i is the section defined by the point p_i fixed by $\hat{\eta} \in \text{Diff}^+(\Sigma_{2g,2})$ for $i = 1, 2$.

The pullback $\text{pr}_1^*\theta$ of $\theta \in \Omega^2(\Sigma_{2g})$ under the projection $\text{pr}_1 : \Sigma_{2g} \times \mathbb{R} \rightarrow \Sigma_{2g}$ is invariant under deck transformations of $\Sigma_{2g} \times \mathbb{R} \rightarrow M_{\hat{\eta}}$, and so induces a closed 2-form $\hat{\theta}$ on $M_{\hat{\eta}}$ that restricts to the symplectic form θ on each fiber of $M_{\hat{\eta}} \rightarrow S^1$.

In order to use the form $\hat{\theta}$ in the Gompf–Thurston construction, we must determine its cohomology class in $H^2(M_{\hat{\eta}}; \mathbb{R})$.

Lemma 6.9. *The cohomology class of the form $\hat{\theta}$ is*

$$[\hat{\theta}] = \text{PD}([\hat{s}_1(S^1)]) \in H^2(M_{\hat{\eta}}; \mathbb{R}).$$

Proof. In order to determine the class $[\hat{\theta}]$, we evaluate $[\hat{\theta}]$ on $H_2(M_{\hat{\eta}}; \mathbb{R})$. Let $\ell_1, \dots, \ell_{2g} \subseteq \Sigma_{2g}$ be simple closed curves which together span $H_1(\Sigma_{2g}; \mathbb{R})^{\langle \hat{\eta} \rangle}$. We may assume that each curve ℓ_i is disjoint from the marked point $p_1 \in \Sigma_{2g}$ which is fixed by $\hat{\eta}$ and defines the section $\hat{s}_1 : S^1 \rightarrow M_{\hat{\eta}}$. For each $1 \leq i \leq 2g$, consider the torus T_{ℓ_i} found in Lemma 6.5. We claim that $\hat{\theta}$ vanishes on T_{ℓ_i} for all $1 \leq i \leq 2g$. Indeed, $T_{\ell_i} \subseteq M_{\hat{\eta}}$ is the image of

$$\ell_i \times \mathbb{R} \subseteq \Sigma_{2g} \times \mathbb{R}$$

under the cover $\Sigma_{2g} \times \mathbb{R} \rightarrow M_{\hat{\eta}}$. Because $\text{pr}_1(\ell_i \times \mathbb{R}) = \ell_i \subseteq \Sigma_{2g}$ is 1-dimensional,

$$(\text{pr}_1^*\theta)|_{\ell_i \times \mathbb{R}} = 0.$$

Therefore $\hat{\theta}|_{T_{\ell_i}} = 0$ as well because $\text{pr}_1^*\theta$ descends to $\hat{\theta}$ and $\ell_i \times \mathbb{R}$ descends to T_{ℓ_i} under $\Sigma_{2g} \times \mathbb{R} \rightarrow M_{\hat{\eta}}$. In other words,

$$\langle [\hat{\theta}], [T_{\ell_i}] \rangle = 0$$

for all $1 \leq i \leq 2g$.

On the other hand,

$$\langle [\hat{\theta}], [\Sigma_{2g}] \rangle = \int_{\Sigma_{2g}} \hat{\theta}|_{\Sigma_{2g}} = \int_{\Sigma_{2g}} \theta = 1$$

for any fiber Σ_{2g} of $M_{\hat{\eta}} \rightarrow S^1$.

Now we analyze $\text{PD}([\hat{s}_1(S^1)])$ evaluated on $H_2(M_{\hat{\eta}}; \mathbb{R})$. Observe that $T_{\ell_i} \subseteq M_{\hat{\eta}}$ and $\hat{s}_1(S^1) \subseteq M_{\hat{\eta}}$ are disjoint for all $1 \leq i \leq 2g$ because $p_1 \in \Sigma_{2g}$ and $\ell_i \subseteq \Sigma_{2g}$ are disjoint. Therefore,

$$\langle \text{PD}([\hat{s}_1(S^1)]), [T_{\ell_i}] \rangle = 0$$

for all $1 \leq i \leq 2g$.

Similarly, Σ_{2g} and $\hat{s}_1(S^1)$ intersect positively once for any fiber Σ_{2g} of $M_{\hat{\eta}} \rightarrow S^1$ because \hat{s}_1 is a section. Therefore,

$$\langle \text{PD}([\hat{s}_1(S^1)]), [\Sigma_{2g}] \rangle = 1.$$

Finally, Lemma 6.5 shows that the classes $[T_{\ell_1}], \dots, [T_{\ell_{2g}}]$ and $[\Sigma_{2g}]$ form a basis of $H_2(M_{\hat{\eta}}; \mathbb{R})$. The forms $[\hat{\theta}]$ and $\text{PD}([\hat{s}_1(S^1)])$ agree on this basis of $H_2(M_{\hat{\eta}}; \mathbb{R})$, and

$$[\hat{\theta}] = \text{PD}([\hat{s}_1(S^1)]) \in H^2(M_{\hat{\eta}}; \mathbb{R}). \quad \square$$

Below, we first build a symplectic form on X_0 whose restriction to a neighborhood of $\pi_0^{-1}(E)$ interacts well with the map $\varphi \in \text{Symp}(\Sigma_{2g}, \theta)$ which will be used in the partial conjugation construction to build X_n . In order for our construction to also interact nicely with the symplectic blowing down procedure, we *standardize* our symplectic form near the sections.

Proposition 6.10 (Gompf–Thurston construction for $\pi_0 : X_0 \rightarrow S^2$). *There exists a symplectic form ω_0 on X_0 such that the smooth loci of the irreducible components of the fibers of $\pi_0 : X_0 \rightarrow S^2$ and the sections $\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0$ of $\pi_0 : X_0 \rightarrow S^2$ are all symplectic submanifolds of (X_0, ω_0) . Moreover, ω_0 satisfies the following properties.*

- (1) (Standard near the sections) *For each $1 \leq j \leq 4$, let U_j be disjoint neighborhoods of σ_j^0 . Then, there exists an open tubular neighborhood $N_j \subset U_j$ of σ_j^0 such that $\pi_0|_{N_j} : N_j \rightarrow S^2$ is isomorphic to the tautological line bundle $\pi_L : L \rightarrow \mathbb{C}P^1$ via an orientation-preserving diffeomorphism $\phi_j : N_j \rightarrow L$. Furthermore, for a small enough neighborhood $\sigma_j^0 \subseteq N'_j \subseteq N_j$,*

$$\omega_0|_{N'_j} = \phi_j^*(a_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2} + \pi_L^* \omega_{S^2})$$

where $\pi_{\mathbb{C}^2} : L \rightarrow \mathbb{C}^2$ is the standard blow-down map, $\omega_{\mathbb{C}^2}$ is the standard symplectic form on \mathbb{C}^2 , ω_{S^2} is the Fubini–Study form on S^2 , and a_j is some positive constant.

- (2) (Standard over the twisting region) *The restriction of ω_0 to $\pi_0^{-1}(A) - \bigcup_{j=1}^4 N_j$ takes the form*

$$\omega_0|_{\pi_0^{-1}(A) - \bigcup_{j=1}^4 N_j} = \varepsilon \hat{\mathcal{G}}^* \hat{\theta}|_{\pi_0^{-1}(A) - \bigcup_{j=1}^4 N_j} + \pi_0^* \omega_A$$

for some $\varepsilon > 0$ and some symplectic form ω_A on $A \subseteq S^2$.

Proof. Recall that a class $[\nu] \in H^2(X_0; \mathbb{R})$ was chosen in Lemma 6.3(c) so that $[\nu] \in H^2(X_0; \mathbb{R})$ pairs positively with the irreducible components of all fibers of $\pi_0 : X_0 \rightarrow S^2$ and so that $\langle [\nu], [F] \rangle = 1$ for any smooth fiber F of π_0 . Given such a class $[\nu]$, the Gompf–Thurston construction ([GS99, Proof of Theorem 10.2.18] or [Gom05, Theorem 2.7]) yields a closed 2-form ζ on X_0 with $[\zeta] = [\nu]$ such that

- (a) $\zeta|_S$ is symplectic for S the smooth locus of an irreducible component of any fiber of π_0 , and
- (b) for any critical point $p \in X_0$ of π_0 and for some charts $U_p \subseteq \mathbb{C}^2$ and $V_{\pi_0(p)} \subseteq \mathbb{C}$ on which π_0 takes the form $(z, w) \mapsto z^2 + w^2$, the restriction $\zeta|_{U_p}$ is the standard symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Furthermore, recall that $[\nu] \in H^2(X_0; \mathbb{R})$ was chosen in Lemma 6.3(c) so that $\langle [\nu], [\sigma_j^0] \rangle = 0$ for all $1 \leq j \leq 4$.

Step 1: Standardizing over the annulus. We will modify the form ζ over the tubular neighborhood $N \subseteq S^2$ of E by the form $\hat{\mathcal{G}}^* \hat{\theta}$. Let $V_0 := N$. For each $i = 1, 2$, consider the open disks $V_i := D_i - (\bar{A} \cap D_i)$ in S^2 . Then

$$S^2 = V_0 \cup V_1 \cup V_2$$

is an open cover of S^2 . Let $\{\rho_0, \rho_1, \rho_2\}$ be a partition of unity subordinate to this open cover. Consider closed 2-forms ξ_i on V_i for each $i = 0, 1, 2$, defined by

$$\xi_0 := \hat{\mathcal{G}}^* \hat{\theta}, \quad \xi_1 = \zeta|_{\pi_0^{-1}(V_1)}, \quad \xi_2 = \zeta|_{\pi_0^{-1}(V_2)}.$$

Then

$$[\xi_1] = [\nu|_{\pi_0^{-1}(V_1)}] \in H^2(\pi_0^{-1}(V_1); \mathbb{R}) \quad \text{and} \quad [\xi_2] = [\nu|_{\pi_0^{-1}(V_2)}] \in H^2(\pi_0^{-1}(V_2); \mathbb{R})$$

by the Gompf–Thurston construction. On the other hand, $[\nu|_{\pi_0^{-1}(E)}] = a \text{PD}([\sigma_1^0(E)]) \in H^2(\pi_0^{-1}(E); \mathbb{R})$ for some $a \in \mathbb{R}$ by Lemma 6.6, and $a = 1$ because $\langle [\nu|_{\pi_0^{-1}(E)}], [F] \rangle = 1$ for any fiber F of $\pi_0^{-1}(E)$. By Lemma 6.9,

$$[(\hat{\mathcal{G}}^* \hat{\theta})|_{\pi_0^{-1}(E)}] = \text{PD}([\sigma_1^0(E)]) \in H^2(\pi_0^{-1}(E); \mathbb{R})$$

because $\hat{\mathcal{G}}$ sends the section $\sigma_1^0(E)$ to the section $\hat{s}_1(S^1)$ of $M_{\hat{\eta}}$. Finally because $H^2(\pi_0^{-1}(E); \mathbb{R}) \cong H^2(\pi_0^{-1}(N); \mathbb{R})$,

$$[\xi_0] = [\nu|_{\pi_0^{-1}(V_0)}] \in H^2(\pi_0^{-1}(V_0); \mathbb{R}).$$

For each $i = 0, 1, 2$, there exists a 1-form β_i on $\pi_0^{-1}(V_i)$ such that

$$\xi_i = \nu|_{\pi_0^{-1}(V_i)} + d\beta_i.$$

Define the closed 2-form ξ on X_0 by

$$\xi := \nu + \sum_{i=0}^2 d((\rho_i \circ \pi_0)\beta_i) = \sum_{i=0}^2 d(\rho_i \circ \pi_0) \wedge \beta_i + (\rho_i \circ \pi_0)\xi_i.$$

We claim that ξ restricts to a symplectic form on smooth loci S of irreducible components of any fiber of π_0 . To see this, note that $d(\rho_i \circ \pi_0)$ vanishes on S for each $i = 0, 1, 2$, so

$$\xi|_S = \sum_{i=0}^2 (\rho_i \circ \pi_0)\xi_i|_S.$$

If $S \subseteq \pi_0^{-1}(V_i)$ then $\xi_i|_S$ is symplectic for $i = 1, 2$ by the Gompf–Thurston construction. For $i = 0$, recall that the form $\hat{\theta}$ restricts to a symplectic form on each fiber of $M_{\bar{\eta}} \rightarrow S^1$ by construction. Because $\hat{\mathcal{G}}$ maps a fiber of $\pi_0^{-1}(N) \rightarrow N$ diffeomorphically onto a fiber of $M_{\bar{\eta}} \rightarrow S^1$, the pullback $\hat{\mathcal{G}}^*\hat{\theta}$ restricted to a fiber of $\pi_0^{-1}(N) \rightarrow N$ is still symplectic. Therefore, $\xi|_S$ is symplectic on S because it is a positive linear combination of volume forms on S .

Step 2: Standardizing near the sections. Observe that $\pi_0 : X_0 \rightarrow S^2$ is a submersion on each section σ_j^0 . Therefore, for a small enough tubular neighborhood $N_j \subseteq U_j$ of σ_j^0 , the restriction $\pi_0|_{N_j} : N_j \rightarrow S^2$ is isomorphic² to a rank 2 vector-bundle over S^2 . Since $[\sigma_j^0]^2 = -1$, there exists a \mathbb{C} -line bundle isomorphism $\phi_j : N_j \rightarrow L$. Define

$$\lambda_j := (\phi_j^{-1})^*\xi|_{N_j} \in \Omega^2(L).$$

Since ϕ_j is orientation-preserving on fibers, the form λ_j tames the standard complex structure i of L in the fiber direction of $\pi_L : L \rightarrow \mathbb{C}\mathbb{P}^1$. Let

$$L(1) := \{v \in L : |\pi_{\mathbb{C}^2}(v)|^2 \leq 1\}$$

where $|\cdot|$ denotes the standard, Kähler metric compatible with $\omega_{\mathbb{C}^2}$. The following is a modification of a lemma of McDuff–Polterovich [MP94, Lemma 5.5.B].

Lemma 6.11 (cf. McDuff–Polterovich [MP94, Lemma 5.5.B]). *There exists a 2-form $\lambda'_j \in \Omega^2(L(1))$ satisfying the following properties:*

- (1) λ'_j agrees with λ_j near the boundary of $L(1)$.
- (2) $\lambda'_j = c_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2}$ near the zero section of $\pi_L : L \rightarrow \mathbb{C}\mathbb{P}^1$ for some constant $c_j > 0$.
- (3) λ'_j tames i in the fiber direction of $\pi_L : L \rightarrow \mathbb{C}\mathbb{P}^1$.

