On the uniqueness of the Prym map

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Abstract

The classical Prym construction associates to a smooth, genus g complex curve X equipped with a nonzero cohomology class $\theta \in H^1(X, \mathbb{Z}/2\mathbb{Z})$, a principally polarized abelian variety (PPAV) $\mathrm{Prym}(X, \theta)$. Denote the moduli space of pairs (X, θ) by $\mathcal{R}_g$, and let $\mathcal{A}_h$ be the moduli space of PPAVs of dimension h. The Prym construction globalizes to a holomorphic map of complex orbifolds $\mathrm{Prym} : \mathcal{R}_g \to \mathcal{A}_{g-1}$. For $g \geq 4$ and $h \leq g-1$, we show that Prym is the unique nonconstant holomorphic map of complex orbifolds $F : \mathcal{R}_g \to \mathcal{A}_h$. This solves a conjecture of Farb [6]. A main component in our proof is a classification of homomorphisms $\pi_1^{\mathrm{orb}}(\mathcal{R}_g) \to \mathrm{Sp}(2h, \mathbb{Z})$ for $h \leq g-1$. This is achieved using arguments from geometric group theory and low-dimensional topology.

Let X be a smooth, genus g complex curve, and let $\Omega^1(X)$ be the space of holomorphic 1-forms on X. The *Jacobian* of X,

$$\mathrm{Jac}(X) := \frac{\Omega^1(X)^\vee}{H_1(X, \mathbb{Z})}$$

is a g-dimensional principally polarized abelian variety (PPAV) canonically associated to X.

Let $\mathcal{M}_g$ be the moduli space of complex smooth genus g curves, and let $\mathcal{A}_g$ be the moduli space of PPAVs of dimension g. The Jacobian induces a holomorphic map, the *period map*

$$J : \mathcal{M}_g \to \mathcal{A}_g \quad , \quad X \to \mathrm{Jac}(X).$$

In a recent paper [6], Farb showed that if $g \geq 3$ and $h \leq g$ then J is the *unique* non-constant holomorphic map of complex orbifolds $\mathcal{M}_g \to \mathcal{A}_h$. In particular, extra data needs to be attached to smooth curves of genus g in order to associate, in a way that respects orbifold structures, a PPAV of dimension less than g to each such curve. An example of such a construction has been known to exist since over 100 years [7], as we now explain.

The Prym construction. Prym varieties [7], named as such by Mumford in honor of Friedrich Prym (1841-1915), provide a classical example of a way to obtain PPAVs of dimension $g-1$ from smooth curves of genus g. Any nonzero $\theta \in H^1(X, \mathbb{Z}/2\mathbb{Z})$ defines an unbranched double cover

$$p : Y \to X,$$

with deck transform σ, and where Y is a curve of genus $2g-1$.

The *Prym variety* associated to (X, θ) is defined as (for more details see Section 3.2.1)

$$\mathrm{Prym}(X, \theta) := \frac{\mathrm{Jac}(Y)}{p^*(\mathrm{Jac}(X))} \in \mathcal{A}_{g-1}.$$

Moduli space of Prym varieties. The Prym construction globalizes as follows. Let

$$\mathcal{R}_g := \{(X, \theta_X) : X \text{ smooth complex curve of genus } g, \text{ and } \theta_X \in H^1(X, \mathbb{Z}/2\mathbb{Z})^*\}/\sim$$

be the space of equivalence classes of pairs (X, θ_X), where $(X_1, \theta_1) \sim (X_2, \theta_2)$ if there exists a biholomorphism $f : X_1 \to X_2$, with $f^*(\theta_2) = \theta_1$. As we explain in more detail below in this introduction, $\mathcal{R}_g$ is a complex orbifold (warning: there are two closely related orbifold structures on $\mathcal{R}_g$, for the details see Section 1), and the Prym construction globalizes to a map of orbifolds

$$\mathrm{Prym} : \mathcal{R}_g \to \mathcal{A}_{g-1} \quad , \quad (X, \theta_X) \mapsto \mathrm{Prym}(X, \theta_X).$$

Our main result shows that, as conjectured by Farb in [6], Prym is rigid.

Theorem 1 (**Rigidity of** Prym)**.** *Let $g \geq 4$ and let $h \leq g-1$. Let $F : \mathcal{R}_g \to \mathcal{A}_h$ be a nonconstant holomorphic map of complex orbifolds*[1]*. Then $h = g-1$ and $F = \mathrm{Prym}$.*

The proof of Theorem 1 uses in a fundamental way that $g \geq 4$. I do not known if the statement holds true also for $g = 2, 3$.

Two orbifold structures on $\mathcal{R}_g$. There exist two natural orbifold structures on $\mathcal{R}_g$, which give very different results with respect to maps to $\mathcal{A}_h$ (see Theorem 2). Here we provide a brief description of the two orbifold structures and refer to Section 1 for the details.

Let S_g be a closed surface of genus g, let $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$, and let $p : S_{2g-1} \to S_g$ be the associated double cover with deck transform σ. Let $\mathrm{Mod}(S_g)$ be the mapping class group of S_g, and define

$$\mathrm{Mod}(S_g, [\beta]) := \mathrm{Stab}_{\mathrm{Mod}(S_g)}([\beta])$$

as the stabilizer of $[\beta]$, with respect to the action of $\mathrm{Mod}(S_g)$ on $H_1(S_g, \mathbb{Z}/2\mathbb{Z})$. Similarly, define

$$\mathrm{Mod}(S_{2g-1}, \sigma) := C_{\mathrm{Mod}(S_{2g-1})}([\sigma])$$

as the centralizer of $[\sigma]$.

Both $\mathrm{Mod}(S_g, [\beta])$ and $\mathrm{Mod}(S_{2g-1}, \sigma)$ act on Teichmüller space $\mathrm{Teich}(S_g)$, and the two orbifold structures on $\mathcal{R}_g$ come from considering $\mathrm{Mod}(S_g, [\beta])$ or $\mathrm{Mod}(S_{2g-1}, \sigma)$ as the orbifold fundamental group of $\mathcal{R}_g$. If we consider $\pi_1^{\mathrm{orb}}(\mathcal{R}_g) = \mathrm{Mod}(S_{2g-1}, \sigma)$, then *every* point of $\mathcal{R}_g$ is an orbifold point of order at least 2. This phenomenon is akin to both $\mathrm{Sp}(2g, \mathbb{Z})$ and $\mathrm{PSp}(2g, \mathbb{Z})$ acting on Siegel upper half-space $\mathfrak{h}_g$ and giving the same quotient $\mathcal{A}_g$, but different orbifold structures on $\mathcal{A}_g$.

The difference between these two orbifold structures on $\mathcal{R}_g$ seems to be elided in the literature, yet as the following Theorem shows, the inclusion of the involution σ is fundamental to our results. Let $\hat{\mathcal{R}}_g$ denote the orbifold structure on $\mathcal{R}_g$ with $\pi_1^{\mathrm{orb}}(\hat{\mathcal{R}}_g) = \mathrm{Mod}(S_g, [\beta])$.

Theorem 2. *Fix $g \geq 4$ and $h \leq g-1$. Then, any holomorphic map $F : \hat{\mathcal{R}}_g \to \mathcal{A}_h$ of complex orbifolds is constant.*

Remark 0.1. If one considers only effective group actions in Definition 1.1 then Theorem 2 is not correct. The action of $\mathrm{Mod}(S_{2g-1}, \sigma)$ on $\mathrm{Teich}(S_g)$ factors through $\mathrm{Mod}(S_g, [\beta])$, and similarly the action of $\mathrm{Sp}(2h, \mathbb{Z})$ on $\mathfrak{h}_h$ factors through $\mathrm{PSp}(2h, \mathbb{Z})$. Hence, for effective actions there is no obstruction at the level of homomorphisms $\mathrm{Mod}(S_g, [\beta]) \to \mathrm{PSp}(2h, \mathbb{Z})$, and the Prym construction globalizes to a holomorphic map of complex orbifolds. I do not know if the analogous statement to Theorem 1 holds in this setting but it will entail answering the following.

Question 1. *Fix g and $h \leq g-1$. Classify homomorphisms $\phi : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{PSp}(2h, \mathbb{Z})$.*

[1] See Definition 1.1

Prym representation. In the same way as the standard symplectic representation of $\mathrm{Mod}(S_g)$ is associated to the period map, the Prym map has an associated representation

$$\mathrm{Prym}_* : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{Sp}(2(g-1), \mathbb{Z}).$$

The first step in the proof of Theorem 1 is the following purely group theoretic result. It shows that Prym_* exhibits a similar level of rigidity as that of the standard symplectic representation for $\mathrm{Mod}(S_g)$.

Theorem 3 (**Rigidity of** Prym_***).** *Let $g \geq 4$ and $m \leq 2(g-1)$. Let $\phi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{GL}(m, \mathbb{C})$. The following holds,*

1. *If $m < 2(g-1)$ then $\mathrm{Im}(\phi)$ is cyclic of order at most 4.*
2. *Let $\chi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathbb{C}^*$ be a group homomorphism. If $m = 2(g-1)$ then $\mathrm{Im}(\phi)$ is either cyclic of order at most 4 or ϕ is conjugate to the map:*

$$f \to \chi(f)\, \mathrm{Prym}_*(f),$$

where $\chi(f)^4 = 1$.