Proof. This is the same computation as [MP94, Lemma 5.5.B]; for completeness, we recall the proof. Let $B(r) \subseteq \mathbb{C}^2$ denote the open ball of radius r in \mathbb{C}^2 with respect to the Kähler metric compatible with $\omega_{\mathbb{C}^2}$. By McDuff–Polterovich [MP94, Proof of Lemma 5.5.B], there exists a Kähler form τ_k on $B(1)$ such that $\tau_k = \varepsilon^2 \omega_{\mathbb{C}^2}$ near the boundary of $B(1)$ and $\tau_k = k^2 \omega_{\mathbb{C}^2}$ on $B(\varepsilon/(2k))$, for any $k > 1$ and $0 < \varepsilon < 1$.

Because $[\xi] = [\nu] \in H^2(X_0; \mathbb{R})$ and because $\langle [\xi|_{N_j}], [\sigma_j^0] \rangle = 0$ by construction of ν in Lemma 6.3(c), the form $\lambda_j \in \Omega^2(L)$ is exact and $\lambda_j = d\beta_j$ for some $\beta_j \in \Omega^1(L)$. Let $0 < \varepsilon < 1$ be such that $\lambda_j - \varepsilon^2 \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2}$ tames i in the fiber direction of $\pi_L : L(1) \rightarrow \mathbb{C}\mathbb{P}^1$. Consider a bump function $\rho : \mathbb{C}^2 \rightarrow \mathbb{R}_{\geq 0}$ supported in $B(1) \subseteq \mathbb{C}^2$ such that $\rho \equiv 1$ near $0 \in B(1)$ and $\rho \equiv 0$ near $\partial B(1)$. Define $\rho_k : L \rightarrow \mathbb{R}_{\geq 0}$ to be $\rho_k(z) = \rho((2k/\varepsilon)\pi_{\mathbb{C}^2}(z))$. Furthermore, define

$$\lambda'_j = \lambda_j + \pi_{\mathbb{C}^2}^*(\tau_k - \varepsilon^2 \omega_{\mathbb{C}^2}) - d(\rho_k \beta_j).$$

For $k \gg 1$ large enough, the lemma follows. \square

²In fact, $N_j \rightarrow S^2$ can be chosen to be isomorphic to the bundle $TF \rightarrow \sigma_j^0$, where $TF \subset TU_j$ is the subbundle parallel to the fibers of π_0 .

Since the U_j are disjoint, it follows that we can modify ξ only over $\bigsqcup_{j=1}^4 N_j$ such that ξ remains symplectic on the fibers and

$$\xi|_{N'_j} = \phi_j^*(c_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2})$$

on a small enough neighborhood $\sigma_j^0 \subseteq N'_j \subseteq N_j$.

Step 3: Finishing the proof. Let ω_{S^2} be the Fubini–Study form on S^2 . For any $\varepsilon > 0$ small enough,

$$\omega_0 = \varepsilon \xi + \pi_0^* \omega_{S^2}$$

is symplectic and the sections σ_j^0 are symplectic by compactness [GS99, Proposition 10.2.20]. Moreover, ω_0 restricts to a symplectic form on the smooth locus S of any fiber of π_0 because $\varepsilon \xi$ does and $\pi_0^* \omega_{S^2}|_S = 0$. Near each section σ_j^0 ,

$$\omega_0|_{N'_j} = \varepsilon \xi|_{N'_j} + \pi_0^* \omega_{S^2}|_{N'_j} = \varepsilon \phi_j^*(c_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2}) + \phi_j^*(\pi_L^* \omega_{S^2})$$

where the last equality follows because $\pi_L \circ \phi_j = \pi_0$ restricted to N'_j . This proves (1), with $a_j = \varepsilon c_j$.

Finally, observe that $\rho_1|_A = \rho_2|_A \equiv 0$ and $\rho_0|_A \equiv 1$ because A is disjoint from both V_1 and V_2 . Therefore,

$$\xi|_{\pi_0^{-1}(A)} = \xi_0|_{\pi_0^{-1}(A)} = \hat{\mathcal{G}}^* \hat{\theta}|_{\pi_0^{-1}(A)},$$

and so $\omega_0|_{\pi_0^{-1}(A) - \bigcup_{j=1}^4 N_j} = \varepsilon \hat{\mathcal{G}}^* \hat{\theta}|_{\pi_0^{-1}(A) - \bigcup_{j=1}^4 N_j} + \pi_0^* \omega_A$ where $\omega_A := \omega_{S^2}|_A$. This proves (2). \square

Proof of Proposition 6.7. We will build an appropriate symplectic form ω_n on X_n by a partial conjugation construction (Section 2.2).

Step 1: Explicitly constructing $\pi_n : X_n \rightarrow S^2$ and $\sigma_j^n : S^2 \rightarrow X_n$. To find the gluing diffeomorphism used in the partial conjugation construction, consider the symplectomorphism $\varphi \in \text{Symp}(\Sigma_{2g}, \theta)$ fixing the four marked points p_1, p_2, p_3, p_4 with $[\varphi] = \tilde{f} \in \text{Mod}(\Sigma_{2g,4})$ found in Proposition 6.8. Recall that there exist disk neighborhoods O_j of each p_j on which both φ and $\tilde{\eta}$ act trivially. In particular, for each $1 \leq j \leq 4$ the set $O_j \times [0, 1] \times (-2, 2)$ induces a neighborhood \tilde{O}_j of the section $\tilde{s}_i : E \times (-2, 2) \rightarrow M_{\tilde{\eta}} \times (-2, 2)$ defined in (8).

With the isotopy κ_t found in Proposition 6.8, build a new isotopy

$$\kappa_t^{-1} \circ \varphi \circ \kappa_t : \Sigma_{2g,4} \times [0, 1] \rightarrow \Sigma_{2g,4},$$

where we note that $\kappa_t^{-1} \circ \varphi \circ \kappa_t$ acts trivially on O_1, O_2, O_3, O_4 for all $t \in [0, 1]$ because $\kappa_t(O_j)$ is not contained in the support of φ for all $t \in [0, 1]$ and for all $1 \leq j \leq 4$. Moreover, compute at $t = 0, 1$ that

$$\begin{aligned} \kappa_0^{-1} \circ \varphi \circ \kappa_0 &= \text{Id}_{\Sigma_{2g}}^{-1} \circ \varphi \circ \text{Id}_{\Sigma_{2g}} = \varphi \\ \kappa_1^{-1} \circ \varphi \circ \kappa_1 &= (\hat{\eta} \circ \tilde{\eta})^{-1} \circ \varphi \circ (\hat{\eta} \circ \tilde{\eta}) = \tilde{\eta}^{-1} \circ \varphi \circ \tilde{\eta}, \end{aligned}$$

where the last equality uses the fact that $\hat{\eta}$ and φ commute. This isotopy induces a Σ_{2g} -bundle automorphism

$$\begin{aligned} \tilde{\mathcal{F}} : M_{\tilde{\eta}} \times (-2, 2) &\rightarrow M_{\tilde{\eta}} \times (-2, 2) \\ ((x, t), s) &\mapsto ((\kappa_t^{-1} \circ \varphi \circ \kappa_t(x), t), s) \end{aligned}$$

that fixes pointwise each neighborhood \tilde{O}_j of each section \tilde{s}_j of $M_{\tilde{\eta}} \times (-2, 2)$.

Recall the isomorphism $\hat{\mathcal{G}} : \pi_0^{-1}(N) \rightarrow M_{\tilde{\eta}} \times (-2, 2)$ fixed in (9). Define the bundle isomorphism $\mathcal{F} : \pi_0^{-1}(N) \rightarrow \pi_0^{-1}(N)$ by $\tilde{\mathcal{F}}$, using the identification $\hat{\mathcal{G}}$

$$\mathcal{F} := \hat{\mathcal{G}}^{-1} \circ \tilde{\mathcal{F}} \circ \hat{\mathcal{G}} : \pi_0^{-1}(N) \rightarrow \pi_0^{-1}(N).$$

By construction, \mathcal{F} fixes each section $\sigma_j^0 : \pi_0^{-1}(N) \rightarrow \pi_0^{-1}(N)$ for $1 \leq j \leq 4$.

Let $U_1 := D_1 \cup A \subseteq S^2$ and $U_2 := D_2 \cup A \subseteq S^2$ be two open disks covering S^2 such that

$$U_1 \cap U_2 = A.$$

By identifying any $x \in \pi_0^{-1}(A) \subseteq \pi_0^{-1}(U_1)$ with $\mathcal{F}^n(x) \in \pi_0^{-1}(A) \subseteq \pi_0^{-1}(U_2)$, form the Lefschetz fibration

$$\pi_n : X_n := \pi_0^{-1}(U_1) \cup_{\mathcal{F}^n: \pi_0^{-1}(A) \rightarrow \pi_0^{-1}(A)} \pi_0^{-1}(U_2) \rightarrow S^2$$

where π_n is defined to agree with π_0 on each $\pi_0^{-1}(U_1)$, $\pi_0^{-1}(U_2)$. Let σ_j^n be the section of π_n defined to agree with σ_j^0 on each U_1, U_2 , for $1 \leq j \leq 4$. By the partial conjugation construction (Section 2.2), the Lefschetz fibration π_n and sections $\sigma_1^n, \dots, \sigma_4^n$ have monodromy factorization given in (5) as desired.

Step 2: A common neighborhood for σ_j^n : Via $\tilde{\mathcal{G}}$ the neighborhoods \tilde{O}_j induce neighborhoods \mathfrak{D}_j of each section σ_j^0 over $\pi_0^{-1}(A)$ for each $1 \leq j \leq 4$ on which \mathcal{F} acts trivially. By Proposition 6.10(1), for each $1 \leq j \leq 4$ there exists a tubular neighborhood $N_j|_{\pi_0^{-1}(A)} \subset \mathfrak{D}_j$ of σ_j^0 over which the symplectic form ω_0 is standard near σ_j^0 . In particular, N_j embeds in X_n for all n and $1 \leq j \leq 4$; denote this natural embedding by $g_j^n : N_j \rightarrow X_n$. Furthermore, note that $\pi_n \circ g_j^n = \pi_0$, so g_j^n is a map of bundles. Denote by N_j^n the embedded copy of N_j in X_n .

Step 3: Constructing ω_n on X_n . Consider the symplectic forms

$$\omega_0|_{\pi_0^{-1}(U_1)} \in \Omega^2(\pi_0^{-1}(U_1)), \quad \omega_0|_{\pi_0^{-1}(U_2)} \in \Omega^2(\pi_0^{-1}(U_2))$$

found in Proposition 6.10. We claim that

$$\omega_n := \begin{cases} \omega_0|_{\pi_0^{-1}(U_1)} & \text{on } \pi_0^{-1}(U_1) \subseteq X_n \\ \omega_0|_{\pi_0^{-1}(U_2)} & \text{on } \pi_0^{-1}(U_2) \subseteq X_n \end{cases}$$

is a well-defined symplectic form on X_n . It suffices to check that ω_n is well-defined on $\pi_0^{-1}(A)$, i.e. that

$$\mathcal{F}^* \omega_0|_{\pi_0^{-1}(A)} = \omega_0|_{\pi_0^{-1}(A)}.$$

This follows from Proposition 6.8, Proposition 6.10(2), and the definition of \mathcal{F} .

Step 4: Restricting ω_n to the fibers of π_n . Any fiber of π_n is contained in $\pi_0^{-1}(U_1) \subseteq X_n$ or $\pi_0^{-1}(U_2) \subseteq X_n$. So for any S the smooth locus of an irreducible component of any fiber of π_n ,

$$\omega_n|_S = \omega_0|_S$$

and so S is a symplectic submanifold of (X_n, ω_n) by Proposition 6.10. Furthermore, restricted to the components F_1^n, F_2^n, F_3^n , and F_4^n of the reducible fibers of π_n ,

$$\langle [\omega_n], [F_j^n] \rangle = \int_{F_j^n \subseteq X_n} \omega_n|_{F_j^n} = \int_{F_j^0 \subseteq X_0} \omega_0|_{F_j^0} = \langle [\omega_0], [F_j^0] \rangle.$$

This finishes the proof of (a) and (b).

Step 5: Restricting ω_n near the sections σ_j^n . Since \mathcal{F} acts trivially on $N_j|_{\pi_0^{-1}(A)}$ it follows that

$$g_j^n : (N_j, \omega_0) \rightarrow (N_j^n, \omega_n)$$

is a symplectomorphism. For each $1 \leq j \leq 4$, let $\phi_j : N_j \rightarrow L$ be the diffeomorphism found in Proposition 6.10(1), so that for $\sigma_j^0(S^2) \subseteq N_j' \subseteq N_j$ we have

$$\omega_0|_{N_j'} = \phi_j^*(a_j \pi_{\mathbb{C}^2}^* \omega_{\mathbb{C}^2} + \pi_L^* \omega_{S^2}).$$

The map $\phi_j^n := \phi_j \circ (g_j^n)^{-1}$ is the desired isomorphism $N_j^n \rightarrow L$, with $V_j^n := g_j^n(N_j')$, making ω_n standard near σ_j^n . Note that under ϕ_j^n , the section σ_j^n corresponds to the zero section of L , which we denote by S . Thus σ_j^n is symplectic and

$$\langle [\omega_n], [\sigma_j^n(S^2)] \rangle = \langle [\omega_{S^2}], [S] \rangle = \pi.$$

This finishes the proof of (c). □

6.3. On the cohomology classes $[\omega_n] \in H^2(X_n; \mathbb{R})$. Finally, we combine the results of the previous subsections and determine the cohomology classes $[\omega_n] \in H^2(X_n; \mathbb{R})$.

Lemma 6.12. *For all $n \in \mathbb{Z}_{\geq 0}$, there exists a diffeomorphism*

$$\Phi_n : X_n \rightarrow M \# 4\overline{\mathbb{C}\mathbb{P}^2}$$

satisfying the following properties.

(a) *On homology,*

$$(\Phi_n)_*[\sigma_j^n] = E_j \quad \text{and} \quad (\Phi_n)_*([F^n]) = (\Phi_0)_*([F^0]) \in H_2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$$

for all $1 \leq j \leq 4$, where $E_1, \dots, E_4 \in H_2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$ denote the exceptional divisors coming from each summand $\overline{\mathbb{C}\mathbb{P}^2}$ and F^n denotes any regular fiber of $\pi_n : X_n \rightarrow S^2$.

(b) *The cohomology class $(\Phi_n^{-1})^*([\omega_n])$ is independent of n , i.e.*

$$(\Phi_n^{-1})^*([\omega_n]) = (\Phi_0^{-1})^*([\omega_0]) \in H^2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R}).$$

Proof. For each $n \in \mathbb{Z}_{\geq 0}$, let Φ_n be the diffeomorphism found in Lemma 6.3. Let $F_1^n \cup F_2^n$ and $F_3^n \cup F_4^n$ denote the two reducible singular fibers of $\pi_n : X_n \rightarrow S^2$ with F_1^n, F_2^n, F_3^n , and F_4^n denoting the irreducible components as defined in Lemma 6.3. By construction of Φ_n ,

$$(\Phi_n)_*([F_j^n]) = (\Phi_0)_*([F_j^0]), \quad (\Phi_n)_*([\sigma_j^n]) = (\Phi_0)_*([\sigma_j^0]) = E_j$$

for all $1 \leq j \leq 4$. It also follows that

$$(\Phi_n)_*([F^n]) = (\Phi_n)_*([F_1^n] + [F_2^n]) = (\Phi_0)_*([F_1^0] + [F_2^0]) = (\Phi_0)_*([F^0])$$

as desired.

By Proposition 6.7,

$$\begin{aligned} \langle (\Phi_n^{-1})^*([\omega_n]), (\Phi_n)_*([F_j^n]) \rangle &= \langle [\omega_n], [F_j^n] \rangle = \langle [\omega_0], [F_j^0] \rangle = \langle (\Phi_0^{-1})^*([\omega_0]), (\Phi_0)_*([F_j^0]) \rangle \\ \langle (\Phi_n^{-1})^*([\omega_n]), (\Phi_n)_*([\sigma_j^n]) \rangle &= \langle [\omega_n], [\sigma_j^n] \rangle = \langle [\omega_0], [\sigma_j^0] \rangle = \langle (\Phi_0^{-1})^*([\omega_0]), (\Phi_0)_*([\sigma_j^0]) \rangle \end{aligned}$$

for each $1 \leq j \leq 4$. By Lemma 6.3(b), the cohomology classes $(\Phi_n^{-1})^*([\omega_n])$ and $(\Phi_0^{-1})^*([\omega_0])$ agree on $H_2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R})$, and so

$$(\Phi_n^{-1})^*([\omega_n]) = (\Phi_0^{-1})^*([\omega_0]) \in H^2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R}). \quad \square$$

Proofs of Theorems 6.1 and 6.2. By Proposition 6.7, the smooth loci of the irreducible components of the fibers and the sections $\sigma_1^n, \sigma_2^n, \sigma_3^n$, and σ_4^n of $\pi_n : X_n \rightarrow S^2$ are all symplectic submanifolds of (X_n, ω_n) . Let $\Phi_n : X_n \rightarrow M \# 4\overline{\mathbb{C}\mathbb{P}^2}$ be the diffeomorphisms found in Lemma 6.12 so that

$$(\Phi_n^{-1})^*([\omega_n]) = (\Phi_n^{-1})^*([\omega_0]) \in H^2(M \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{R})$$

and

$$(\Phi_n)_*([F^n]) = (\Phi_0)_*([F^0])$$

for all $n \in \mathbb{Z}_{\geq 0}$. Because each $\sigma_1^n, \sigma_2^n, \sigma_3^n$, and σ_4^n are symplectic submanifolds of self-intersection -1 , the form ω_n induces a symplectic form (which we also denote by ω_n) on the blowdown M_n of $\sigma_1^n, \sigma_2^n, \sigma_3^n$, and σ_4^n in X_n . Denote also by $\pi_n : M_n - B_n \rightarrow S^2$ the induced Lefschetz pencil. Note that since each ω_n is standard (see Proposition 6.7(c)) near each section σ_j^n , we can ensure that the smooth loci of the irreducible components of the fibers of π_n are symplectic submanifolds [MP94, Section 5].

Let $F_n \subset M_n$ denote a regular fiber of π_n . Because Φ_n sends $[\sigma_j^n]$ to E_j , it induces a diffeomorphism $\Psi_n : M_n \rightarrow M$ such that

$$(\Psi_n^{-1})^*([\omega_n]) = (\Psi_0^{-1})^*([\omega_0]) \in H^2(M; \mathbb{R})$$

and

$$(\Psi_n)_*([F_n]) = (\Psi_0)_*([F_m]) \in H_2(M; \mathbb{Z})$$

for all $n \in \mathbb{Z}_{\geq 0}$. Work of Lalonde–McDuff [LM96, Theorem 1.1] shows that cohomologous symplectic forms on ruled surfaces are diffeomorphic, and so there is an equality of forms

$$(\Psi_n^{-1})^* \omega_n = (\Psi_0^{-1})^* \omega_0$$

for all $n \in \mathbb{Z}_{\geq 0}$ after possibly post-composing Ψ_n with a diffeomorphism of M . Any diffeomorphism of M that fixes the cohomology class of a symplectic form acts by the identity on $H_2(M; \mathbb{Z})$ (cf. [LL97, Theorem 3]), and so the equality

$$(\Psi_n)_*([F_n]) = (\Psi_n)_*([F_m]) \in H_2(M; \mathbb{R})$$

still holds. Letting

$$\omega = (\Psi_0^{-1})^* \omega_0$$

concludes the proof of Theorem 6.1.

Recall by construction in Proposition 6.7 that the sections σ_j^n and σ_j^0 have equal area in (X_n, ω_n) and (X_0, ω_0) for all $n \in \mathbb{Z}_{\geq 0}$, i.e.

$$\langle [\omega_n], [\sigma_j^n] \rangle = \langle [\omega_0], [\sigma_j^0] \rangle.$$

Symplectic blowups of ruled surfaces are determined up to isotopy by the areas of the exceptional divisors by work of McDuff [McD96, Corollary 1.3]. Therefore,

$$(\Phi_n^{-1})^* \omega_n = (\Phi_0^{-1})^* \omega_0$$

for all $n \in \mathbb{Z}_{\geq 0}$ as symplectic forms, after possibly post-composing Φ_n with a diffeomorphism (isotopic to the identity) of $M \# 4\overline{\mathbb{C}\mathbb{P}^2}$. Finally, letting

$$\omega = (\Phi_0^{-1})^* \omega_0$$

concludes the proof of the symplectic portion of Theorem 6.2. \square

Finally, Theorems 1.2 and 1.3 follow as immediate corollaries.

Proof of Theorem 1.2. If X is a ruled surface with $\chi(X) < 0$, it is diffeomorphic to either $M^1 = \Sigma_g \times S^2$ or $M^2 = \Sigma_g \tilde{\times} S^2$ for $g \geq 2$. Let $i = 1$ or $i = 2$ such that $X \cong M^i$ and fix the notation of Theorem 6.1. For all $n \in \mathbb{Z}_{\geq 0}$,

$$\pi_{n,i} \circ (\Psi_n^i)^{-1} : M^i - \Psi_n^i(B_n^i) \rightarrow S^2$$

are all Lefschetz pencils of genus $2g$ with four base points that are compatible with ω , whose regular fibers are all homologous in $H_2(M^i; \mathbb{Z})$. \square

Proof of Theorem 1.3. If X is a ruled surface with $\chi(X) < 0$, it is diffeomorphic to either $M^1 = \Sigma_g \times S^2$ or $M^2 = \Sigma_g \tilde{\times} S^2$ for $g \geq 2$, and $X \# 4\overline{\mathbb{C}\mathbb{P}^2}$ is diffeomorphic to $(\Sigma_g \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}$ in either case. Fix the notation of Theorem 6.2. For all $n \in \mathbb{Z}_{\geq 0}$,

$$\pi_n \circ \Phi_n^{-1} : (\Sigma_g \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2} \rightarrow S^2$$

are all Lefschetz fibrations of genus $2g$ that are compatible with ω , whose regular fibers are all homologous in $H_2((\Sigma_g \times S^2) \# 4\overline{\mathbb{C}\mathbb{P}^2}; \mathbb{Z})$. \square

7. INFINITELY MANY HOMEOMORPHIC LEFSCHETZ FIBRATIONS

Our construction of infinitely many pairwise inequivalent Lefschetz fibrations can apply to settings outside of ruled surfaces. In this section we demonstrate this flexibility through an example of infinitely many Lefschetz fibrations of every genus $g \geq 3$ that are pairwise inequivalent but are pairwise *homeomorphic*.

Consider the isotopy classes $c_1, \dots, c_{2g} \subseteq \Sigma_g^1$ of curves shown in Figure 6. Let δ denote the boundary of Σ_g^1 .

Lemma 7.1 (Chain Relation [FM12, Proposition 4.12]). *For any $g \geq 2$,*

$$(T_{c_1} \dots T_{c_{2g}})^{4g+2} = T_\delta \in \text{Mod}(\Sigma_g^1).$$

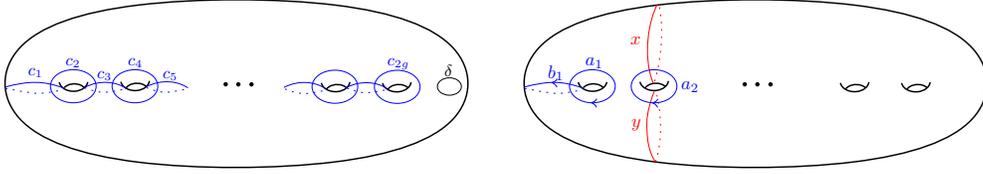


FIGURE 6. Left: A chain of curves in Σ_g^1 ; Right: Homologous curves $x, y \subseteq \Sigma_g$ and homology classes $a_1, a_2, b_1 \in H_1(\Sigma_g; \mathbb{Z})$

Let $f = T_x T_y^{-1} \in \text{Mod}(\Sigma_g)$ be a bounding pair of genus 1, where $x, y \subseteq \Sigma_g$ are as shown in Figure 6. Conjugating the factorization given by Lemma 7.1 by f^n for any $n \in \mathbb{Z}_{\geq 0}$, shows that

$$(T_{f^n(c_1)} \dots T_{f^n(c_{2g})})^{4g+2} = T_\delta \in \text{Mod}(\Sigma_g^1).$$

By combining the above with the factorization of Lemma 7.1, we define the genus- g Lefschetz fibrations $\pi_n : Z_n \rightarrow S^2$.

Definition 7.2. Let $g \geq 3$. For any $n \in \mathbb{Z}_{\geq 0}$, let $\pi_n : Z_n \rightarrow S^2$ denote the Lefschetz fibration of genus g with monodromy factorization

$$(T_{c_1} \dots T_{c_{2g}})^{2(4g+2)} (T_{f^n(c_1)} \dots T_{f^n(c_{2g})})^{4g+2} = 1 \in \text{Mod}(\Sigma_g).$$

Note that $\pi_n : Z_n \rightarrow S^2$ has a section of self-intersection -3 for all $n \geq 0$ determined by the following lift of the monodromy factorization of π_n to $\text{Mod}(\Sigma_g^1)$ given by

$$(11) \quad (T_{c_1} \dots T_{c_{2g}})^{2(4g+2)} (T_{f^n(c_1)} \dots T_{f^n(c_{2g})})^{4g+2} = T_\delta^3 \in \text{Mod}(\Sigma_g^1).$$

The following theorem is the main result of this section.

Theorem 7.3. For every $g \geq 3$, there exist genus- g Lefschetz fibrations $\pi_n : Z_n \rightarrow S^2$ for every $n \in \mathbb{Z}_{\geq 0}$ such that that π_n and π_m are inequivalent if $n \neq m$ and such that Z_n is homeomorphic to

$$(6g^2 - 2g + 1)\mathbb{CP}^2 \# (18g^2 + 10g + 1)\overline{\mathbb{CP}^2}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Similarly as in Section 4, consider the following groups for any $n \in \mathbb{Z}_{\geq 0}$:

$$\hat{G}_n := \langle T_{c_i}, T_{f^n(c_i)} : 1 \leq i \leq 2g \rangle,$$

$$\hat{G}_n^{\mathcal{I}} := G_n \cap \mathcal{I}_g,$$

$$\hat{A}_n := \langle [T_{c_i}^{-1}, f^n] : 1 \leq i \leq 2g \rangle.$$

We use the hat notation for groups in this section to distinguish them from the groups of Section 4.

As in Section 4, we first study the image $\tau(G_n^{\mathcal{I}})$ of $G_n^{\mathcal{I}}$ under the Johnson homomorphism. Let a_1, a_2, b_1 be as in Figure 6 and define

$$w := a_1 \wedge a_2 \wedge b_1 \in (\wedge^3 H) / H$$

where $H := H_1(\Sigma_g; \mathbb{Z})$.

Lemma 7.4 (cf. Lemmas 4.2, 4.3, and 4.5). For any $n \in \mathbb{Z}_{\geq 0}$, the image $\tau(\hat{G}_n^{\mathcal{I}})$ is contained in

$$\langle \text{Mod}(\Sigma_g) \cdot (nw) \rangle \leq n(\wedge^3 H) / H,$$

the group generated by the $\text{Mod}(\Sigma_g)$ -orbit of $nw \in (\wedge^3 H) / H$. Moreover, the class nw is nonzero and is contained in $\tau(\hat{G}_n^{\mathcal{I}})$.

Proof. For any $i \neq 4$, note that $[T_{c_i}^{-1}, f^n] = 1$ because c_i is disjoint from x and y . Therefore,

$$\hat{A}_n = \langle [T_{c_4}^{-1}, f^n] \rangle.$$

By [FM12, p. 195-196],

$$\tau(f) = \tau(T_x T_y^{-1}) = a_1 \wedge b_1 \wedge b_2,$$

and compute using naturality (4) of τ that

$$\tau([T_{c_4}^{-1}, f^n]) = n(T_{c_4}^{-1} \cdot \tau(f) - \tau(f)) = n(a_1 \wedge b_1 \wedge T_{c_4}^{-1}(b_2) - a_1 \wedge b_1 \wedge b_2) = n(a_1 \wedge a_2 \wedge b_1) = nw.$$

Because $[T_{c_4}^{-1}, f^n]$ is contained in $\hat{G}_n^{\mathcal{I}}$, the class nw is contained in $\tau(\hat{G}_n^{\mathcal{I}})$. The fact that $nw \neq 0$ follows from the existence of a \mathbb{Z} -basis of the free abelian group $(\wedge^3 H_1(\Sigma_g)) / H_1(\Sigma_g)$ containing w , shown in Lemma 4.4.

On the other hand, the proof of Lemma 4.2 shows that there is an equality of subgroups of \mathcal{I}_g

$$\hat{G}_n^{\mathcal{I}} = \langle \hat{G}_0^{\mathcal{I}}, k[T_{c_4}^{-1}, f^n]k^{-1} : k \in \hat{G}_0 \rangle.$$

Each generator of \hat{G}_0 is hyperelliptic, and hence $\hat{G}_0^{\mathcal{I}}$ is contained in the hyperelliptic mapping class group $\text{SMod}(\Sigma_g)$. By Corollary 2.4, $\hat{G}_0^{\mathcal{I}}$ is contained in $\ker(\tau)$, and by naturality (4) of τ ,

$$\tau(\hat{G}_n^{\mathcal{I}}) = \langle \hat{G}_0 \cdot \tau([T_{c_4}^{-1}, f^n]) \rangle \leq \langle \text{Mod}(\Sigma_g) \cdot (nw) \rangle. \quad \square$$

The following non-conjugacy result forms the main part of the proof of Proposition 7.6.