Note that, unlike the case of $\mathrm{Mod}(S_g)$ (as shown in [8, 10]), the group $\mathrm{Mod}(S_{2g-1}, \sigma)$ has a *nontrivial* (infinite-image) linear representation of dimension less than g, and the same is true for $\mathrm{Mod}(S_g, [\beta])$ when g is even (see Theorem 5).

Since $\mathcal{A}_h$ for $h \geq 1$ is a $K(\pi, 1)$ in the category of orbifolds, Theorem 3 implies the following (see also Corollary 12).

Corollary 4. *Fix $g \geq 4$ and $h \leq g-1$. Let $F : \mathcal{R}_g \to \mathcal{A}_h$ be a continuous map of orbifolds. If $h = g-1$, then F is homotopic to* Prym*. Otherwise, there exists a cyclic cover $\tilde{\mathcal{R}}_g$ of $\mathcal{R}_g$ of order at most* 4*, so that the induced map $\tilde{F} : \tilde{\mathcal{R}}_g \to \mathcal{A}_h$ is homotopic to a constant map.*

Strategy of proof of Theorems 1 and 2. Our proof follows the general strategy laid out in [6]. The two main aspects of the proof are the topological and holomorphic sides of the story.

1. In Section 2, we classify low-dimensional linear and symplectic representations of $\mathrm{Mod}(S_g, [\beta])$ and $\mathrm{Mod}(S_{2g-1}, \sigma)$. Our approach is based on (and extends) the results of Franks-Handel, and Korkmaz [8, 10], which classify linear representations for the full mapping class group $\mathrm{Mod}(S_g)$. A key ingredient in our proof is to prove connectedness of the complex of curves $\mathcal{N}_1(S_g)$ (see Section 2.2).

2. In Section 3, we add the assumption of holomorphicity for the map $F : \mathcal{R}_g \to \mathcal{A}_h$ to deduce Theorem 2 and reduce the proof of Theorem 1 to the case of $h = g-1$ and F homotopic to Prym. In order to avoid orbifold issues when dealing with the $h = g-1$ case, we will pass to a suitable (smooth) cover $\mathcal{R}_g[\psi]$ of $\mathcal{R}_g$. Steps 2-4 in Farb's proof [6] for the rigidity of the period map $J : \mathcal{M}_g \to \mathcal{A}_g$, extend to our case without modifications. Step 5, the existence of $\mathcal{A}_{g-1}$-rigid curves, requires some minor modifications. They arise due to our use of finite non-Galois covers of $\overline{\mathcal{M}_g}$.

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1 Orbifold structures on $\mathcal{R}_g$

In this section we show how to give $\mathcal{R}_g$ the structure of a complex orbifold. First, let us briefly recall the definition of orbifold and maps between orbifolds [6, Remark 2.1].

Definition 1.1 (**Orbifolds and maps between orbifolds**)**.** Let X be a simply connected manifold (resp. complex manifold) and let Γ be a group acting properly discontinuously on X by homeomorphisms (resp. biholomorphisms), but not necessarily freely nor effectively. Then the quotient X/Γ is a topological (resp. complex) *orbifold*. Define $\pi_1^{\mathrm{orb}}(X/\Gamma) := \Gamma$ as the *orbifold fundamental group* of X/Γ. Let Y/Λ be another orbifold, and $\rho : \Gamma \to \Lambda$ a group homomorphism. A continuous (resp. holomorphic) map in the category of orbifolds $F : X/\Gamma \to Y/\Lambda$ is a map so that there exists a continuous (resp. holomorphic) lift $\tilde{F} : X \to Y$ that *intertwines* ρ:

$$\tilde{F}(\gamma.x) = \rho(\gamma).\tilde{F}(x) \quad \text{ for all } x \in X, \gamma \in \Gamma.$$

If this is the case we denote ρ by $F_* : \Gamma \to \Lambda$. Note that postcomposition of F_* with an inner automorphism c_ℓ of Λ changes $\tilde{F} \to \ell \circ \tilde{F}$, so that F_* is defined up to postcomposition with inner automorphisms of Λ.

Remark 1.1. If Γ acts effectively, our definition agrees with Thurston's definition of *good* orbifold [15, Ch.13].

Let S_g be a closed surface of genus g. The mapping class group $\mathrm{Mod}(S_g)$ is defined as

$$\mathrm{Mod}(S_g) := \pi_0(\mathrm{Diff}^+(S_g)).$$

Let $\mathrm{Teich}(S_g)$ denote the *Teichmüller* space of S_g, the space of holomorphic structures on S_g up to isotopy. $\mathrm{Mod}(S_g)$ acts on $\mathrm{Teich}(S_g)$ properly discontinously, but not freely, by biholomorphisms. Let $[\beta] \in H_1(S, \mathbb{Z}/2\mathbb{Z})$ and define

$$\mathrm{Mod}(S_g, [\beta]) := \mathrm{Stab}_{\mathrm{Mod}(S_g)}([\beta]),$$

as the stabilizer of $[\beta]$ in $\mathrm{Mod}(S_g)$. Then define

$$\hat{\mathcal{R}}_g := \mathrm{Teich}(S_g)/\,\mathrm{Mod}(S_g, [\beta]).$$

In particular, $\hat{\mathcal{R}}_g$ has the structure of a complex orbifold with $\pi_1^{\mathrm{orb}}(\hat{\mathcal{R}}_g) = \mathrm{Mod}(S_g, [\beta])$. Furthermore, $\hat{\mathcal{R}}_g$ is in bijective correspondence with $\mathcal{R}_g$ and thus endows $\mathcal{R}_g$ with an orbifold structure.

One of the goals of this paper is to classify all holomorphic maps of complex orbifolds $\hat{\mathcal{R}}_g \to \mathcal{A}_h$ for $h \leq g - 1$. Define the map,

$$\widehat{\mathrm{Prym}} : \hat{\mathcal{R}}_g \to \mathcal{A}_{g-1} \quad , \quad (X, \theta) \to \mathrm{Prym}(X, \theta).$$

Theorem 2 shows that $\widehat{\mathrm{Prym}}$ *cannot* be a map in the category of complex orbifolds.

Obstruction. The obstruction to realize $\widehat{\mathrm{Prym}}$ as map of orbifolds is the *non-existence* of non-finite representations $\phi : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{Sp}(2(g-1), \mathbb{Z})$. As we explain in more detail in Section 2, the Prym construction defines a representation:

$$\widehat{\mathrm{Prym}}_* : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{PSp}(2(g-1), \mathbb{Z}),$$

which does not lift to a symplectic representation. Thus, there is an associated non-split central $\mathbb{Z}/2\mathbb{Z}$ extension:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to H \to \mathrm{Mod}(S_g, [\beta]) \to 1$$

By definition, there is a representation $H \to \mathrm{Sp}(2(g-1), \mathbb{Z})$ and H acts on $\mathrm{Teich}(S_g)$ via $\mathrm{Mod}(S_g, [\beta])$ so that every point is an orbifold point of order at least 2. Thus, $\mathcal{R}_g$ can be endowed with an orbifold structure for which the Prym construction does define a holomorphic map Prym in the category of complex orbifolds. In fact, there is a concrete description of H and this alternative orbifold structure, as we now explain.

Moduli space of double covers. Let Y be a complex smooth genus $2g-1$ curve, and $\sigma_Y : Y \to Y$ a fixed-point free biholomorphic involution. Say that two such pairs (Y_1, σ_{Y_1}) and (Y_2, σ_2) are equivalent if there is a biholomorphism $f : Y_1 \to Y_2$ such that $f^{-1}\sigma_2 f = \sigma_1$. Then, there is a bijection

$$\phi : \{[(Y, \sigma_Y)]\} \to \mathcal{R}_g \quad , \quad [(Y, \sigma_Y)] \to [(Y/\sigma_Y, \theta_Y)]$$

where θ_Y is given by the monodromy of the covering $p : Y \to Y/\sigma_Y$.

Let σ be a fixed-point free involution on the closed surface S_{2g-1}, and let $[\sigma]$ be its class in $\mathrm{Mod}(S_{2g-1})$. Let $\mathrm{Fix}([\sigma]) := \mathrm{Teich}(S_{2g-1})^{[\sigma]}$ and define

$$\mathrm{Mod}(S_{2g-1}, \sigma) := C_{\mathrm{Mod}(S_{2g-1})}([\sigma]).$$

Then, there is an exact sequence

$$1 \to \langle \sigma \rangle \to \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{Mod}(S_g, [\beta]) \to 1,$$

and, via the bijection ϕ,

$$\mathcal{R}_g = \mathrm{Fix}([\sigma]) / \mathrm{Mod}(S_{2g-1}, \sigma)$$

so that $\pi_1^{\mathrm{orb}}(\mathcal{R}_g) = \mathrm{Mod}(S_{2g-1}, \sigma)$.