Proposition 7.5 (cf. Proposition 4.6). *If $n \neq m \in \mathbb{Z}_{\geq 0}$ then \hat{G}_n and \hat{G}_m are not conjugate as subgroups of $\text{Mod}(\Sigma_g)$.*

Proof. Suppose that there exists $k \in \text{Mod}(\Sigma_g)$ so that $k\hat{G}_n k^{-1} = \hat{G}_m$ as subgroups of $\text{Mod}(\Sigma_g)$. Because \mathcal{I}_g is normal in $\text{Mod}(\Sigma_g)$, this implies that $k\hat{G}_n^{\mathcal{I}} k^{-1} = \hat{G}_m^{\mathcal{I}}$ as subgroups of \mathcal{I}_g . By naturality (4) of τ and Lemma 7.4,

$$\tau(\hat{G}_m^{\mathcal{I}}) = k \cdot \tau(\hat{G}_n^{\mathcal{I}}) \leq \langle \text{Mod}(\Sigma_g) \cdot (nw) \rangle$$

By Lemma 7.4, the class nw is nonzero and is contained in $\tau(\hat{G}_m^{\mathcal{I}})$. Since w is primitive (Lemma 4.4), this implies that n divides m . By symmetry, m also divides n , i.e. $n = m$. \square

Rephrasing Proposition 4.6 in terms of the monodromy of $\pi_n : Z_n \rightarrow S^2$ yields the following proposition.

Proposition 7.6. *For any $n \neq m \in \mathbb{Z}_{\geq 0}$, the Lefschetz fibrations $\pi_n : Z_n \rightarrow S^2$ and $\pi_m : Z_m \rightarrow S^2$ are inequivalent.*

Proof. If $n \neq m \in \mathbb{Z}_{\geq 0}$ then \hat{G}_n and \hat{G}_m are not conjugate as subgroups of $\text{Mod}(\Sigma_g)$. By construction, \hat{G}_n and \hat{G}_m are the images $\text{im}(\rho_n)$ and $\text{im}(\rho_m)$ of the monodromy representations ρ_n and ρ_m of $\pi_n : Z_n \rightarrow S^2$ and $\pi_m : Z_m \rightarrow S^2$ respectively. By Corollary 2.2, π_n and π_m are inequivalent Lefschetz fibrations. \square

It remains to show that the manifolds Z_n are pairwise homeomorphic. To do so, we compute the algebraic topology invariants of Z_n .

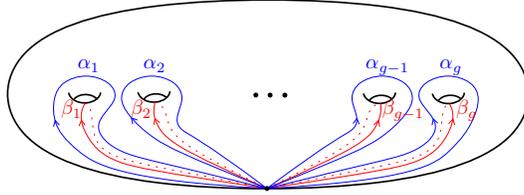
Lemma 7.7. *For any $n \in \mathbb{Z}_{\geq 0}$, the manifold Z_n is simply-connected and*

$$b_2^+(Z_n) = 6g^2 - 2g + 1, \quad b_2^-(Z_n) = 18g^2 + 10g + 1.$$

Proof. Because $\pi_n : Z_n \rightarrow S^2$ admits a section, there is an isomorphism

$$\pi_1(Z_n) \cong \pi_1(\Sigma_g) / N(c_1, \dots, c_{2g}, f^n(c_1), \dots, f^n(c_{2g}))$$

where $N(c_1, \dots, f^n(c_{2g}))$ denotes the subgroup of $\pi_1(\Sigma_g)$ normally generated by the vanishing cycles of $\pi_n : Z_n \rightarrow S^2$. On the other hand, let $\alpha_i, \beta_i \in \pi_1(\Sigma_g)$ for $1 \leq i \leq g$ be the generators of $\pi_1(\Sigma_g)$ as shown below:



The loop $\alpha_i \in \pi_1(\Sigma_g)$ is freely homotopic to c_{2i} for all $1 \leq i \leq g$, the loop $\beta_1 \in \pi_1(\Sigma_g)$ is freely homotopic to c_1 , and the loop $\beta_i^{-1}\beta_{i+1} \in \pi_1(\Sigma_g)$ is freely homotopic to c_{2i+1} for all $1 \leq i \leq g-1$. Therefore,

$$N(c_1, \dots, c_{2g}) = N(c_1, \dots, c_{2g}, f^n(c_1), \dots, f^n(c_{2g})) = \pi_1(\Sigma_g),$$

and Z_n is simply-connected for all $n \geq 0$.

There are $12g(2g+1)$ -many vanishing cycles in $\pi_n : Z_n \rightarrow S^2$, and so

$$\chi(Z_n) = 4 - 4g + 12g(2g+1) = 24g^2 + 8g + 4.$$

In other words, $b_2(Z_n) = 24g^2 + 8g + 2$.

In the case of $n = 0$, the Lefschetz fibration $\pi_0 : Z_0 \rightarrow S^2$ is hyperelliptic with $12g(2g+1)$ -many non-separating vanishing cycles. Therefore by [End00, Theorems 4.4(2), 4.8],

$$\sigma(Z_0) = -\left(\frac{g+1}{2g+1}\right)(12g(2g+1)) = -12g(g+1).$$

To compute $\sigma(Z_n)$ for $n \geq 1$, consider the Lefschetz fibrations $Z' \rightarrow D^2$ and $Z'' \rightarrow D^2$ with monodromy factorizations

$$(T_{c_1} \dots T_{c_{2g}})^{2(4g+2)} \in \text{Mod}(\Sigma_g), \quad (T_{f^n(c_1)} \dots T_{f^n(c_{2g})})^{4g+2} \in \text{Mod}(\Sigma_g)$$

respectively. Then Z_n is formed by gluing Z' to Z'' along their boundaries by some diffeomorphism $\partial Z' \rightarrow \partial Z''$ (which varies with n) for any $n \geq 0$ (cf. Section 2.2). By Novikov additivity,

$$\sigma(Z_n) = \sigma(Z') + \sigma(Z'') = \sigma(Z_0) = -12g(g+1)$$

for all $n \geq 0$. Finally, compute for all $n \geq 0$ that

$$b_2^+(Z_n) = \frac{b_2(Z_n) + \sigma(Z_n)}{2} = 6g^2 - 2g + 1, \quad b_2^-(Z_n) = \frac{b_2(Z_n) - \sigma(Z_n)}{2} = 18g^2 + 10g + 1. \quad \square$$

By Freedman's theorem [Fre82], the algebraic topology invariants of Z_n determines its homeomorphism type.

Proposition 7.8. *For any $n \in \mathbb{Z}_{\geq 0}$, the manifold Z_n is homeomorphic to*

$$(6g^2 - 2g + 1)\mathbb{C}\mathbb{P}^2 \# (18g^2 + 10g + 1)\overline{\mathbb{C}\mathbb{P}^2}.$$

Proof. By Lemma 7.7,

$$\sigma(Z_n) = -12g(g+1), \quad b_2(Z_n) = 24g^2 + 8g + 2$$

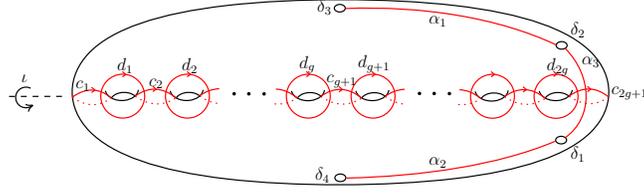
for all $n \geq 0$. In particular, the rank $b_2(Z_n)$ and signature $\sigma(Z_n)$ of the intersection form Q_{Z_n} of Z_n only depend on g and are independent of n .

To determine the parity of Q_{Z_n} , recall that π_n has a section of self-intersection -3 given by (11) for all $n \geq 0$. In other words, the intersection form Q_{Z_n} is odd for all $n \geq 0$. Because Q_{Z_n} is indefinite and unimodular, [GS99, Theorem 1.2.21] shows that

$$Q_{Z_n} \cong (6g^2 - 2g + 1)\langle 1 \rangle \oplus (18g^2 + 10g + 1)\langle -1 \rangle.$$

Finally, Freedman's theorem [Fre82, Theorem 1.5] says that the homeomorphism type of a simply-connected, smooth 4-manifold is determined by its intersection form. Because Z_n is simply-connected for all $n \geq 0$ by Lemma 7.7, Freedman's theorem implies that Z_n is homeomorphic to

$$(6g^2 - 2g + 1)\mathbb{C}\mathbb{P}^2 \# (18g^2 + 10g + 1)\overline{\mathbb{C}\mathbb{P}^2}$$

FIGURE 7. Filling curves and arcs, and the hyperelliptic involution ι .

for all $n \geq 0$. □

Combining Propositions 7.8 and 7.6 concludes the proof of the main theorem.

Proof of Theorem 7.3. For any $n \in \mathbb{Z}_{\geq 0}$, the manifold Z_n is homeomorphic to $(6g^2 - 2g + 1)\mathbb{C}\mathbb{P}^2 \# (18g^2 + 10g + 1)\mathbb{C}\mathbb{P}^2$ by Proposition 7.8. If $n \neq m$ then the Lefschetz fibrations $\pi_n : Z_n \rightarrow S^2$ and $\pi_m : Z_m \rightarrow S^2$ are inequivalent by Proposition 7.6. □

APPENDIX A. CALCULATIONS IN $\text{Mod}(\Sigma_{2g}^4)$

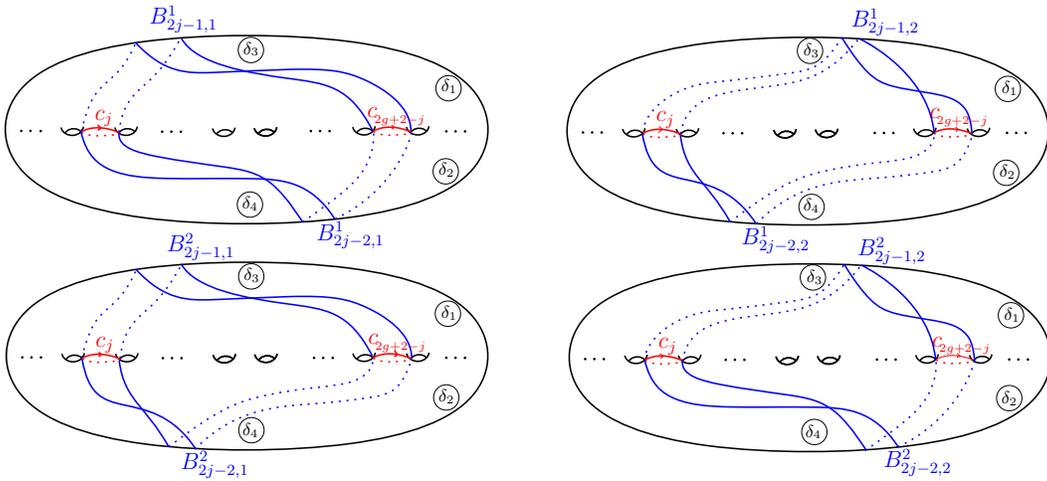
In this appendix we collect some routine calculations in $\text{Mod}(\Sigma_{2g}^4)$, $\text{Mod}(\Sigma_{2g,2})$, and $\text{Mod}(\Sigma_{2g})$, including proofs of Theorems 3.1 and 3.3. For $i = 1, 2$, consider the curves $B_{j,k}^i$ and C_1^i, C_2^i as shown in Figure 4. Let c_1, \dots, c_{2g+1} and d_1, \dots, d_{2g} denote isotopy classes of curves in Σ_{2g}^4 as shown in Figure 7. Let $\alpha_1, \alpha_2, \alpha_3$ denote the isotopy classes of arcs in Σ_{2g}^4 as shown in Figure 7.

The following two lemmas compute a composition of Dehn twists applied to the curves c_j and d_j .

Lemma A.1. *For $1 \leq j \leq 2g + 1$ and $i = 1, 2$, there are equalities of isotopy classes of oriented curves*

$$(T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i})(T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i})(c_j) = c_j, \quad T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}(c_j) = c_{2g+2-j}.$$

Proof. Suppose $j \neq g + 1$. The curves $B_{2j-2,k}^i$ and $B_{2j-1,k}^i$ for $k, i = 1, 2$ and c_j, c_{2g+2-j} are shown below:



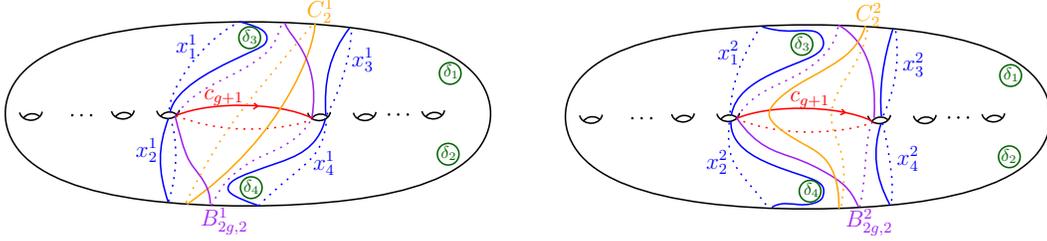
Direct computations show that as isotopy classes of oriented curves,

$$T_{B_{2j-2,2}^i} T_{B_{2j-1,2}^i}(c_j) = c_{2g+2-j}, \quad T_{B_{2j-2,1}^i} T_{B_{2j-1,1}^i}(c_{2g+2-j}) = c_j.$$

Note that the curves c_j and c_{2g+2-j} are disjoint from the curves $B_{\ell,k}^i$ and C_1^i, C_2^i if $\ell \neq 2j - 2, 2j - 1$ and compute as isotopy classes of oriented curves:

$$\begin{aligned} T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i}(c_j) &= c_{2g+2-j} \\ T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i} T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i}(c_j) &= T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i}(c_{2g+2-j}) = c_j. \end{aligned}$$

The case $j = g + 1$ follows from the lantern relation [FM12, Proposition 5.1]. The curves $B_{2g,2}^i$ and C_2^i are shown below:

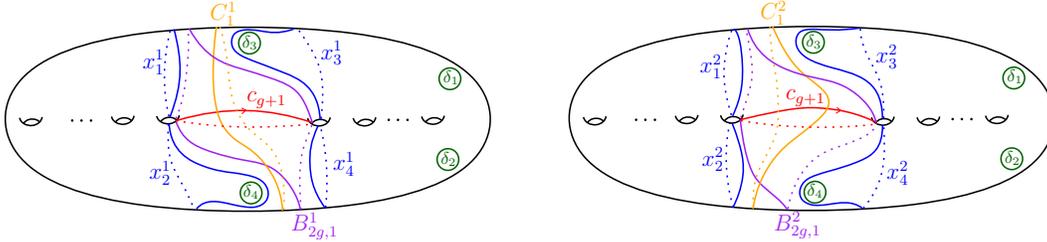


For both $i = 1, 2$, compute with the depicted curves $x_1^i, x_2^i, x_3^i, x_4^i$ that

$$T_{B_{2g,2}^i} T_{C_2^i}(c_{g+1}) = T_{B_{2g,2}^i} T_{C_2^i} T_{c_{g+1}}(c_{g+1}) = T_{x_1^i} T_{x_2^i} T_{x_3^i} T_{x_4^i}(c_{g+1}) = c_{g+1}$$

where the second equality follows from the lantern relation $T_{B_{2g,2}^i} T_{C_2^i} T_{c_{g+1}} = T_{x_1^i} T_{x_2^i} T_{x_3^i} T_{x_4^i}$.