Furthermore, ϕ induces a $2:1$ map of complex orbifolds (but a biholomorphism in the complex category)

$$\begin{tikzcd} \mathrm{Fix}([\sigma]) \arrow[r, "\tilde{\phi}"] \arrow[d] & \mathrm{Teich}(S_g) \arrow[d] \\ \mathcal{R}_g \arrow[r, "\phi"] & \hat{\mathcal{R}}_g \end{tikzcd}$$

Thus, viewing $\mathcal{R}_g$ as equivalence classes of curves with an involution is precisely the alternative orbifold structure stated at the end of the previous section, and the prym construction induces a holomorphic map of complex orbifolds,

$$\mathrm{Prym} : \mathcal{R}_g \to \mathcal{A}_{g-1} \quad , \quad (Y, \sigma_Y) \to \mathrm{Prym}(Y/\sigma_Y, \theta_Y).$$

The difference between $\mathcal{R}_g$ and $\hat{\mathcal{R}}_g$ is precisely the difference between having covers of $\mathcal{M}_g$ given by G-structures or by G-covers (cf. [1, Ch 16, p 525-526]), in our case $G = \mathbb{Z}/2\mathbb{Z}$.

2 Topological results

Let S_g be a closed surface of genus $g \geq 1$, and $[\beta] \in H_1(S, \mathbb{Z}/2\mathbb{Z})^*$. Then, there is a (unique up to isomorphism) double cover

$$p : S_{2g-1} \to S_g,$$

with deck transform σ, and monodromy given by intersection with $[\beta]$. Define

$$\mathrm{Mod}(S_g, [\beta]) := \mathrm{Stab}_{\mathrm{Mod}(S_g)}([\beta]), \text{ and } \mathrm{Mod}(S_{2g-1}, \sigma) := C_{\mathrm{Mod}(S_{2g-1})}(\sigma)$$

the stabilizer of $[\beta]$ in $\mathrm{Mod}(S_g)$, and the centralizer of σ in $\mathrm{Mod}(S_{2g-1})$ respectively. By the work of Birman-Hilden [2],

$$\mathrm{Mod}(S_{2g-1},\sigma) = \pi_0(\mathrm{Diff}^+(S_{2g-1},\sigma)),$$

which gives an exact sequence,

$$1 \to \langle\sigma\rangle \to \mathrm{Mod}(S_{2g-1},\sigma) \to \mathrm{Mod}(S_g,[\beta]) \to 1.$$

Remark 2.1. Note that $\mathrm{Mod}(S_g)$ acts transitively on $H_1(S_g,\mathbb{Z}/2\mathbb{Z})$, and so any choice of $[\beta] \in H_1(S_g,\mathbb{Z}/2\mathbb{Z})$ gives conjugate subgroups $\mathrm{Mod}(S_g,[\beta])$ within $\mathrm{Mod}(S_g)$. The same remark applies to $\mathrm{Mod}(S_{2g-1},\sigma)$ in $\mathrm{Mod}(S_{2g-1})$, for different choices of σ.

Prym representation. For any $f \in \mathrm{Mod}(S_{2g-1},\sigma)$, denote by f_* its induced action on $H_1(S_g,\mathbb{Z})$. As $f\sigma = \sigma f$, f_* preserves the eigenspaces of σ_*. In particular, f_* preserves $H_1(S_{2g-1},\mathbb{Z})^-$, which consists of σ-anti-invariant elements.

Let $\hat{i}_- := \frac{1}{2}\hat{i}$, for $\hat{i}$ the restriction of the intersection pairing on $H_1(S_{2g-1},\mathbb{Z})$ to $H_1(S_{2g-1},\mathbb{Z})^-$. Then f_* will further preserve $\hat{i}_-$; thus by choosing a symplectic basis we obtain a representation

$$\mathrm{Prym}_* : \mathrm{Mod}(S_{2g-1},\sigma) \to \mathrm{Sp}(2(g-1),\mathbb{Z}),$$

called the *Prym representation* of $\mathrm{Mod}(S_{2g-1},\sigma)$.

Let $f \in \mathrm{Mod}(S_g,[\beta])$, then there is a lift $\tilde{f} \in \mathrm{Mod}(S_{2g-1},\sigma)$, well-defined up to composition with σ. As σ_* acts as -1 on $H_1(S_g,\mathbb{Z})^-$, the Prym representation induces a *projective Prym representation*,

$$\widehat{\mathrm{Prym}}_* : \mathrm{Mod}(S_g,[\beta]) \to \mathrm{PSp}(2(g-1),\mathbb{Z}).$$

In this section we build on the results of Franks-Handel and Korkmaz[8, 10], to classify low-dimensional linear and symplectic representations of $\mathrm{Mod}(S_g,[\beta])$ and $\mathrm{Mod}(S_{2g-1},\sigma)$.

Remark 2.2. The existence of $\chi : \mathrm{Mod}(S_{2g-1},\sigma) \to \mathbb{C}^*$ in Theorem 3 is possible due to the fact that

$$\mathrm{Mod}(S_{2g-1},\sigma)^{\mathrm{Ab}} \cong \mathbb{Z}/4\mathbb{Z}.$$

Similarly,

$$\mathrm{Mod}(S_g,[\beta])^{\mathrm{Ab}} \cong \mathbb{Z}/d\mathbb{Z}$$

where $d = 2$ for g even and 4 otherwise (see Sato [14], or the appendix for an alternate proof of the even case).

A similar rigidity result as of Theorem 3 holds for $\mathrm{Mod}(S_g,[\beta])$,

Theorem 5. *Let $g \geq 4$ and $m \leq 2(g-1)$. Let $\phi : \mathrm{Mod}(S_g,[\beta]) \to \mathrm{GL}(m,\mathbb{C})$. Then the following holds,*

1. *If $m < 2(g-1)$ or $m = 2(g-1)$ and g odd, or $m = 2(g-1)$, and g even and $\mathrm{Im}(\phi) \subset \mathrm{SL}(m,\mathbb{C})$. Then, $\mathrm{Im}(\phi)$ is abelian, so it is a quotient of $\mathbb{Z}/4\mathbb{Z}$.*
2. *Otherwise, ϕ is induced from a representation $\tilde{\phi} : \mathrm{Mod}(S_{2g-1},\sigma) \to \mathrm{GL}(m,\mathbb{C})$ such that $\tilde{\phi}(\sigma) = 1$. In particular, $\phi(T_a^2) = \pm i\,\mathrm{Id}$, for $\hat{i}_2([a],[\beta]) = 1$.*

In particular, this shows that $\widehat{\mathrm{Prym}}_*$ does *not* lift to a linear representation. In fact, let

$$1 \to \mathbb{Z}/2\mathbb{Z} \to H \to \mathrm{Mod}(S_g, [\beta]) \to 1$$

be the central extension determined by $\widehat{\mathrm{Prym}}_* : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{PSp}(2(g-1), \mathbb{Z})$. Then

$$\mathrm{Mod}(S_{2g-1}, \sigma) \cong H,$$

where the isomorphism is given by $\tilde{f} \to (f, \tilde{f}_*)$. Thus,

Corollary 6. *The sequence,*

$$1 \to \langle\sigma\rangle \to \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{Mod}(S_g, [\beta]) \to 1,$$

does not split.

Proof outline for Theorems 3 and 5. Here we briefly sketch the main ideas used in the proofs of Theorems 3 and 5, the details will be given in the subsequent sections. First observe that any $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$ can be represented by a simple closed curve b. Then, there exists a subsurface (with boundary) R of genus $g-1$ so that

$$\mathrm{Mod}(R) \subset \mathrm{Mod}(S_g, [\beta]).$$

Results of Franks-Handel and Korkmaz applied to $\mathrm{Mod}(R)$, then give constraints on the restriction of ϕ to $\mathrm{Mod}(R)$.

Moreover, as any element in $\mathrm{Mod}(R)$ fixes a point of S_g, $\mathrm{Mod}(R)$ lifts to $\widetilde{\mathrm{Mod}(R)} \subset \mathrm{Mod}(S_{2g-1}, \sigma)$ and one can check that $\mathrm{Prym}\,|_{\widetilde{\mathrm{Mod}(R)}}$ is precisely the symplectic representation of $\mathrm{Mod}(R)$.

In order to extend our knowledge of ϕ to the whole of $\mathrm{Mod}(S_g, [\beta])$, we find good generating sets for $\mathrm{Mod}(S_g, [\beta])$. This is accomplished in two ways:

1. A key property of $\mathrm{Mod}(S_g)$ is the fact that all Dehn twists T_a are conjugate to each other. This is no longer true in $\mathrm{Mod}(S_g, [\beta])$ and the results in Section 2.1 give a classification of (powers of) Dehn twists in $\mathrm{Mod}(S_g, [\beta])$ up to conjugation. As a corollary, there exists a normal generating set for $\mathrm{Mod}(S_g, [\beta])$ composed of only three types of Dehn Twists.

2. Section 2.2 describes properties of the action of $\mathrm{Mod}(S_g, [\beta])$ on a modified complex of curves $\mathcal{N}(S_g)$. $\mathcal{N}(S_g)$ is connected and $\mathrm{Mod}(S_g, [\beta])$ acts transitively on the edges and vertices of $\mathcal{N}(S_g)$. Thus, via a geometric group theory argument, there exists an additional generating set for $\mathrm{Mod}(S_g, [\beta])$.