The curves $B_{2g,1}^i$ and C_1^i are shown below:



For both $i = 1, 2$, compute with the depicted curves $x_1^i, x_2^i, x_3^i, x_4^i$ that

$$T_{B_{2g,1}^i} T_{C_1^i}(c_{g+1}) = T_{B_{2g,1}^i} T_{C_1^i} T_{c_{g+1}}(c_{g+1}) = T_{x_1^i} T_{x_2^i} T_{x_3^i} T_{x_4^i}(c_{g+1}) = c_{g+1}$$

where the second equality follows from the lantern relation $T_{B_{2g,1}^i} T_{C_1^i} T_{c_{g+1}} = T_{x_1^i} T_{x_2^i} T_{x_3^i} T_{x_4^i}$.

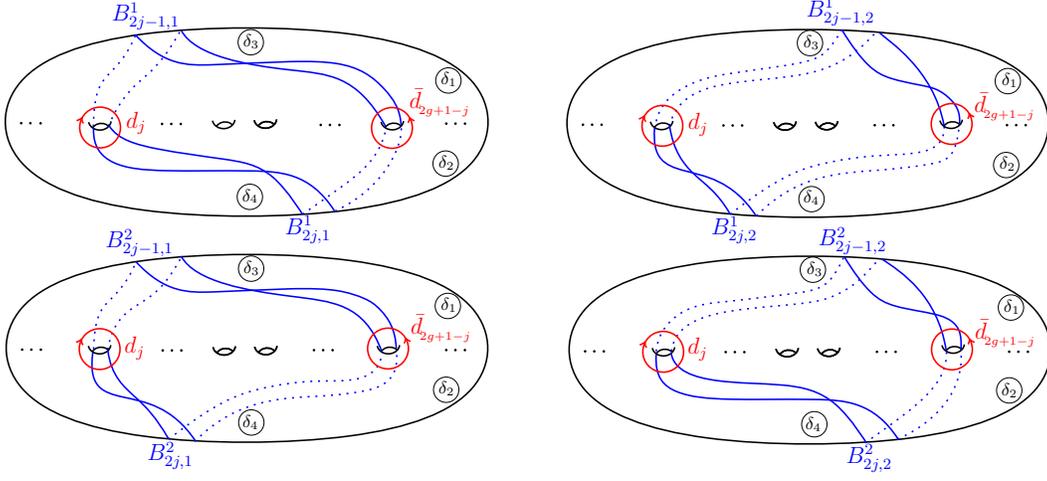
Finally, note that c_{g+1} and $B_{\ell,k}^i$ are disjoint if $\ell \neq 2g$ and compute

$$\begin{aligned} T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i}(c_{g+1}) &= c_{g+1} \\ T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i} T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i}(c_{g+1}) &= T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i}(c_{g+1}) = c_{g+1}. \quad \square \end{aligned}$$

Lemma A.2. For $1 \leq j \leq 2g$ and $i = 1, 2$, there are equalities of isotopy classes of oriented curves

$$(T_{B_{0,1}^i} T_{B_{1,1}^i} \cdots T_{B_{2g,1}^i} T_{C_1^i})(T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i})(d_j) = d_j, \quad T_{B_{0,2}^i} T_{B_{1,2}^i} \cdots T_{B_{2g,2}^i} T_{C_2^i}(d_j) = \bar{d}_{2g+1-j}.$$

Proof. Consider the curves $B_{2j-1,k}^i, B_{2j,k}^i$ for $k, i = 1, 2$ and d_j, \bar{d}_{2g+1-j} shown below:



Direct computations show that

$$T_{B_{2j-1,2}^i} T_{B_{2j,2}^i}(d_j) = \bar{d}_{2g+1-j}, \quad T_{B_{2j-1,1}^i} T_{B_{2j,1}^i}(\bar{d}_{2g+1-j}) = d_j.$$

Note that the curves d_j and d_{2g+1-j} are disjoint from the curves $B_{\ell,k}^i$ if $\ell \neq 2j-1, 2j$ and compute

$$\begin{aligned} T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}(d_j) &= \bar{d}_{2g+1-j} \\ T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i} T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}(d_j) &= T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i}(\bar{d}_{2g+1-j}) = d_j. \quad \square \end{aligned}$$

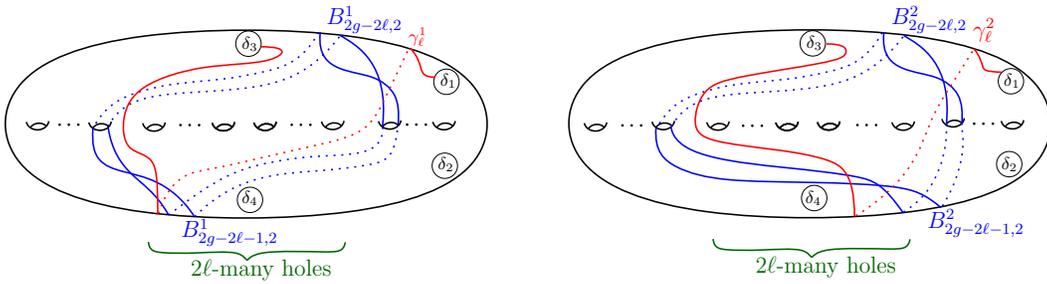
We now compute $\tilde{h}_i(\alpha_1)$ and $\tilde{h}_i(\alpha_2)$.

Lemma A.3. For $i = 1, 2$ and $j = 1, 2$,

$$(T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i})(T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i})(\alpha_j) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_j)$$

as isotopy classes of arcs in Σ_{2g}^4 .

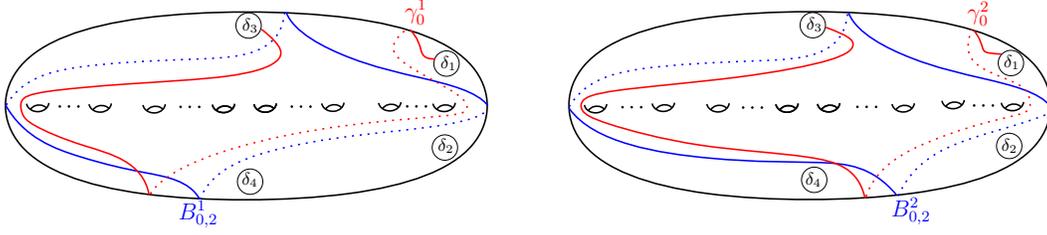
Proof. We first consider the arc α_1 . For each $i = 1, 2$ and any $0 \leq \ell \leq g$, consider the arc γ_ℓ^i depicted below:



Direct computations show that the following equalities of isotopy classes of arcs in Σ_{2g}^4 hold:

- (a) $\gamma_0^i = T_{C_2^i}(\alpha_1)$, and
- (b) $T_{B_{2g-2\ell-1}^i} T_{B_{2g-2\ell}^i}(\gamma_\ell^i) = \gamma_{\ell+1}^i$ for all $0 \leq \ell \leq g-1$.

In particular, $\gamma_g^i = T_{B_{1,2}^i} T_{B_{2,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}(\alpha_1)$ by induction; the arc γ_g^i and the curve $B_{0,2}^i$ are depicted below.



Using the above, direct computation shows that

$$T_{B_{0,2}^i}(\gamma_g^i) = T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i}(\alpha_1) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_1).$$

Because $T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_1)$ is disjoint from $B_{j,1}^i$ and C_1^i for all $0 \leq j \leq 2g$,

$$(T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i})(T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i})(\alpha_1) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_1).$$

For the arc α_2 , consider the hyperelliptic involution $\iota \in \text{Diff}^+(\Sigma_{2g}, \delta_1 \cup \delta_2 \cup \delta_3 \cup \delta_4)$ with $\iota(\delta_1) = \delta_2$ and $\iota(\delta_3) = \delta_4$ as shown in Figure 7. Then $\iota(\alpha_1) = \alpha_2$.

- (a) If $i = 1$ then for any $0 \leq j \leq 2g$, there are equalities of isotopy classes $\iota(B_{j,1}^1) = B_{j,2}^1$ and $\iota(C_1^1) = C_2^1$. Therefore,

$$\begin{aligned} T_{B_{0,1}^1} \dots T_{B_{2g,1}^1} T_{C_1^1} T_{B_{0,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1}(\alpha_2) &= \iota^2(T_{B_{0,1}^1} \dots T_{B_{2g,1}^1} T_{C_1^1} T_{B_{0,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1})(\iota\alpha_1) \\ &= \iota(T_{B_{0,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1} T_{B_{0,1}^1} \dots T_{B_{2g,1}^1} T_{C_1^1})(\alpha_1) \\ &= \iota(T_{B_{0,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1})(\alpha_1) \\ &= T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_2). \end{aligned}$$

- (b) If $i = 2$ then for any $0 \leq j \leq 2g$, there are equalities of isotopy classes $\iota(B_{j,1}^2) = B_{j,1}^2$, $\iota(B_{j,2}^2) = B_{j,2}^2$, $\iota(C_1^2) = C_1^2$, and $\iota(C_2^2) = C_2^2$. Therefore,

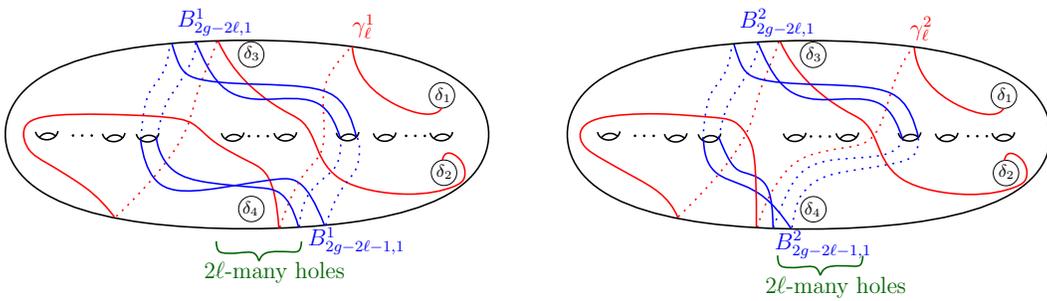
$$\begin{aligned} T_{B_{0,1}^2} \dots T_{B_{2g,1}^2} T_{C_1^2} T_{B_{0,2}^2} \dots T_{B_{2g,2}^2} T_{C_2^2}(\alpha_2) &= \iota^2(T_{B_{0,1}^2} \dots T_{B_{2g,1}^2} T_{C_1^2} T_{B_{0,2}^2} \dots T_{B_{2g,2}^2} T_{C_2^2})(\iota\alpha_1) \\ &= \iota(T_{B_{0,1}^2} \dots T_{B_{2g,1}^2} T_{C_1^2} T_{B_{0,2}^2} \dots T_{B_{2g,2}^2} T_{C_2^2})(\alpha_1) \\ &= T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_2). \end{aligned} \quad \square$$

Lemma A.4. For $i = 1, 2$,

$$(T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i})(T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i})(\alpha_3) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\alpha_3)$$

as isotopy classes of arcs in Σ_{2g}^4 .

Proof. For each $i = 1, 2$ and any $0 \leq \ell \leq g - 1$, consider the arc γ_ℓ^i in Σ_{2g}^4 depicted below:



Direct computation shows that the following equalities of isotopy classes of arcs hold:

- (a) $T_{C_1^i} T_{B_{0,2}^i}(\alpha_3) = \gamma_0^i$,
 (b) $T_{B_{2g-2\ell-1,1}^i} T_{B_{2g-2\ell,1}^i}(\gamma_\ell^i) = \gamma_{\ell+1}^i$ for every $0 \leq \ell \leq g - 1$, and

$$(c) \quad T_{B_{0,1}^i}(\gamma_g^i) = T_{\delta_1} T_{\delta_2}(\alpha_3).$$

Combining these facts (and the fact that α_3 is disjoint from the curves $B_{1,2}^i, \dots, B_{2g,2}^i, C_2^i$) together shows that

$$T_{B_{0,1}^i}(T_{B_{1,1}^i} T_{B_{2,1}^i} \dots T_{B_{2g,1}^i})(T_{C_1^i} T_{B_{0,2}^i})(T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i})(\alpha_3) = T_{\delta_1} T_{\delta_2}(\alpha_3).$$

Noting that α_3 is disjoint from δ_3 and δ_4 concludes the proof. \square

Below, we prove Theorem 3.3. Our proof is different from Hamada's original proof but instead is similar to Gurtas' proof of [Gur04, Theorem 2.0.1]. Note, however, that our version of Theorem 3.3 does not recover the full computations of Hamada's work [Ham17].

Proof of Theorem 3.3. Consider the curves $c_1, \dots, c_{2g+1}, d_1, \dots, d_{2g}$ and arcs $\alpha_1, \alpha_2, \alpha_3$ shown in Figure 7; these curves and arcs fill Σ_{2g}^4 and satisfy the conditions of the Alexander method [FM12, Proposition 2.8]. Let

$$k_i := T_{\delta_1}^{-1} T_{\delta_2}^{-1} T_{\delta_3}^{-1} T_{\delta_4}^{-1} T_{B_{0,1}^i} T_{B_{1,1}^i} \dots T_{B_{2g,1}^i} T_{C_1^i} T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i} \in \text{Mod}(\Sigma_{2g}^4).$$

By Lemmas A.1 and A.2, there are equalities of oriented isotopy classes of curves

$$k_i(c_j) = c_j, \quad k_i(d_j) = d_j$$

for all j . By Lemmas A.3 and A.4, $k_i(\alpha_j) = \alpha_j$ for $j = 1, 2, 3$. These two facts combined imply that the mapping class k_i fixes each vertex and each edge (with orientations) of the graph determined by $\bigcup_{j=1}^{2g+1} c_j \cup \bigcup_{j=1}^{2g} d_j \cup \bigcup_{j=1}^3 \alpha_j$. Therefore, the Alexander method [FM12, Proposition 2.8] shows that $k_i = 1 \in \text{Mod}(\Sigma_{2g}^4)$ for both $i = 1, 2$. \square

Similarly using the Alexander method, we deduce Theorem 3.1.

Proof of Theorem 3.1. Consider the curves $c_1, \dots, c_{2g+1}, d_1, \dots, d_{2g}$ as shown in Figure 7, under the inclusion $\Sigma_{2g}^4 \hookrightarrow \Sigma_{2g}$ given by capping off the four boundary components with disks. Under this inclusion, the images of the curves $B_{j,1}^i$ and $B_{j,2}^i$ are both isotopic to the curve B_j in Σ_{2g} for all $0 \leq j \leq 2g$ and $i = 1, 2$ and the images of the curves C_1^i and C_2^i are both isotopic to the curve C in Σ_{2g} for $i = 1, 2$. By Lemmas A.1 and A.2,

$$T_{B_0} \dots T_{B_{2g}} T_C(c_j) = c_{2g+2-j}, \quad T_{B_0} \dots T_{B_{2g}} T_C(d_j) = \bar{d}_{2g+1-j}$$

for all j as oriented isotopy classes of curves. Therefore, the mapping class

$$[\eta] T_{B_0} \dots T_{B_{2g}} T_C \in \text{Mod}(\Sigma_{2g})$$

acts as the identity on the graph determined by $\bigcup_{j=1}^{2g+1} c_j \cup \bigcup_{j=1}^{2g} d_j$. Because the curves $c_1, \dots, c_{2g+1}, d_1, \dots, d_{2g}$ fill Σ_{2g} and satisfy the conditions of the Alexander method [FM12, Proposition 2.8], it applies to show that

$$[\eta] T_{B_0} \dots T_{B_{2g}} T_C = 1 \in \text{Mod}(\Sigma_{2g}).$$

Finally, note that $[\eta]$ has order 2 because η has order 2 and is not isotopic to the identity. \square

We also record a corollary of Theorem 3.3 which studies the mapping class $\hat{h} \in \text{Mod}(\Sigma_{2g,2})$ first defined in (7).

Corollary A.5. *For $i = 1, 2$, recall the mapping classes*

$$\tilde{h}_i = T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i} \in \text{Mod}(\Sigma_{2g}^4)$$

as defined in Lemma 3.4. Let

$$\hat{h}_1, \hat{h}_2 \in \text{Mod}(\Sigma_{2g,2})$$

denote the images of $\tilde{h}_1, \tilde{h}_2 \in \text{Mod}(\Sigma_{2g}^4)$ respectively under the capping and forgetful homomorphisms

$$\text{Mod}(\Sigma_{2g}^4) \rightarrow \text{Mod}(\Sigma_{2g,2})$$

which caps each boundary component $\delta_1, \delta_2, \delta_3, \delta_4$ of Σ_{2g}^4 with a disk with a marked point p_1, p_2, p_3, p_4 respectively, and then forgets two marked points p_3, p_4 . Then $\hat{h}_1 = \hat{h}_2 \in \text{Mod}(\Sigma_{2g,2})$ and has order 2.

Proof. Consider the inclusion of surfaces

$$\Sigma_{2g}^4 \hookrightarrow \Sigma_{2g,2}$$

given by capping each boundary component δ_1, δ_2 with a disk with a marked point p_1, p_2 respectively and each boundary component δ_3, δ_4 of Σ_{2g}^4 with a disk. We now take isotopy classes of curves in Σ_{2g}^4 and consider their images in $\Sigma_{2g,2}$ under this inclusion. From Figure 4, observe that there are equalities of isotopy classes of curves

$$B_{j,1}^1 = B_{j,2}^1 = B_{j,1}^2 = B_{j,2}^2 \subseteq \Sigma_{2g,2}$$

for all $0 \leq j \leq 2g$ and

$$C_1^1 = C_2^1 = C_1^2 = C_2^2 \subseteq \Sigma_{2g,2}.$$

In other words,

$$T_{B_{j,1}^1} = T_{B_{j,2}^1} = T_{B_{j,1}^2} = T_{B_{j,2}^2} \in \text{Mod}(\Sigma_{2g,2}), \quad T_{C_1^1} = T_{C_2^1} = T_{C_1^2} = T_{C_2^2} \in \text{Mod}(\Sigma_{2g,2})$$

for all $0 \leq j \leq 2g$. Therefore, there is an equality in $\text{Mod}(\Sigma_{2g,2})$

$$\hat{h}_1 = T_{B_{0,2}^1} T_{B_{1,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1} = T_{B_{0,2}^2} T_{B_{1,2}^2} \dots T_{B_{2g,2}^2} T_{C_2^2} = \hat{h}_2 \in \text{Mod}(\Sigma_{2g,2}).$$

The equalities of isotopy classes of curves above also show that

$$\hat{h}_1 = T_{B_{0,1}^1} T_{B_{1,1}^1} \dots T_{B_{2g,1}^1} T_{C_1^1} \in \text{Mod}(\Sigma_{2g,2}).$$

Now to see that \hat{h}_1 (and hence \hat{h}_2) has order 2, compute using Theorem 3.3 that

$$\hat{h}_1^2 = \left(T_{B_{0,1}^1} T_{B_{1,1}^1} \dots T_{B_{2g,1}^1} T_{C_1^1} \right) \left(T_{B_{0,2}^1} T_{B_{1,2}^1} \dots T_{B_{2g,2}^1} T_{C_2^1} \right) = 1 \in \text{Mod}(\Sigma_{2g,2}). \quad \square$$

Finally, the following lemma is crucial in the construction of the partial conjugations of the MCK Lefschetz fibration in Section 3.

Lemma A.6. *For $i = 1, 2$, recall the mapping classes*

$$\tilde{h}_i = T_{B_{0,2}^i} T_{B_{1,2}^i} \dots T_{B_{2g,2}^i} T_{C_2^i} \in \text{Mod}(\Sigma_{2g}^4)$$

as defined in Lemma 3.4.

- (a) *The curves $\tilde{x}, \tilde{y}, \tilde{h}_i(\tilde{x})$, and $\tilde{h}_i(\tilde{y})$ are pairwise disjoint in Σ_{2g}^4 and are as shown in Figure 3. In particular,*

$$\tilde{h}_1(\tilde{x}) = \tilde{h}_2(\tilde{x}), \quad \tilde{h}_1(\tilde{y}) = \tilde{h}_2(\tilde{y}).$$

- (b) *There are equalities of isotopy classes of curves $\tilde{h}_i(\tilde{x}) = \tilde{h}_i^{-1}(\tilde{x})$ and $\tilde{h}_i(\tilde{y}) = \tilde{h}_i^{-1}(\tilde{y})$.*

Proof. Let \tilde{z} and \tilde{w} be the isotopy classes of curves depicted in Figure 3 labelled $\tilde{h}_i(\tilde{x})$ and $\tilde{h}_i(\tilde{y})$ respectively. First, observe that $B_{j,2}^i$ and C_2^i are disjoint from \tilde{x} and \tilde{z} for any $j > 3$ and both $i = 1, 2$. Therefore,

$$\tilde{h}_i(\tilde{x}) = T_{B_{0,2}^i} T_{B_{1,2}^i} T_{B_{2,2}^i} T_{B_{3,2}^i}(\tilde{x}), \quad \tilde{h}_i(\tilde{z}) = T_{B_{0,2}^i} T_{B_{1,2}^i} T_{B_{2,2}^i} T_{B_{3,2}^i}(\tilde{z})$$

for both $i = 1, 2$. A direct computation shows that

$$T_{B_{0,2}^i} T_{B_{1,2}^i} T_{B_{2,2}^i} T_{B_{3,2}^i}(\tilde{x}) = \tilde{z}, \quad T_{B_{0,2}^i} T_{B_{1,2}^i} T_{B_{2,2}^i} T_{B_{3,2}^i}(\tilde{z}) = \tilde{x}$$

for both $i = 1, 2$. This proves (a) and (b) for \tilde{x} .

To simplify the computations for \tilde{y} , note that the hyperelliptic involution ι (as shown on the right side of Figure 3) acts by

$$\iota(\tilde{x}) = \tilde{y}, \quad \iota(\tilde{z}) = \tilde{w}.$$

Letting $\tilde{g}_i := \iota^{-1} \tilde{h}_i \iota$, compute that

$$\tilde{h}_i(\tilde{y}) = \iota \tilde{g}_i(\tilde{x}), \quad \tilde{h}_i(\tilde{w}) = \iota \tilde{g}_i(\tilde{z}).$$

We now compute for $i = 1$ and $i = 2$ separately. For $i = 1$, observe in Figure 4 that

$$\iota(B_{j,2}^1) = B_{j,1}^1, \quad \iota(C_2^1) = C_1^1$$

for all $0 \leq j \leq 2g$. Therefore,

$$\tilde{g}_1 = T_{B_{0,1}^1} T_{B_{1,1}^1} \cdots T_{B_{2g,1}^1} T_{C_1^1} \in \text{Mod}(\Sigma_{2g}^4).$$

Theorem 3.3 shows that

$$\tilde{g}_1 \tilde{h}_1(\tilde{x}) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\tilde{x}) = \tilde{x}, \quad \tilde{g}_1 \tilde{h}_1(\tilde{z}) = T_{\delta_1} T_{\delta_2} T_{\delta_3} T_{\delta_4}(\tilde{z}) = \tilde{z}.$$

Now (a) for \tilde{y} follows because

$$\tilde{h}_1(\tilde{y}) = \iota \tilde{g}_1(\tilde{x}) = \iota \tilde{g}_1(\tilde{h}_1^2(\tilde{x})) = \iota \tilde{g}_1(\tilde{h}_1(\tilde{z})) = \iota(\tilde{z}) = \tilde{w},$$

where the second and third equalities follow from (b) and (a) for \tilde{x} respectively. To see (b) for \tilde{y} , apply the same computation as above for \tilde{z} :

$$\tilde{h}_1^2(\tilde{y}) = \tilde{h}_1(\tilde{w}) = \iota \tilde{g}_1(\tilde{z}) = \iota \tilde{g}_1(\tilde{h}_1^2(\tilde{z})) = \iota \tilde{g}_1(\tilde{h}_1(\tilde{x})) = \iota(\tilde{x}) = \tilde{y},$$

and hence $\tilde{h}_1^{-1}(\tilde{y}) = \tilde{h}_1(\tilde{y})$.

For $i = 2$, observe in Figure 4 that

$$\iota(B_{j,2}^2) = B_{j,2}^2, \quad \iota(C_2^2) = C_2^2$$

for all $0 \leq j \leq 2g$, and so

$$\tilde{g}_2 = \tilde{h}_2 \in \text{Mod}(\Sigma_{2g}^4).$$

Now (a) for \tilde{y} follows because

$$\tilde{h}_2(\tilde{y}) = \tilde{h}_2(\iota(\tilde{x})) = \iota \tilde{g}_2(\tilde{x}) = \iota \tilde{h}_2(\tilde{x}) = \iota(\tilde{z}) = \tilde{w},$$

where the second equality follows from the definition of \tilde{g}_2 , the third follows from the fact that $\tilde{g}_2 = \tilde{h}_2$, and the fourth follows from (a) for \tilde{x} . To see (b) for \tilde{y} , apply the same computation as above for \tilde{z} :

$$\tilde{h}_2^2(\tilde{y}) = \tilde{h}_2(\tilde{w}) = \tilde{h}_2(\iota(\tilde{z})) = \iota \tilde{g}_2(\tilde{z}) = \iota \tilde{h}_2(\tilde{z}) = \iota(\tilde{x}) = \tilde{y},$$

and hence $\tilde{h}_2^{-1}(\tilde{y}) = \tilde{h}_2(\tilde{y})$. □

APPENDIX B. PROOF OF PROPOSITION 6.8

The goal of this appendix is to prove Proposition 6.8. Throughout this section we follow the notation of Remark 6.4. By Lemma A.6, there are equalities of isotopy classes of curves in $\Sigma_{2g,4}$

$$\tilde{h}^2(\tilde{x}) = \tilde{x}, \quad \tilde{h}^2(\tilde{y}) = \tilde{y}.$$

Therefore there exists a representative $\tilde{\eta}_0 \in \text{Diff}^+(\Sigma_{2g,4})$ of $\tilde{h} \in \text{Mod}(\Sigma_{2g,4})$ that preserves the set $\{\alpha, \beta, \gamma, \delta\}$ of curves in $\Sigma_{2g,4}$ (cf. [FM12, Section 13.2.2]).

Let T_1, S, T_2 denote the closures of the three connected components of

$$\Sigma_{2g,4} - (\alpha \sqcup \beta \sqcup \tilde{\eta}_0(\alpha) \sqcup \tilde{\eta}_0(\beta))$$

as shown in Figure 8, where we regard $S \cong \Sigma_{2g-2,4}^4$ as a surface with four marked points and four boundary components. Observe that $\tilde{\eta}_0$ permutes the subsets T_1 and T_2 and preserves S . Moreover, $\tilde{\eta}_0^2$ preserves each subset T_1, S , and T_2 .

In the following lemma, we modify $\tilde{\eta}_0$ to a new diffeomorphism $\tilde{\eta}_1 \in \text{Diff}^+(\Sigma_{2g,4})$ by an isotopy supported compactly in \tilde{T}_1 such that $\tilde{\eta}_1$ has order 2 away from some collar neighborhoods in $T_1 \cup T_2$. The isotopies and diffeomorphisms of the proof of Lemma B.1 are depicted in Figure 9.

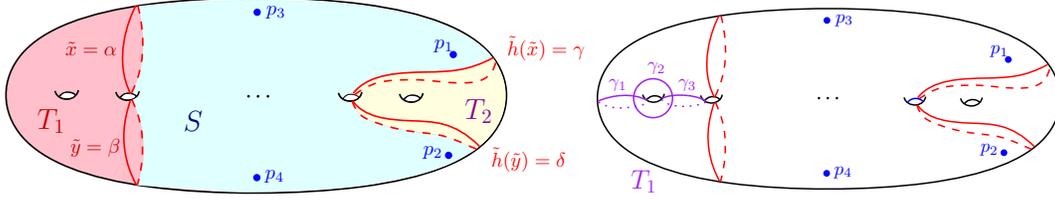


FIGURE 8. Left: Curves $\alpha, \beta, \gamma, \delta$ in $\Sigma_{2g,4}$ decomposing the surface as a union of compact subsurfaces T_1, S, T_2 ; Right: Curves $\gamma_1, \gamma_2, \gamma_3$ that fill T_1° and are fixed up to isotopy by \tilde{h} .

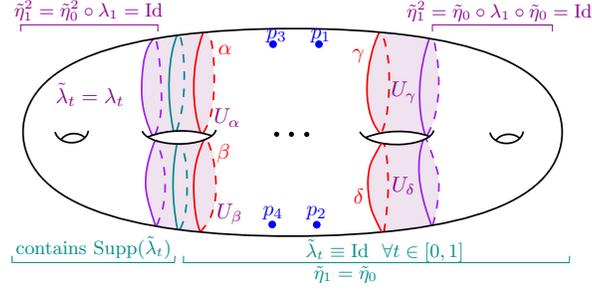


FIGURE 9. A schematic of the constructed diffeomorphism $\tilde{\eta}_1$ and the support of the isotopy $\tilde{\lambda}_t$ in the proof of Lemma B.1

Lemma B.1 (Modification in $T_1 \cup T_2$). *There exist collar neighborhoods $U_\alpha \subseteq T_1$ and $U_\beta \subseteq T_1$ of α and β respectively and a diffeomorphism $\tilde{\eta}_1 \in \text{Diff}^+(\Sigma_{2g,4})$ with $[\tilde{\eta}_1] = \tilde{h} \in \text{Mod}(\Sigma_{2g,4})$ such that*

$$\tilde{\eta}_1^2|_{T_1 - (U_\alpha \sqcup U_\beta)} = \text{Id}|_{T_1 - (U_\alpha \sqcup U_\beta)}$$

and such that

$$\tilde{\eta}_1|_{S \cup T_2} = \tilde{\eta}_0|_{S \cup T_2}.$$

In particular, $\tilde{\eta}_1$ permutes the subsets $T_1, S,$ and T_2 , and $\tilde{\eta}_1^2$ preserves each subset

$$T_1 - (U_\alpha \sqcup U_\beta), \quad U_\alpha, \quad U_\beta, \quad S, \quad T_2 - (U_\gamma \sqcup U_\delta), \quad U_\gamma, \quad U_\delta$$

where $U_\gamma := \tilde{\eta}_1(U_\alpha)$ and $U_\delta := \tilde{\eta}_1(U_\beta)$.