By using these two distinct generating sets, we are then able to constrain all low-dimensional representations of $\mathrm{Mod}(S_g, [\beta])$ (except for the last item of Theorem 5). In Section 2.4, we lift the results from Sections 2.1 and 2.2 to $\mathrm{Mod}(S_{2g-1}, \sigma)$ and are able to conclude all but the second item of Theorem 3. The final ingredient in the proof is an explicit (finite) generating set for $\mathrm{Mod}(S_g, [\beta])$, found by [4], on which one can check that ϕ has the desired form.

2.1 Conjugation in $\mathrm{Mod}(S_g, [\beta])$

Let $\hat{i}_2$ be the algebraic intersection pairing mod 2 on $H_1(S_g, \mathbb{Z}/2\mathbb{Z})$. In what follows all homology classes are mod 2. Let $\mathcal{S}(S_g)$ denote the set of isotopy classes of simple closed curves (SCC) in S_g. The action of $\mathrm{Mod}(S_g, [\beta])$ splits $\mathcal{S}(S_g)$ into three components,

Lemma 2.1. *Let a_1, a_2 be a pair of isotopy classes of nonseparating simple closed curves in S_g. The following are necessary and sufficient conditions for there to be an $f \in \mathrm{Mod}(S_g, [\beta])$ such that $f(a_1) = a_2$.*

1. *$\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 1$.*

2. *$\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 0$, and either both $[a_i] \neq [\beta]$ or both $[a_i] = [\beta]$.*

Proof. Let α be a nonseparating simple closed curve in S_g. Observe that if $\hat{i}_2([\alpha], [\beta]) = c$, there exists[2] a simple closed curve b representing $[\beta]$ and intersecting α transversely c times. Let α_1 and α_2 be two simple closed curves in S_g with

$$\hat{i}_2([\alpha], [\beta]) = \hat{i}_2([\alpha_2], [\beta]) = 1.$$

By the previous observation, there exist two 2-chains (α_i, b_i) with $[b_i] = [\beta]$. Thus, by the change of coordinates principle [5, Ch 1, sec 3], there is a $\phi \in \mathrm{Homeo}^+(S_g)$ so that $\phi(\alpha_1) = \alpha_2$ and $\phi(b_1) = b_2$. In particular, $\phi_*([\beta]) = [\beta]$ and so $[\phi] \in \mathrm{Mod}(S_g, [\beta])$ and the first claim follows.

Now suppose that $\hat{i}_2([\alpha], [\beta]) = 0$. If $[\alpha] = [\beta]$, the statement follows since $\mathrm{Mod}(S_g)$ acts transitively on $\mathcal{S}(S_g)$ and any f with $f(\alpha) = \beta$ is in $\mathrm{Mod}(S_g, [\beta])$. Suppose that $[\alpha] \neq [\beta]$. Let b be a simple closed curve representing $[\beta]$ and *not intersecting* α. In particular, α is *nonseparating* in $S_g - b$. Let α_1, α_2 be two simple closed curves such that

$$\hat{i}_2([\alpha], [\beta]) = \hat{i}_2([\alpha_2], [\beta]) = 0.$$

Then, there are two b_i representing $[\beta]$ such that $\alpha_i \cap b_i = 0$. Let $\phi \in \mathrm{Homeo}^+(S_g)$ with $\phi(b_1) = b_2$. Then $\phi(\alpha_1)$ is nonseparating in the cut-surface S_{b_2} obtained by cutting along b_2. Applying the change of coordinates again, there is a $\psi \in \mathrm{Homeo}^+(S_{b_2}, b_2)$ such that $\psi(\phi(\alpha_1)) = \alpha_2$. Composing ϕ with the map $\overline{\psi} \in \mathrm{Homeo}^+(S_g)$, induced by ψ, the claim follows. □

Corollary 7 (Conjugation in $\mathrm{Mod}(S_g, [\beta])$). *Let a_1, a_2 be a pair of isotopy classes of nonseparating simple closed curves in S_g.*

1. *If $\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 1$ then $T^2_{a_1}$ and $T^2_{a_2}$ are conjugate in $\mathrm{Mod}(S_g, [\beta])$.*

2. *If $\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 0$ and either for each i $[a_i] \neq [\beta]$ or for each i $[a_i] = [\beta]$, then T_{a_1} and T_{a_2} are conjugate in $\mathrm{Mod}(S_g, [\beta])$.*

The importance of Corollary 7 lies on the following.

Lemma 2.2 (Generating set-Twists). *Let $g \geq 2$, and $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$. $\mathrm{Mod}(S_g, [\beta])$ is generated by*

$$\{T_c^{\xi(c)} : c \text{ nonseparating SCC in } S_g \text{ and } \xi(c) \in \{0,1\}, \text{ with } \xi(c) = \hat{i}_2([c], [\beta]) + 1 \mod 2\}.$$

Proof. Let $\Lambda_g[\beta]$ be the stabilizer of $[\beta]$ in $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$, and consider the following exact sequence, given by reducing the symplectic representation mod 2.

$$1 \longrightarrow \mathrm{Mod}(S_g)[2] \longrightarrow \mathrm{Mod}(S_g, [\beta]) \xrightarrow{\Psi_2} \Lambda_g[\beta] \longrightarrow 1,$$

$\Lambda_g[\beta]$ is generated by transvections of the form $\psi_2(T_{c_i})$ for $\hat{i}_2([c_i], [\beta]) = 0$ (cf.[11, Lemma 3.4]). Similarly $\mathrm{Mod}(S_g)[2]$ is generated by squares of Dehn twists [9, Thm 1], thus the claim follows. □

[2] Extend α to a geometric simplectic basis. Locally, there are only 3 choices for a representative of $[\beta]$ and they can be glued together as needed.

2.2 Complex of curves

The generating set given by Lemma 2.2 is enough for providing bounds for the abelianization of $\mathrm{Mod}(S_g, [\beta])$ (see appendix). Yet, in order to stablish Theorem 5, we need to make use of another generating set. For this purpose, we examine the action of $\mathrm{Mod}(S_g, [\beta])$ on $\mathcal{S}(S_g)$.

As above, fix $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$. Let $\mathcal{S}_1(S_g)$ be the set of isotopy classes a of simple closed curves on S_g such that $\hat{i}_2([a], [\beta]) = 1$. For two isotopy classes a, b of simple closed curves, let $i(a, b)$ denote their geometric intersection number.

Definition 2.1 (Complex of curves). Let $\mathcal{N}_1(S_g)$ be the 1-complex with vertex set $\mathcal{S}_1(S_g)$. An edge (a, c) between $a, c \in S_1(S_g)$ exists iff $i(a, c) = 1$ and $[a] + [c] \neq [\beta]$.

The most important property of $\mathcal{N}_1(S_g)$ for our purposes is the following.

Lemma 2.3. *For $g \geq 3$, $\mathcal{N}_1(S_g)$ is connected.*

The proof of Lemma 2.3 follows the same idea as when dealing with the standard complex of curves [5, Chapter 4]. We first define two associated 1-complexes, the second of which contains $\mathcal{N}_1(S_g)$. We prove connectivity for each of them and then refine the paths to be in $\mathcal{N}_1(S_g)$.

Define $\mathcal{C}_1(S_g)$ to be the 1-complex with the same vertex set as $\mathcal{N}_1(S_g)$ and edges between vertices a, c if and only if $i(a, c) = 0$.

Lemma 2.4. *$\mathcal{C}_1(S_g)$ is connected for $g \geq 2$.*

Proof. Let $a, c \in \mathcal{S}_1(S_g)$. We proceed by induction on $i(a, c)$, the case $i(a, c) = 0$ being clear. For $i(a, c) = 1$. Let α and γ be representatives of a, c in minimal position. It follows that α and γ are part of a geometric symplectic basis ν for $H_1(S_g, \mathbb{Z})$. Thus, there exist a multi-curve representative of $[\beta]$ intersecting α and γ only once. If $[a] + [c] = [\beta]$, then there is a a curve δ with isotopy class d, with the following properties:

1. $\alpha \cap \delta = \emptyset$
2. $[d] + [c] \neq [\beta]$.
3. $i(c, d) = 1$.

Indeed, δ can be found by applying the change of coordinates principle. Hence it is enough to assume that $[a] + [c] \neq [\beta]$. In this case there is a component of $[\beta]$ intersecting one of the other curves in the basis ν, say γ', once. The isotopy class of γ' provides the path between a and c in $\mathcal{C}_1(S_g)$.