Proof. Consider the curves $\gamma_1, \gamma_2, \gamma_3$ in T_1 as shown in Figure 8. By Lemmas A.1 (for the isotopy classes $[\gamma_1] = c_1$ and $[\gamma_2] = c_2$) and A.2 (for the isotopy class $[\gamma_2] = d_1$), the curve $\tilde{\eta}_0^2(\gamma_i)$ is isotopic to γ_i in $\Sigma_{2g,4}$ for each $i = 1, 2, 3$. By [FM12, Lemma 3.16], the curve $\tilde{\eta}_0^2(\gamma_i)$ is then also isotopic to γ_i in T_1 for each $i = 1, 2, 3$.

Consider the interior \mathring{T}_1 of T_1 . The curves $\gamma_1, \gamma_2, \gamma_3 \subseteq \mathring{T}_1$ together fill \mathring{T}_1 . By the Alexander method [FM12, Proposition 2.8],

$$[\tilde{\eta}_0^2|_{\mathring{T}_1}] = 1 \in \text{Mod}(\mathring{T}_1).$$

Let $\lambda_t : \mathring{T}_1 \times [0, 1] \rightarrow \mathring{T}_1$ denote an isotopy with $\lambda_0 = \text{Id}_{\mathring{T}_1}$ and $\lambda_1 = \tilde{\eta}_0^{-2}|_{\mathring{T}_1}$. Fix any collar neighborhoods U_α, U_β of α and β in T_1 . By the isotopy extension theorem [Hir94, Theorem 8.1.4], there exists an isotopy $\tilde{\lambda}_t : \Sigma_{2g,4} \times [0, 1] \rightarrow \Sigma_{2g,4}$ with $\tilde{\lambda}_0 = \text{Id}_{\Sigma_{2g,4}}$ such that

$$\tilde{\lambda}_t|_{T_1 - (U_\alpha \sqcup U_\beta)} = \lambda_t|_{T_1 - (U_\alpha \sqcup U_\beta)}$$

and such that $\tilde{\lambda}_t$ is compactly supported in \mathring{T}_1 . Finally, let

$$\tilde{\eta}_1 := \tilde{\eta}_0 \circ \tilde{\lambda}_1.$$

Because $\tilde{\lambda}_t$ is supported in \mathring{T}_1 ,

$$\tilde{\eta}_1|_{S \cup T_2} = \tilde{\eta}_0 \circ \tilde{\lambda}_1|_{S \cup T_2} = \tilde{\eta}_0|_{S \cup T_2}.$$

Finally,

$$(\tilde{\eta}_0 \circ \tilde{\lambda}_1)^2|_{T_1 - (U_\alpha \sqcup U_\beta)} = (\tilde{\eta}_0 \circ \tilde{\lambda}_1)|_{T_2} \circ (\tilde{\eta}_0 \circ \lambda_1)|_{T_1 - (U_\alpha \sqcup U_\beta)} = \tilde{\eta}_0|_{T_2} \circ \tilde{\eta}_0^{-1}|_{T_1 - (U_\alpha \sqcup U_\beta)} = \text{Id}_{T_1 - (U_\alpha \sqcup U_\beta)}$$

as desired. \square

In the following lemma, we view the subsurface $S \subseteq \Sigma_{2g,4}$ as the compact subsurface with four boundary components (coming from $\alpha, \beta, \gamma, \delta$) containing the four marked points p_1, p_2, p_3, p_4 . Let \mathring{S} denote the interior of S ; in particular, \mathring{S} has four punctures (coming from the four boundary components of S). We write $\text{Diff}^+(\mathring{S}, p_1, p_2)$ or $\text{Mod}(\mathring{S}, p_1, p_2)$ below to denote the group of diffeomorphisms or mapping classes respectively of \mathring{S} that fix each point p_1, p_2 .

Lemma B.2 (Modification in S). *There exists a diffeomorphism $\psi \in \text{Diff}^+(\mathring{S}, p_1, p_2)$ of order 2 such that*

$$[\psi] = [\tilde{\eta}_1|_{\mathring{S}}] \in \text{Mod}(\mathring{S}, p_1, p_2).$$

Proof. Recall that $\tilde{\eta}_1 \in \text{Diff}^+(\Sigma_{2g,4})$ preserves the subsurface $S \subseteq \Sigma_{2g}$ and fixes each marked point p_1, p_2, p_3, p_4 of $\Sigma_{2g,4}$. We claim that the restriction $\tilde{\eta}_1^2|_{\mathring{S}} \in \text{Diff}^+(\mathring{S}, p_1, p_2)$ satisfies

$$[\tilde{\eta}_1^2|_{\mathring{S}}] = 1 \in \text{Mod}(\mathring{S}, p_1, p_2).$$

Because $[\tilde{\eta}_1^2] = 1 \in \text{Mod}(\Sigma_{2g,2})$ by Corollary A.5, the curve $\tilde{\eta}_1^2(\gamma_0)$ is isotopic to γ_0 in $\Sigma_{2g,2}$ for any curve $\gamma_0 \subseteq \mathring{S}$ that is not isotopic to α, β, γ , or δ in $\Sigma_{2g,2}$. By [FM12, Lemma 3.16], $\tilde{\eta}_1^2(\gamma_0)$ and γ_0 are also isotopic through an isotopy supported in \mathring{S} fixing p_1, p_2 for all time t . By applying the Alexander method [FM12, Proposition 2.8] to a set of filling curves of \mathring{S} and its image under $\tilde{\eta}_1^2$, we conclude that

$$[\tilde{\eta}_1^2|_{\mathring{S}}] = 1 \in \text{Mod}(\mathring{S}, p_1, p_2)$$

By the Nielsen realization theorem [FM12, Theorem 7.1], there exists a diffeomorphism $\psi \in \text{Diff}^+(\mathring{S}, p_1, p_2)$ of order 2 such that

$$[\psi] = [\tilde{\eta}_1|_{\mathring{S}}] \in \text{Mod}(\mathring{S}, p_1, p_2). \quad \square$$

Using the previous two lemmas, we now build an order-2 representative $\hat{\eta}$ of $\hat{h} \in \text{Mod}(\Sigma_{2g,2})$ with a controlled isotopy to a representative of $\tilde{h} \in \text{Mod}(\Sigma_{2g,4})$. See Figure 10 for a schematic of the resulting diffeomorphism $\hat{\eta} \in \text{Diff}^+(\Sigma_{2g,2})$ and isotopy $\kappa_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$.

Lemma B.3 (Correction near $\alpha \cup \beta \cup \gamma \cup \delta$). *There exist*

- a diffeomorphism $\hat{\eta} \in \text{Diff}^+(\Sigma_{2g,2})$ of order 2 with $[\hat{\eta}] = \hat{h} \in \text{Mod}(\Sigma_{2g,2})$,
- an isotopy $\kappa_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ with $\kappa_0 = \text{Id}_{\Sigma_{2g,2}}$,
- disjoint tubular neighborhoods $W_\alpha, W_\beta, W_\gamma$, and W_δ of α, β, γ , and δ in $\Sigma_{2g,2}$, and
- disjoint neighborhoods O_i of p_i for $i = 1, 2, 3, 4$

such that

(a) the diffeomorphism $\hat{\eta} \circ \kappa_1$ fixes pointwise O_1, O_2, O_3, O_4 and

$$[\hat{\eta} \circ \kappa_1] = \tilde{h} \in \text{Mod}(\Sigma_{2g,4}),$$

(b) the diffeomorphism $\hat{\eta}$ permutes the sets $W_\alpha, W_\beta, W_\gamma$, and W_δ by

$$\hat{\eta}(W_\alpha) = W_\gamma, \quad \hat{\eta}(W_\beta) = W_\delta,$$

and

(c) for all $i = 1, 2, 3, 4$ and all $t \in [0, 1]$,

$$\kappa_t(O_i) \cap (W_\alpha \sqcup W_\beta \sqcup W_\gamma \sqcup W_\delta) = \emptyset.$$

Proof. In this proof we construct $\hat{\eta}$ by appropriately splicing together the order-2 pieces found in Lemmas B.1 and B.2.

Step 0: Finding appropriate neighborhoods of p_i . Since $\tilde{\eta}_1 \in \text{Diff}^+(\Sigma_{2g,4})$, there exist (apply [Hir94, Theorem 8.3.1] to small enough neighborhoods of each marked point)

- (1) an isotopy $g_t : \Sigma_{2g,4} \rightarrow \Sigma_{2g,4}$ with $g_0 = \text{Id}_{\Sigma_{2g,4}}$ and compact support contained in \mathring{S} , and
- (2) neighborhoods O_1, O_2, O_3, O_4 of p_1, p_2, p_3, p_4 such that $\overline{O_i} \subseteq \mathring{S}$, $\overline{O_i}$ is compact for all $i = 1, 2, 3, 4$, and $\overline{O_i} \cap \overline{O_j} = \emptyset$ for all $i \neq j$

such that $\tilde{\eta}_1 \circ g_1$ fixes pointwise each O_i . In particular

$$[\tilde{\eta}_1 \circ g_1|_{\mathring{S}}] = [\tilde{\eta}_1] \in \text{Mod}(\mathring{S}, p_1, p_2).$$

Thus, replacing $\tilde{\eta}_1$ with $\tilde{\eta}_1 \circ g_1$ we can assume in Lemma B.2 that $\tilde{\eta}_1$ acts trivially on each O_i for $i = 1, 2, 3, 4$.

Throughout, let $\lambda_t : (\mathring{S}, p_1, p_2) \times [0, 1] \rightarrow (\mathring{S}, p_1, p_2)$ be the isotopy with $\lambda_0 = \text{Id}_{\mathring{S}}$ and $(\tilde{\eta}_1|_{\mathring{S}}) \circ \lambda_1 = \psi$ found in Lemma B.2.

Step 1: Fixing two tubular neighborhoods $V \subseteq W$ of $\alpha \cup \beta \cup \gamma \cup \delta$. Recall the collar neighborhoods U_α, U_β of α, β in T_1 and U_γ, U_δ of γ, δ in T_2 found in Lemma B.1. Let W_α and W_β be tubular neighborhoods of α and β in $\Sigma_{2g,4}$ respectively such that

$$\bar{U}_\alpha \subseteq W_\alpha, \quad \bar{U}_\beta \subseteq W_\beta$$

and such that the disks $\lambda_t^{-1}(O_i) \in S$ are disjoint from

$$(W_\alpha \cap \mathring{S}), \quad (W_\beta \cap \mathring{S}), \quad \psi(W_\alpha \cap \mathring{S}), \quad \psi(W_\beta \cap \mathring{S})$$

for all $t \in [0, 1]$ and for all $i = 1, 2, 3, 4$. (Such choices of W_α and W_β exist by compactness of the paths $\lambda_t^{-1}(\overline{O_i})$.) Then let

$$W_\gamma := \psi(W_\alpha \cap \mathring{S}) \cup \tilde{\eta}_1(W_\alpha \cap T_1), \quad W_\delta := \psi(W_\beta \cap \mathring{S}) \cup \tilde{\eta}_1(W_\beta \cap T_1).$$

By construction of ψ and $\tilde{\eta}_1$, the open annuli W_γ and W_δ are tubular neighborhoods of γ and δ respectively in $\Sigma_{2g,4}$ such that

$$\bar{U}_\gamma \subseteq W_\gamma, \quad \bar{U}_\delta \subseteq W_\delta.$$

Finally, let W be the tubular neighborhood of $\alpha \cup \beta \cup \gamma \cup \delta$ in $\Sigma_{2g,4}$ defined by

$$W := W_\alpha \cup W_\beta \cup W_\gamma \cup W_\delta.$$

To specify $V \subseteq W$, choose collar neighborhoods V_α, V_β of α and β in S such that

$$\bar{V}_\alpha \subseteq W_\alpha, \quad \bar{V}_\beta \subseteq W_\beta$$

and let

$$V := (U_\alpha \cup V_\alpha) \cup (U_\beta \cup V_\beta) \cup (U_\gamma \cup \psi(V_\alpha \cap \mathring{S})) \cup (U_\delta \cup \psi(V_\beta \cap \mathring{S})) \subseteq W.$$

Recall that $\tilde{\eta}_1(U_\alpha) = U_\gamma$ and $\tilde{\eta}_1(U_\beta) = U_\delta$ and that $\tilde{\eta}_1^2|_{T_1 \cup T_2}$ acts as the identity outside of $U_\alpha \sqcup U_\beta \sqcup U_\gamma \sqcup U_\delta$ by construction in Lemma B.1. Therefore, $\tilde{\eta}_1$ preserves both $W \cap (T_1 \cup T_2)$ and $V \cap (T_1 \cup T_2)$. On the other hand, ψ has order 2 and so ψ preserves both $W \cap \mathring{S}$ and $V \cap \mathring{S}$. See Figure 10.

Step 2: Isotoping $\tilde{\eta}_1$ in S to have order 2 away from V . By the isotopy extension theorem [Hir94, Theorem 8.1.4], there exists an isotopy $\tilde{\lambda}_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ with $\tilde{\lambda}_0 = \text{Id}_{\Sigma_{2g,2}}$ such that

$$\tilde{\lambda}_t|_{\mathring{S} - (S \cap V)} = \lambda_t|_{\mathring{S} - (S \cap V)}$$

and such that $\tilde{\lambda}_t$ has compact support contained in $\mathring{S} \subseteq \Sigma_{2g,2}$. In particular, the disks $\tilde{\lambda}_t^{-1}(O_i) = \lambda_t^{-1}(O_i)$ are contained in $S - (S \cap W)$ for all $t \in [0, 1]$ and for all $i = 1, 2, 3, 4$.