Now assume $i(a, c) \geq 2$, and let α, γ be as above. As before, there is a representative β of $[\beta]$ intersecting γ only once and intersecting α transversely. Take two consecutive intersection points of γ and α. There are two cases, depending on the orientation at the intersections:

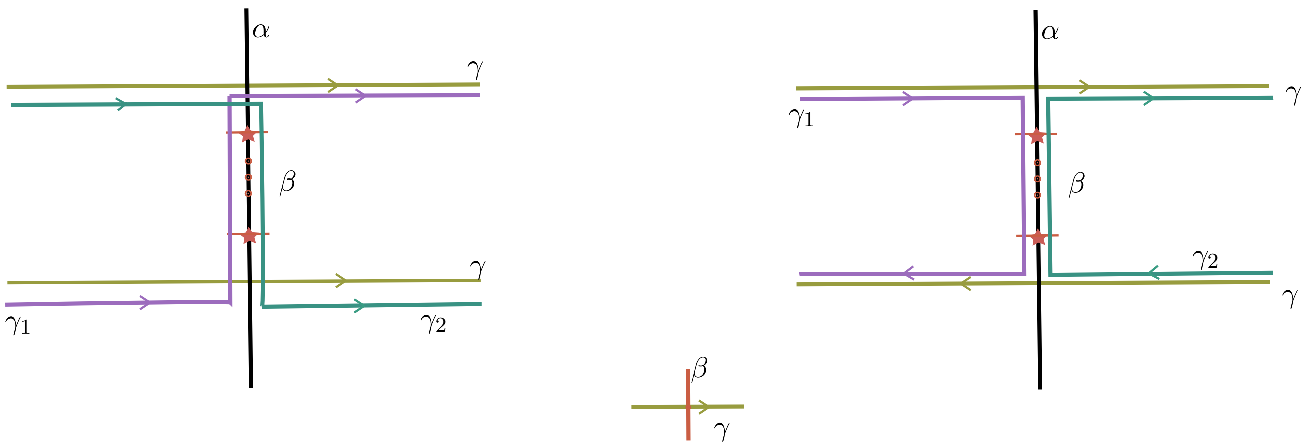

Figure 1: Path surgery.

In either case, let γ_1 and γ_2 be SCC constructed by the surgery described Figure 1. As these curves travel parallel to γ and the union gives all of γ outside the neighboorhood of α depicted above, only one of the curves crosses β along γ, say it is γ_1. Furthermore, γ_1 and γ_2 have a segment parallel to α, and this segment will meet β in either an even or odd number of points. Depending on the parity, either γ_1 (even case) or γ_2 (odd case) will meet β an odd number of times, and intersect both α and γ in fewer than $i(a, c)$ points. By induction, there is a path between a and c and the claim follows. □

Next, define $\mathcal{NC}_1(S_g)$, to be the 1-complex with vertex set $\mathcal{S}_1(S_g)$ and where two classes a, c in $\mathcal{S}_1(S_g)$ are connected by an edge if and only if $i(a, c) = 1$.

Lemma 2.5. *For $g \geq 2$, $\mathcal{NC}_1(S_g)$ is connected.*

Proof. Let $a, c \in \mathcal{S}_1(S_g)$ with $i(a, c) = 0$. By Lemma 2.4, it is enough to show that there is a class $d \in \mathcal{S}_1(S_g)$ such that $i(a, d) = i(d, c) = 1$. There exist representatives α and γ of a and c, with $\alpha \cap \gamma = \emptyset$. To find such a curve d, there are two cases to consider. If $\alpha \cup \gamma$ is non-separating, by the change of coordinates, there is a curve δ intersecting both α and γ once, and intersecting a (multicurve) representative of $[\beta]$ an odd number of times. Indeed, just note that α and γ can be extended to a geometric symplectic basis ν for S_g. Let α' and γ' be the curves intersecting α and γ once respectively. The multicurve representative of $[\beta]$ is given by a union of g curves β_i around each torus neighborhood of a pair $\{\alpha_i, \alpha'_i\}$ of ν with $i(\alpha_i, \alpha'_i) = 1$. Call each such curve β_i a *local representative* for $[\beta]$. Thus local representatives of $[\beta]$ around $\{\alpha, \alpha'\}$ and $\{\gamma, \gamma'\}$ are given by $T^k_\alpha(\alpha')$ and $T^j_\gamma(\gamma')$, where $k, j \in \{0, 1\}$ depend on $[\beta]$ intersecting α' or γ'. Define δ by connecting $T^{k'}_\alpha(\alpha')$ and $T^j_\gamma(\gamma')$, where $k' \in \{0, 1\}$ satisfy $k' = k + 1 \mod 2$.

If $\alpha \cup \gamma$ is separating, then $\{a, c\}$ is a bounding pair. Applying the change of coordinates principle, there is a d with $i(a, d) = 1 = i(a, c)$ and $d \in \mathcal{S}_1(S_g)$. □

Proof of Lemma 2.3. The goal is to modify the path given by Lemma 2.5 to conclude the proof. It is enough to show that if $a, c \in \mathcal{S}_1(S_g)$, with $i(a, c) = 1$, then there are $b_1, b_2 \in \mathcal{S}_1(S_g)$ so that $i(a, b_1) = i(b_1, b_2) = i(c, b_2) = 1$ and whose pair-wise sum in $H_1(S_g, \mathbb{Z}/2\mathbb{Z})$ is not $[\beta]$.
Assume then that $[a] + [c] = [\beta]$. Figure 2 shows the curves b_1, b_2.

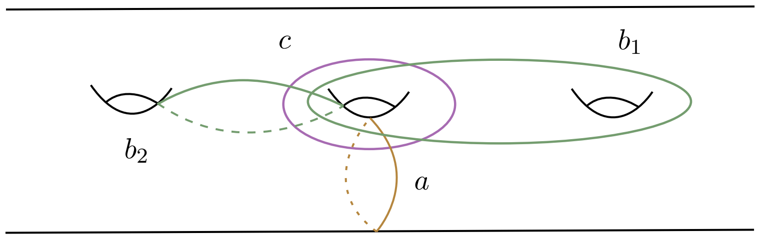

Figure 2: Refining the path in $\mathcal{NC}_1(S_g)$ to lie on $\mathcal{N}_1(S_g)$.

□

Our next result characterizes the action of $\mathrm{Mod}(S_g, [\beta])$ on $\mathcal{N}_1(S_g)$, and thus gives us a new generating set for $\mathrm{Mod}(S_g, [\beta])$.

Lemma 2.6 (Generating set-Stabilizer). *Let $g \geq 3$ and $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$. $\mathrm{Mod}(S_g, [\beta])$ acts transitively on $\mathcal{S}_1(S_g)$ and on pairs of edges (a_i, c_i) of $\mathcal{N}_1(S_g)$. In particular, for any $a \in \mathcal{S}_1(S_g)$, $\mathrm{Mod}(S_g, [\beta])$ is generated by the stabilizer of a in $\mathrm{Mod}(S_g, [\beta])$ and any $h \in \mathrm{Mod}(S_g, [\beta])$, so that $(a, h^{-1}(a)) \in \mathcal{N}_1(S_g)$.*

Proof. Let α_i, γ_i be representatives for a_i, c_i in minimal position, and let δ_i be the boundary curve of the closed torus neighborhood T_i of $\alpha_i \cup \gamma_i$. Let P_i be the complementary subsurface bounded by δ_i. By assumption, there exist multi-curve representatives $\{\beta_1^i, \beta_2^i\}$ of $[\beta]$, supported to both sides of δ_i, furthermore β_1^i intersects both α_i and γ_i only once. Let $f \in \mathrm{Mod}(S_g)$ with $f(\delta_1) = \delta_2$, and inducing homeomorphisms $f_T : T_1 \to T_2$ and $f_P : P_1 \to P_2$. As the symplectic representation mod 2 is surjective, there exists $g_P \in \mathrm{Mod}(P_2)$ so that $(g_P f_P)[\beta_2^1] = [\beta_2^2]$. On the other hand, note that f_T maps (α_1, γ_1) to a 2-chain in T_2. Thus, as $\mathrm{Mod}(T_2)$ acts transitively on 2-chains, there is $g_T \in \mathrm{Mod}(T_2)$ such that $g_T f_T(\alpha_1) = \alpha_2$ and $g_T f_T(\gamma_1) = \gamma_2$. It follows that $g_T f_T$ maps β_1^i to a curve intersecting α_2 and γ_2 only once each, and so $g_T f_T[\beta_1^1] = [\beta_1^2]$. The first claim follows by composing f with the extensions of g_T and g_P.

Let $a \in \mathcal{S}_1(S_g)$ and $h \in \mathrm{Mod}(S_g, [\beta])$ so that a and $h^{-1}(a)$ are connected by an edge in $\mathcal{N}_1(S_g)$. Then, the hypothesis of Lemma 4.10 of [5] are satisfied and the second claim follows. □

2.3 Low-dimensional representations of $\mathrm{Mod}(S_g, [\beta])$

The interplay between the two generating sets of $\mathrm{Mod}(S_g, [\beta])$ found in Sections 2.1 and 2.2 allows us to conclude all but the last item of Theorem 5.

Proof of (1)-Theorem 5. Represent $[\beta]$ by a simple closed curve b, and let a be a simple closed curve intersecting b transversely at one point. Let R be the complement of an open annular neighborhood of b, then $R \cong S_{g-1}^2$, $\mathrm{Mod}(R) \to \mathrm{Mod}(S_g, [\beta])$ and ϕ induces a representation $\phi_R : \mathrm{Mod}(R) \to \mathrm{GL}(m, \mathbb{C})$.