Let

$$\tilde{\eta}_2 := \tilde{\eta}_1 \circ \tilde{\lambda}_1 \in \text{Diff}^+(\Sigma_{2g,2}).$$

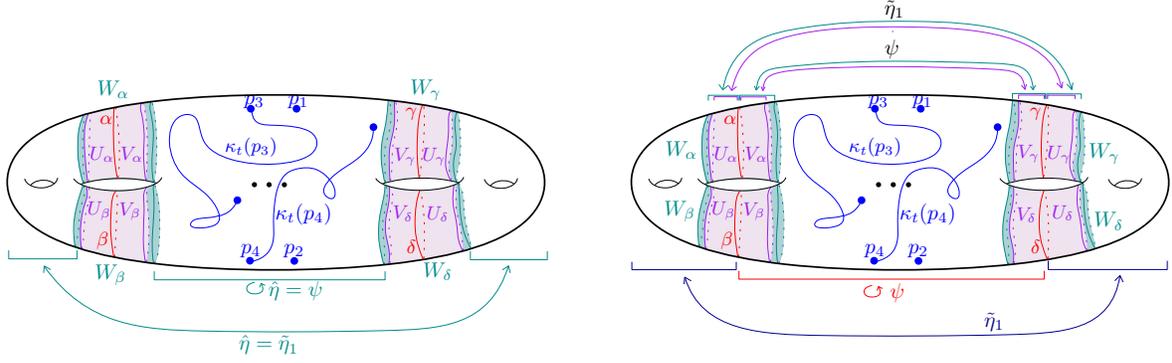


FIGURE 10. Left: A schematic for the order-2 diffeomorphism $\hat{\eta} \in \text{Diff}^+(\Sigma_{2g,2})$ and isotopy $\kappa_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ in the conclusion of Lemma B.3. Away from some tubular neighborhoods $W_\alpha, W_\beta, W_\gamma, W_\delta$ of $\alpha, \beta, \gamma, \delta$, the diffeomorphism $\hat{\eta}$ restricts to ψ on S and to $\tilde{\eta}_1$ on $T_1 \sqcup T_2$. Right: Choice of tubular neighborhoods W and V and the action of ψ and $\tilde{\eta}_1$.

We claim that $\tilde{\eta}_2$ preserves the two sets V, W and that $\tilde{\eta}_2^2$ has support contained in V . To see that $\tilde{\eta}_2^2$ has support contained in V , we consider $T_1 \cup T_2$ and S separately. In $T_1 \cup T_2$,

$$\tilde{\eta}_2^2|_{(T_1 \cup T_2) - V} = \tilde{\eta}_1^2|_{(T_1 \cup T_2) - V} = \text{Id}_{(T_1 \cup T_2) - V}$$

where first equality follows because $\tilde{\lambda}_1$ has compact support in \mathring{S} and the second equality follows by Lemma B.1 because $(T_1 \cup T_2) \cap V = U_\alpha \sqcup U_\beta \sqcup U_\gamma \sqcup U_\delta$. In S ,

$$\tilde{\eta}_2^2|_{S - V} = (\tilde{\eta}_1 \circ \tilde{\lambda}_1)^2|_{S - V} = \psi^2|_{S - V} = \text{Id}_{S - V},$$

where the second equality follows because $\lambda_1|_{S - V} = \tilde{\lambda}_1|_{S - V}$ and the third equality follows because $\psi \in \text{Diff}^+(\mathring{S})$ has order 2 by construction in Lemma B.2.

To see that $\tilde{\eta}_2$ preserves V and W , first recall that $W \cap (T_1 \cup T_2)$ and $V \cap (T_1 \cup T_2)$ are preserved by $\tilde{\eta}_1$, and that $\tilde{\eta}_1|_{T_1 \cup T_2} = \tilde{\eta}_2|_{T_1 \cup T_2}$. On the other hand, note that $\tilde{\eta}_2|_{S - V} = \psi|_{S - V}$, and that $W \cap \mathring{S}$ and $V \cap \mathring{S}$ are both preserved by ψ by construction of W and V .

Step 3: Cutting and pasting in W . By [FM12, Proposition 2.4 and Lemma 3.17], the restriction $\tilde{\eta}_2^2|_{\bar{W}}$ is topologically isotopic (rel $\partial\bar{W}$) to $\text{Id}_{\bar{W}}$, because

$$[\tilde{\eta}_2^2] = [(\tilde{\eta}_1 \circ \tilde{\lambda}_1)^2] = [\tilde{\eta}_1^2] = \hat{h}^2 = 1 \in \text{Mod}(\Sigma_{2g,2}),$$

where the last equality follows by Corollary A.5. Therefore, there exists a topological isotopy $\mu_t : (\bar{W}_\alpha \sqcup \bar{W}_\beta) \times [0, 1] \rightarrow \bar{W}_\alpha \sqcup \bar{W}_\beta$ rel $\partial(\bar{W}_\alpha \sqcup \bar{W}_\beta)$ with $\mu_0 = \text{Id}_{\bar{W}_\alpha \sqcup \bar{W}_\beta}$ such that

$$\tilde{\eta}_2^{-1}|_{\bar{W}_\alpha \sqcup \bar{W}_\beta} \circ \mu_1 = \tilde{\eta}_2|_{\bar{W}_\alpha \sqcup \bar{W}_\beta} : \bar{W}_\alpha \sqcup \bar{W}_\beta \rightarrow \bar{W}_\gamma \sqcup \bar{W}_\delta.$$

With the isotopy μ_t in hand, consider the map

$$\hat{\eta} := \begin{cases} \tilde{\eta}_2 & \text{on } \Sigma_{2g,2} - (\bar{W}_\alpha \sqcup \bar{W}_\beta) \\ \tilde{\eta}_2^{-1} & \text{on } \bar{W}_\alpha \sqcup \bar{W}_\beta. \end{cases}$$

The map $\hat{\eta}$ is smooth because $\tilde{\eta}_2^{-1}|_{\Sigma_{2g,2} - V} = \tilde{\eta}_2|_{\Sigma_{2g,2} - V}$ by construction in Step 2, since $\Sigma_{2g,2} - V$ contains an open neighborhood of $\partial(\bar{W}_\alpha \sqcup \bar{W}_\beta)$. See Figure 10.

The extension $\tilde{\mu}_t : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ of μ_t by the identity outside of $\bar{W}_\gamma \sqcup \bar{W}_\delta$ is a topological isotopy of $\Sigma_{2g,2}$ with $\tilde{\mu}_0 = \text{Id}_{\Sigma_{2g,2}}$. Therefore,

$$[\hat{\eta}] = [\hat{\eta} \circ \tilde{\mu}_1] = [\tilde{\eta}_2] = \hat{h} \in \text{Mod}(\Sigma_{2g,2}).$$

Furthermore, we claim that $\hat{\eta}$ has order 2. Because $\tilde{\eta}_2^2$ has support contained in $V \subseteq W$, it suffices to check that $\hat{\eta}^2|_W = \text{Id}_W$, which is true by construction. Moreover, $\hat{\eta}$ permutes the components of W because $\tilde{\eta}_2$ does, and

$$\hat{\eta}(W_\alpha) = \tilde{\eta}_2(W_\alpha) = W_\gamma, \quad \hat{\eta}(W_\beta) = \tilde{\eta}_2(W_\beta) = W_\delta.$$

We have thus shown that $\hat{\eta}$ satisfies all the desired properties listed in the statement of the lemma.

Step 4: Finding the isotopy κ_t . Consider $\hat{\eta} \circ \tilde{\lambda}_1^{-1} \in \text{Diff}^+(\Sigma_{2g,2})$. Observe that $\hat{\eta} \circ \tilde{\lambda}_1^{-1}$ fixes pointwise the neighborhoods O_i for $i = 1, 2, 3, 4$: For any $x \in O_i$,

$$(\hat{\eta} \circ \tilde{\lambda}_1^{-1})(x) = (\tilde{\eta}_2 \circ \tilde{\lambda}_1^{-1})(x) = (\tilde{\eta}_1 \circ \tilde{\lambda}_1)(\tilde{\lambda}_1^{-1}(x)) = \tilde{\eta}_1(x) = x,$$

where the first equality holds because $\tilde{\lambda}_1^{-1}(O_i)$ is not contained in \bar{W} by construction of W , and the last equality holds because $\tilde{\eta}_1$ fixes pointwise O_i by Step 0. This shows that $\hat{\eta} \circ \tilde{\lambda}_1^{-1}$ is an element of $\text{Diff}^+(\Sigma_{2g,4})$ fixing each O_i pointwise for $i = 1, 2, 3, 4$.

The topological isotopy $\tilde{\lambda}_1 \circ \tilde{\mu}_t \circ \tilde{\lambda}_1^{-1} : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$ fixes the two points p_3, p_4 for all $t \in [0, 1]$ because $\tilde{\mu}_t$ is the identity outside $\bar{W}_\gamma \sqcup \bar{W}_\delta$. Therefore,

$$[\hat{\eta} \circ \tilde{\lambda}_1^{-1}] = [(\hat{\eta} \circ \tilde{\lambda}_1^{-1}) \circ (\tilde{\lambda}_1 \circ \tilde{\mu}_t \circ \tilde{\lambda}_1^{-1})] = [(\hat{\eta} \circ \tilde{\mu}_t) \circ \tilde{\lambda}_1^{-1}] = [(\tilde{\eta}_1 \circ \tilde{\lambda}_1) \circ \tilde{\lambda}_1^{-1}] = [\tilde{\eta}_1] = \tilde{h}$$

as mapping classes in $\text{Mod}(\Sigma_{2g,4})$. Finally, letting

$$\kappa_t := \tilde{\lambda}_t^{-1} : \Sigma_{2g,2} \times [0, 1] \rightarrow \Sigma_{2g,2}$$

concludes the proof. \square

We now collect the lemmas above to prove the main proposition of this appendix.

Proof of Proposition 6.8. Fix the notation of the statement of Lemma B.3. Let $W = W_\alpha \sqcup W_\beta \sqcup W_\gamma \sqcup W_\delta$ and $O = O_1 \sqcup O_2 \sqcup O_3 \sqcup O_4$. The diffeomorphisms $\hat{\eta}$ and $\tilde{\eta}$ and the isotopy κ_t found in Lemma B.3 satisfy the condition that $[\hat{\eta}] = \tilde{h} \in \text{Mod}(\Sigma_{2g,2})$ and conditions (c) and (d). It therefore suffices to construct the desired symplectic form θ and symplectomorphism $\varphi \in \text{Diff}^+(\Sigma_{2g,4})$ with $[\varphi] = \tilde{f} \in \text{Mod}(\Sigma_{2g,4})$ satisfying (a) and (b).

Let θ_α be a symplectic form on $W_\alpha \subseteq \Sigma_{2g}$ and let $\varphi_\alpha \in \text{Symp}(W_\alpha, \theta_\alpha)$ be a compactly supported, right-handed Dehn twist about α . (For example, we can take $\theta_\alpha = d\theta \wedge dt$, where θ, t are the two coordinates of $W_\alpha \cong S^1 \times (0, 1)$, and $\varphi_\alpha(\theta, t) = (\theta - \rho(t), t)$ where $\rho : (0, 1) \rightarrow [0, 2\pi]$ is a smooth, increasing function with $\rho \equiv 0$ near $t = 0$ and $\rho \equiv 2\pi$ near $t = 2\pi$.) Similarly, let θ_β be a symplectic form on $W_\beta \subseteq \Sigma_{2g}$ and let $\varphi_\beta \in \text{Symp}(W_\beta, \theta_\beta)$ be a compactly supported, right-handed Dehn twist about β .

Furthermore, we define similar symplectic forms and Dehn twists in W_δ and W_γ by

$$\begin{aligned} \varphi_\gamma &:= \hat{\eta} \circ \varphi_\alpha \circ \hat{\eta}, & \theta_\gamma &:= \hat{\eta}^* \theta_\alpha, \\ \varphi_\delta &:= \hat{\eta} \circ \varphi_\beta \circ \hat{\eta}, & \theta_\delta &:= \hat{\eta}^* \theta_\beta. \end{aligned}$$

Here, recall from Lemma B.3 that $\hat{\eta}(W_\alpha) = W_\gamma$ and $\hat{\eta}(W_\beta) = W_\delta$ by construction.

With the above symplectic forms in hand, let θ be a symplectic form on Σ_{2g} such that

$$\hat{\eta}^* \theta = \theta, \quad \theta|_{W'_\alpha} = \theta_\alpha, \quad \theta|_{W'_\beta} = \theta_\beta, \quad \theta|_{W'_\gamma} = \theta_\gamma, \quad \theta|_{W'_\delta} = \theta_\delta.$$

Here $W'_\alpha, W'_\beta, W'_\gamma, W'_\delta$ are suitable open subsets of $W_\alpha, W_\beta, W_\gamma, W_\delta$ respectively, each containing the support of $\varphi_\alpha, \varphi_\beta, \varphi_\gamma$ and φ_δ respectively. One way to find such a form θ is to first extend the forms $\theta_\alpha, \theta_\beta, \theta_\gamma$, and θ_δ to all of Σ_{2g} using a partition of unity argument, and then averaging this form under the action of $\langle \hat{\eta} \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Now let

$$\varphi := \varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\gamma \circ \varphi_\delta^{-1}.$$

Then φ is supported in W because $\varphi_\alpha, \varphi_\beta, \varphi_\gamma$, and φ_δ are, and hence φ satisfies (a). Because as elements of $\text{Mod}(\Sigma_{2g,4})$,

$$[\varphi_\alpha] = T_x, \quad [\varphi_\beta] = T_y, \quad [\varphi_\gamma] = T_{\tilde{h}(x)}, \quad [\varphi_\delta] = T_{\tilde{h}(y)},$$

there is also an equality of mapping classes $[\varphi] = \tilde{f} \in \text{Mod}(\Sigma_{2g,4})$ as desired.

To see that φ preserves the form θ , it suffices to check this on W , which contains the support of φ . On $W_\alpha \sqcup W_\beta$, φ preserves θ because

$$(\varphi^*\theta)|_{W_\alpha} = \varphi_\alpha^*\theta_\alpha = \theta_\alpha, \quad (\varphi^*\theta)|_{W_\beta} = (\varphi_\beta^{-1})^*\theta_\beta = \theta_\beta.$$

On $W_\gamma \sqcup W_\delta$, φ preserves θ because

$$\begin{aligned} (\varphi^*\theta)|_{W_\gamma} &= \varphi_\gamma^*\theta_\gamma = (\hat{\eta}\varphi_\alpha\hat{\eta})^*(\hat{\eta}^*\theta_\alpha) = \hat{\eta}^*\varphi_\alpha^*\theta_\alpha = \hat{\eta}^*\theta_\alpha = \theta_\gamma, \\ (\varphi^*\theta)|_{W_\delta} &= (\varphi_\delta^{-1})^*\theta_\delta = (\hat{\eta}\varphi_\beta^{-1}\hat{\eta})^*(\hat{\eta}^*\theta_\beta) = \hat{\eta}^*(\varphi_\beta^{-1})^*\theta_\beta = \hat{\eta}^*\theta_\beta = \theta_\delta. \end{aligned}$$

Finally, to see that $\hat{\eta}$ and φ commute, compute

$$\hat{\eta} \circ \varphi = \hat{\eta} \circ (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\gamma \circ \varphi_\delta^{-1}) = \hat{\eta} \circ (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\gamma \circ \varphi_\delta^{-1}) \circ \hat{\eta}^2 = (\varphi_\gamma \circ \varphi_\delta^{-1} \circ \varphi_\alpha \circ \varphi_\beta^{-1}) \circ \hat{\eta} = \varphi \circ \hat{\eta}.$$

This concludes the proof of (b). \square

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