We claim that ϕ_R is trivial. For $m < 2(g-1)$ this follows from the results of Franks-Handel[8], as the genus of R is at least 3. Similarly for $m = 2(g-1)$, by Korkmaz[10], ϕ_R is either trivial or conjugate to the standard symplectic representation $\psi : \mathrm{Mod}(R) \to \mathrm{Sp}(2(g-1), \mathbb{Z})$. Note that in either case, $\phi(T_b) = 1$ as b is separating in R. Let d be the boundary of a regular neighborhood of $a \cup b$. Via the 2-chain relation (see [5, Prop 4.12]),

$$\phi(T_a^2 T_b)^4 = \phi(T_d) = 1,$$

as $d \in R$ is separating. Thus, regardless of ϕ_R, $\phi(T_a^2)$ is of order at most 4 and by conjugation the same applies to any $\phi(T_{a'}^2)$ with $\hat{i}_2([a'],[\beta]) = 1$.

Now suppose that ϕ_R is not trivial, then after conjugating ϕ we can assume $\phi_R = \psi$. Consider two k_i-chains to each side of b with k_i odd, as in Figure 3.

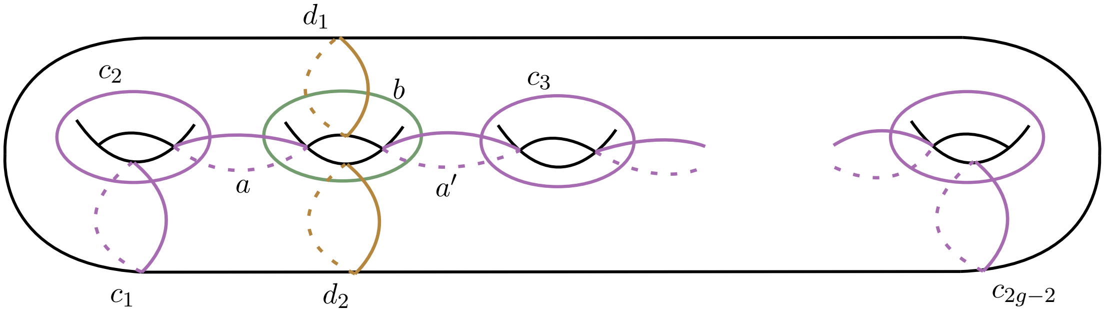

Figure 3: Complementary k-chains around b.

Then, the k-chain relations [5, Prop 4.12] imply that:

$$(T_a^2 T_{c_1} T_{c_2})^3 = (T_{a'}^2 T_{c_3} \dots T_{c_{2g-2}})^{2g-3}$$

Let $\tilde{R} \subset R$, be the complement of a torus neighbohood of $d_1 \cup b$, then

$$\phi(\mathrm{Mod}(\tilde{R})) = \mathrm{Sp}(2g-2, \mathbb{Z}).$$

As $T_{d_1}^2$ commutes with any $f \in \mathrm{Mod}(\tilde{R})$, we find that $\phi(T_{d_1}^2) = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}^*$ and $\lambda^4 = 1$. By conjugation, $\phi(T_a^2) = \phi(T_{d_1}^2)$ for any a with $\hat{i}_2([a],[\beta]) = 1$. Furthermore, by assumption $\lambda^{2g-2} = 1$ for any g (this is were we need to add the extra condition in the g even case).

The k-chain relations, under ϕ, induce the relation,

$$(\psi(T_{c_1})\psi(T_{c_2}))^3 = \lambda^{2g-2}(\psi(T_{c_3}) \dots \psi(T_{c_{2g-2}}))^{2g-3}$$

A direct computation shows that $(\psi(T_{c_1})\psi(T_{c_2}))^3$ acts as $-\mathrm{Id}$ on $\mathrm{span}\{[c_1],[c_2]\}$, while any T_{c_i} for $i > 2$ acts trivially, hence we reach a contradiction.

Consequently, ϕ_R is trivial and so $\phi(T_c) = 1$ for any c with $\hat{i}_2([c],[\beta]) = 0$. Let c be a nonseparating SCC in R meeting a transversely at one point. Let $v = [T_c(a)]$, then $\hat{i}_2(v,[\beta]) = 1$ and $v + [a] \neq [\beta]$. By Lemma 2.6, $\mathrm{Mod}(S_g,[\beta])$ is generated by $T_c^{-1} \in \mathrm{Mod}(R)$ and the stabilizer of a. Hence, for any element $f \in \mathrm{Mod}(S_g,[\beta]))$, $\phi(f)$ commutes with $\phi(T_a^2) = L_a$. For any other curve a' with $\hat{i}_2([a'],[\beta]) = 1$, $T_{a'}^2$ is conjugate to T_a^2 in $\mathrm{Mod}(S_g,[\beta])$. Thus, $\phi(T_{a'}^2) = L_a$ for all such a'. By Lemma 2.2, $\phi(\mathrm{Mod}(S_g,[\beta])) = \langle L_a \rangle$ and the theorem follows. □

2.4 Lifting relations to $\mathrm{Mod}(S_{2g-1},\sigma)$

Let $\rho : S_{2g-1} \to S_g$ be the double cover with deck transform σ and induced by intersection with $[\beta] \in H_1(S_g,\mathbb{Z}/2\mathbb{Z})^*$. By choosing lifts of elements of $\mathrm{Mod}(S_g,[\beta])$ to $\mathrm{Mod}(S_{2g-1},\sigma)$, we can translate the results of Sections 2.1 and 2.2 to $\mathrm{Mod}(S_{2g-1},\sigma)$.

Dehn twists have distinguished lifts: let a be an isotopy class of simple closed curves in S_g. If $\hat{i}_2([a],[\beta]) = 0$, then a has two disjoint and nonisotopic lifts $\tilde{a}$ and $\sigma\tilde{a}$ to S_{2g-1}. A lift of T_a is given by the multi-twist $T_{\tilde{a}}T_{\sigma\tilde{a}}$. Similarly, if $\hat{i}_2([a],[\beta]) = 1$ let $\tilde{a}$ be the union (in any order) of the two simple paths lifting a to S_{2g-1}. Then, a lift of T_a^2 is given by $T_{\tilde{a}}$. The way we join both lifts of a does not affect the lift as both ways give isotopic loops. Furthermore, as σ permute the lifts of a,

$$\sigma(T_{\tilde{a}}) = T_{\tilde{a}}.$$

With this notation, we obtain the following.

Corollary 8 (**Conjugation in** $\mathrm{Mod}(S_{2g-1}, \sigma)$)**.** *Let a_1, a_2 be a pair of isotopy classes of nonseparating simple closed curves in S_g.*

1. *If $\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 1$, then $T_{\tilde{a}_1}$ and $T_{\tilde{a}_2}$ are conjugate in $\mathrm{Mod}(S_{2g-1}, \sigma)$.*
2. *If $\hat{i}_2([a_1], [\beta]) = \hat{i}_2([a_2], [\beta]) = 0$, and either for each i $[a_i] \neq [\beta]$ or for each i $[a_i] = [\beta]$, then $T_{\tilde{a}_1} T_{\sigma(\tilde{a}_1)}$ and $T_{\tilde{a}_2} T_{\sigma(\tilde{a}_2)}$ are conjugate in $\mathrm{Mod}(S_{2g-1}, \sigma)$.*

Proof. Lift the element $f \in \mathrm{Mod}(S_g, [\beta])$ such that $f(a_1) = a_2$. □

Similarly, Lemmas 2.2 and 2.6 imply the following results.

Corollary 9 (**Generating set** $\mathrm{Mod}(S_{2g-1}, \sigma)$**-twists**)**.** *Let $g \geq 2$, and σ the deck transform of the double cover $p : S_{2g-1} \to S_g$ associated to $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$. $\mathrm{Mod}(S_{2g-1}, \sigma)$ is generated by*

$$\{\sigma\} \cup \{T_{\tilde{c}}(T_{\sigma(\tilde{c})})^{\xi(c)} : c \text{ nonseparating SCC in } S_g \text{ and } \xi(c) \in \{0,1\}, \text{ with } \xi(c) = \hat{i}_2([c], [\beta]) + 1 \mod 2\}.$$

Corollary 10 (**Generating set** $\mathrm{Mod}(S_{2g-1}, \sigma)$**-stabilizer**)**.** *Let $a \in \mathcal{S}_1(S_g)$. $\mathrm{Mod}(S_{2g-1}, \sigma)$ is generated by the following elements: σ, f such that $f T_{\tilde{a}} f^{-1} = \sigma^i T_{\tilde{a}}$, and h such that $(\overline{h}^{-1}(a), a) \in \mathcal{N}_1(S_g)$. Where $\overline{h} \in \mathrm{Mod}(S_g, [\beta])$ is the projection of $h \in \mathrm{Mod}(S_{2g-1}, \sigma)$.*

Remark 2.3 (**Relations in** $\mathrm{Mod}(S_{2g-1}, \sigma)$)**.** Note that for any proper subsurface $S \subset S_g$, there is a lift $\mathrm{Mod}(S) \cap \mathrm{Mod}(S_g, [\beta]) \to \mathrm{Mod}(S_{2g-1}, \sigma)$. Indeed, this is because all $f \in \mathrm{Mod}(S)$ fix a point p in $S_g \setminus S$, which implies the existence of a well-defined choice of lift to $\mathrm{Mod}(S_{2g-1}, \sigma)$ by requiring the map to fix a lift of p.

On the other hand, it is not possible to lift all of $\mathrm{Mod}(S_g, [\beta])$. A way to see this is to note that Prym_* surjects onto $\mathrm{Sp}(2(g-1), \mathbb{Z})$ and thus a lift would give an infinite image representation of $\mathrm{Mod}(S_g, [\beta])$ contrary to theorem 5.

In fact, there is an explicit relation that cannot hold in $\mathrm{Mod}(S_{2g-1}, \sigma)$. Let $\tilde{R}$ be the subsurface defined in the proof of theorem 5. Prym_* acts as the symplectic representation on the lift of $\mathrm{Mod}(\tilde{R})$, while $\mathrm{Prym}_*(T_{\tilde{a}}) = 1$ for any a with $\hat{i}_2([a], [\beta]) = 1$. Hence, the k-chain relations used in the proof of Theorem 5 *cannot hold* in $\mathrm{Mod}(S_{2g-1}, \sigma)$ and so

$$(T_{\tilde{a}} T_{\tilde{c}_1} T_{\sigma(\tilde{c}_1)} T_{\tilde{c}_2} T_{\sigma(\tilde{c}_2)})^3 = \sigma (T_{\tilde{a}'} T_{\tilde{c}_3} T_{\sigma(\tilde{c}_3)} \ldots T_{\tilde{c}_{2g-2}} T_{\sigma(\tilde{c}_{2g-2})})^{2g-3}$$

2.5 Low dimensional representations of $\mathrm{Mod}(S_{2g-1}, \sigma)$

The relations in $\mathrm{Mod}(S_{2g-1}, \sigma)$ described on Section 2.4 imply the first item in Theorem 3.xs

Proof of theorem 3-(1). Just apply the same argument as in the proof of theorem 5. Note that, by the lifted k-chain relation, $\phi(\sigma) \in \langle \phi(T_{\tilde{a}}) \rangle$. □

To tackle the $m = 2(g-1)$ case, we use the results of Dey et.al. [4, Theorem 1] to get an explicit finite generating set for $\mathrm{Mod}(S_{2g-1}, \sigma)$.

Corollary 11 (Finite generating set). *Let c_i, a_i, b_i be the curves in the top of Figure 4.* $\mathrm{Mod}(S_{2g-1}, \sigma)$ *is generated by σ and chosen lifts of*

$$S \cup \{F_2, \ldots, F_{g-1}\} \cup \{T_{a_2}, T_{b_2}, \ldots, T_{a_g}, T_{b_g}, T_{c_1}, \ldots, T_{c_{g-1}}\}$$

Where F_i are the bounding pairs given at the bottom of Figure 4, and S is a generating set for the subgroup of $\mathrm{Mod}(N(a_1 \cup b_1))$ *fixing $[\beta] = [b_1]$ mod 2.*

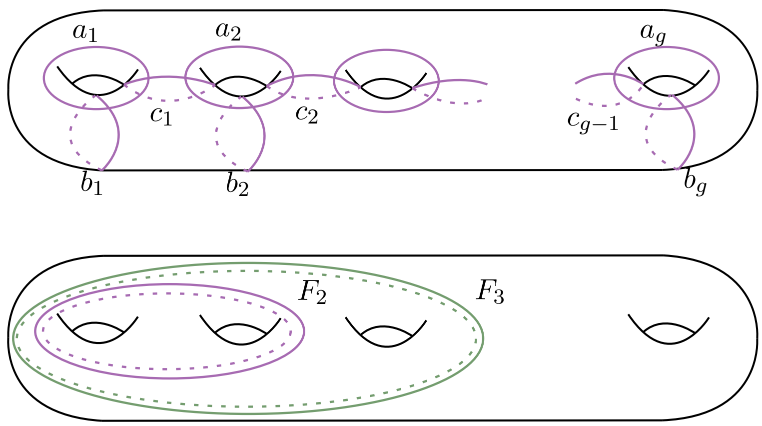

Figure 4: Top: Curve generators for $\mathrm{Mod}(S_g, [\beta])$. Bottom: Torelli generators for $\mathrm{Mod}(S_g, [\beta])$.

Proof of Theorem 3-(2). Let $\phi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{GL}(2(g-1), \mathbb{C})$ be any non-finite representation. Represent $[\beta]$ by $[b_1]$. Let R be the complement of an annular neighborhood of b_1, and lift $\mathrm{Mod}(R)$ to $\mathrm{Mod}(S_{2g-1}, \sigma)$ fixing the lift $\tilde{b}_1$ of b_1 pointwise, in particular any Dehn twist of $\mathrm{Mod}(R)$ lifts to a multi-twist. It follows that ϕ induces a representation $\phi_R : \mathrm{Mod}(R) \to \mathrm{GL}(2(g-1), \mathbb{C})$. As ϕ is non-finite, we must have ϕ_R non-trivial thus after a conjugation we can assume that $\phi_R = \psi$. Prym_* acts as ψ on this lift of $\mathrm{Mod}(R)$, thus $\phi_R = \psi_R$.

It remains to check the action of ϕ on the other generators of $\mathrm{Mod}(S_{2g-1}, \sigma)$ coming from corollary 11. Let $T = N(a_1 \cup b_1)$ be a torus neighborhood of $a_1 \cup b_1$, and $\tilde{R} \subset R$ the the complementary subsurface. There is a lift $\widetilde{\mathrm{Mod}(T)}$ of $\mathrm{Mod}(T)$ to $\mathrm{Mod}(S_{2g-1}, \sigma)$, so that the lift of each element fixes both lifts of $\tilde{R}$ to S_{2g-1} pointwise. In particular, any lift $\tilde{f}$ of $f \in \mathrm{Mod}(T)$ commutes with lifts of $\mathrm{Mod}(\tilde{R})$. As $\phi_R(\tilde{R}) = \mathrm{Sp}(2(g-1), \mathbb{Z})$, it follows that $\phi(\tilde{f})$ is a scalar, for any $f \in \mathrm{Mod}(T)$. In particular $T_{\tilde{a}_1} = \lambda \in \mathbb{C}^*$, and so the same holds for any $T_{\tilde{a}}$ with $\hat{i}_2([a], [\beta]) = 1$. Prym_* acts trivially on this lift of $\mathrm{Mod}(T)$, thus

$$\phi_T . \, \mathrm{Prym}_* |^{-1}_{\widetilde{\mathrm{Mod}(T)}} \in \mathbb{C}^*.$$

Next, note that by the chain-relations each bounding pair F_i can be expressed as,

$$F_i = (T^2_{a_1} T_{c_1} T_{a_2} \ldots T_{c_{i-1}} T_{a_i})^{2i-1} T^{-2}_{d_i}$$

Where T_{d_i} is one of the curves of the bounding pair. So for each lift $\tilde{F}_i$,

$$\phi(\tilde{F}_i) . \, \mathrm{Prym}_*(\tilde{F}_i)^{-1} \in \mathbb{C}^*.$$

Lastly, $\phi(\sigma)$ commutes with any element of $\phi(\mathrm{Mod}(S_{2g-1}, \sigma))$, thus it must be a scalar.

It follows that for any $f \in \mathrm{Mod}(S_{2g-1}, \sigma)$, $\phi(f)\,\mathrm{Prym}_*(f)^{-1} = \lambda(f) \in \mathbb{C}^*$. We claim that $f \to \lambda(f)$ is a homomorphism. Indeed,

$$\phi(fg)\,\mathrm{Prym}_*(fg)^{-1} = \phi(f)\phi(g)\,\mathrm{Prym}_*(g)^{-1}\,\mathrm{Prym}_*(f)^{-1} = \lambda(f).\lambda(g)$$

In particular $\lambda : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathbb{C}^*$ must be cyclic of order at most 4. □

Proof of Theorem 5-(2). Let $g \geq 4$ be even, Theorem 3 gives us an example of an infinite representation $\phi : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{GL}(2(g-1), \mathbb{C})$, induced by

$$\tilde{\phi} : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{GL}(2(g-1), \mathbb{Z}) \quad , \quad f \to \chi(f)\,\mathrm{Prym}_*(f),$$

with $\chi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathbb{C}^*$, satisfying $\chi(\sigma) = -1$.

The same argument as in the of the proof of Theorem 3-(2), replacing Prym_* with $\overline{\phi}$, conclude the proof of Theorem 5. □

2.6 Symplectic representations

Section 2.5 considered representations $\phi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{GL}(m, \mathbb{C})$. The resuts extend easily to cases where the image is contained in $\mathrm{Sp}(2h, \mathbb{Z})$.

Corollary 12. *Let $g \geq 4$ and $h \leq (g-1)$. Let $\phi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \mathrm{Sp}(2h, \mathbb{Z})$ be a homomorphism, then*

1. *If $h < g-1$, then $\mathrm{Im}(\phi)$ is a quotient of $\mathbb{Z}/4\mathbb{Z}$.*
2. *If $h = g-1$, then either $\mathrm{Im}(\phi)$ is a quotient of $\mathbb{Z}/4\mathbb{Z}$ or up to a conjugation in $\Delta(2(g-1), \mathbb{Z})$ is of the form:*
$$f \to \chi(f)\,\mathrm{Prym}_*(f)$$
where $\chi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \{-\mathrm{Id}, +\mathrm{Id}\}$, and $\Delta(2(g-1), \mathbb{Z})$ is the subgroup of $\mathrm{GL}(2(g-1), \mathbb{Z})$ fixing the symplectic form up to sign.

Proof. The first item follows directly from Theorem 3. For the second, assume that the image is not finite as the finite case follows directly. Then, by Theorem 3 there is a matrix $A \in \mathrm{GL}(2(g-1), \mathbb{C})$ such that $\Psi = A\phi A^{-1}$ is of the desired form. Indeed, Let R be the subsurface in the proof of Theorem 3, then note that A is chosen so that $\phi_R := \phi|_{\mathrm{Mod}(R)}$ is the standard symplectic representation. In particular $C_A : \mathrm{Sp}(2(g-1), \mathbb{Z}) \to \mathrm{Sp}(2(g-1), \mathbb{Z})$, conjugation by A, is an automorphism. Reiner [12] showed that all such automorphisms come frome conjugation in $\Delta(2(g-1), \mathbb{Z})$. With this remark in place, the proof of theorem 3 goes through without modifications. Importantly, the image of χ lies on the centralizer of $\mathrm{Im}(\phi)$, hence the result. □

Remark 2.4. Note that any element of $\Delta(2(g-1), \mathbb{Z})$ is of the form $Z^i A$ for $A \in \mathrm{Sp}(2g-1), \mathbb{Z})$, and $Z = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$.

3 Holomorphic results

Fix $g \geq 4$ and $h \leq g-1$, and consider a holomorphic map of complex orbifolds

$$F : \mathcal{R}_g \to \mathcal{A}_h.$$

The aim of this section is to complete the proof of Theorem 1.

Theorem 1 (**Rigidity of** Prym)**.** *Let $g \geq 4$ and let $h \leq g-1$. Let $F : \mathcal{R}_g \to \mathcal{A}_h$ be a nonconstant holomorphic map of complex orbifolds*[3]*. Then $h = g-1$ and $F = \mathrm{Prym}$.*

The results of Section 2 quickly reduce the statement to the case of $h = g-1$ and F homotopic to Prym. By steps 2 to 4 in Farb's proof [6], it is enough to find a curve $C \subset \mathcal{R}_g$ so that $\mathrm{Prym}(C)$ is $\mathcal{A}_g$-rigid. To avoid orbifold issues, all the arguments are done on a suitable finite cover of $\mathcal{R}_g$.

3.1 Case $h < g-1$

By Theorem 3, there is a finite cover $\tilde{\mathcal{R}}_g \to \mathcal{R}_g$ such that $\tilde{F}_* = \mathrm{Id}$. Thus, $\tilde{F} : \tilde{\mathcal{R}}_g \to \mathcal{A}_h$ lifts to a holomorphic map $G : \tilde{\mathcal{R}}_g \to \mathfrak{h}_h$. As $\tilde{\mathcal{R}}_g$ is a finite (branched) cover of $\mathcal{R}_g$ it is also a quasiprojective variety, and as $\mathfrak{h}_h$ is a bounded domain it follows that G is constant. The same argument gives a proof of Theorem 2.

3.2 Case $h = g-1$

3.2.1 The Prym map

Let X be a smooth genus g complex curve. Any nonzero $\theta \in H^1(X, \mathbb{Z}/2\mathbb{Z})$ defines an unbranched double cover

$$p : Y \to X,$$

with deck transform σ, and where Y is a curve of genus $2g-1$. The order 2 action of σ^* on $\Omega^1(Y)$ induces a splitting

$$\Omega^1(Y) = \Omega^1(Y)^+ \oplus \Omega^1(Y)^-$$

corresponding to the ± 1 eigenspaces of σ^*. Similarly, the action of σ_* on $H_1(Y, \mathbb{Z})$ has two distinct subspaces[4], $H_1(Y, \mathbb{Z})^+$ and $H_1(Y, \mathbb{Z})^-$. The *Prym variety* associated to (X, θ) is defined as

$$\mathrm{Prym}(X, \theta) := \frac{(\Omega^1(Y)^-)^\vee}{H_1(Y, \mathbb{Z})^-}.$$

$\mathrm{Prym}(X, \theta)$ is a subtorus of $\mathrm{Jac}(Y)$, and the restriction of the principal polarization from $\mathrm{Jac}(Y)$ (given by the intersection pairing on $H_1(Y, \mathbb{Z})$) to $\mathrm{Prym}(X, \theta)$ induces twice a principal polarization. In particular, $\mathrm{Prym}(X, \theta)$ is a PPAV of dimension $g-1$.

The isomorphism,

$$\mathrm{Jac}(X) \cong \frac{(\Omega^1(Y)^+)^\vee}{H_1(Y, \mathbb{Z})^+}$$

implies that,

$$\mathrm{Prym}(X, \theta) \cong \frac{\mathrm{Jac}(Y)}{\mathrm{Jac}(X)} \in \mathcal{A}_{g-1}.$$

The Prym period matrix. Consider $[(Y, \phi)] \in \mathrm{Fix}(\sigma) \subset \mathrm{Teich}(S_{2g-1})$. Let $\{a_i, b_i\}_{i=0,\ldots,2g-2}$ be a geometric symplectic basis for S_{2g-1} such that

$$\sigma(b_i) = b_{i+g-1} \quad , \ \sigma(a_i) = a_{i+g-1} \quad i = 1, \ldots, g-1$$

[3] See Definition 1.1

[4] But this is *not* a splitting of $H_1(Y, \mathbb{Z})$.

Let ω_i be a basis for $\Omega^1(Y)$ dual to $\{\phi(a_i)\}$, and let $u_i := \frac{\omega_i - \omega_{i+g-1}}{2}$ for $i = 1, \ldots, g-1$. Then, $\{u_i\}$ is a basis for $\Omega^1(Y)^{-1}$. Moreover $\{u_i\}$ is in fact dual to $\{\phi(a_i) - \phi(a_{i+g-1})\}_{i\geq 1}$. Then

$$\tau = (\int_{\phi(b_j)-\phi(b_j+g-1)} u_i) \in \mathfrak{h}_{g-1}$$

Let $\widetilde{\mathrm{Prym}} : \mathrm{Fix}(\sigma) \to \mathfrak{h}_{g-1}$ be $[(Y, \phi)] \to \tau$. Moreover, if the (normalized) period matrix of Y with respect to $\{a_i, b_i\}$ and $\{\omega_i\}$ is given by:

$$(\int_{b_j} \omega_i)_{0\leq i,j\leq 2g-1} = \begin{pmatrix} * & * & * \\ * & B & C^T \\ * & C & D \end{pmatrix}$$

Then, $\tau = B - C$ and in particular $\widetilde{\mathrm{Prym}}$ is holomorphic. Similarly, by a direct computation one can check that $\widetilde{\mathrm{Prym}}$ is Prym_*-equivariant[5] and lifts Prym

$$\begin{array}{ccc} \mathrm{Fix}(\sigma) & \xrightarrow{\widetilde{\mathrm{Prym}}} & \mathfrak{h}_{g-1} \\ \downarrow & & \downarrow \\ \mathcal{R}_g & \xrightarrow{\mathrm{Prym}} & \mathcal{A}_{g-1} \end{array}$$

3.2.2 F homotopic to Prym

Let $F : \mathcal{R}_g \to \mathcal{A}_{g-1}$ be a nonconstant holomorphic map. Then, by Corollary 12 there is an $A \in \Delta(2g, \mathbb{Z})$ and $\chi : \mathrm{Mod}(S_{2g-1}, \sigma) \to \{\pm \mathrm{Id}\}$ such that

$$AF_*A^{-1} = \chi \, \mathrm{Prym}_* \, .$$

If $A \in \mathrm{Sp}(2g, \mathbb{Z})$, it follows that there is a lift $\tilde{F} : \mathrm{Fix}(\sigma) \to \mathfrak{h}_{g-1}$ which is equivariantly homotopic to $\widetilde{\mathrm{Prym}}$. Indeed, this is because χ acts trivially on $\mathfrak{h}_{g-1}$ so both F_* and Prym_* factor trough the *same* representation $\widehat{\mathrm{Prym}}_* : \mathrm{Mod}(S_g, [\beta]) \to \mathrm{PSp}(2(g-1), \mathbb{Z})$. In fact, the homotopy can be chosen to be a straight-line homotopy as $\mathfrak{h}_{g-1}$ has a Kähler metric of nonpositive curvature under which the action of $\mathrm{Sp}(2(g-1), \mathbb{Z})$ is by isometries.

The case in which $A = ZB$ for $B \in \mathrm{Sp}(2(g-1), \mathbb{Z})$, and $Z = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$ can be ruled out as follows: Consider the map $G : \mathfrak{h}_{g-1} \to \mathfrak{h}_{g-1}$, given by $\tau \to -\overline{\tau}$, then G is Z-equivariant and anti-holomorphic. In particular there is a lift $\tilde{F}$ of F such that $F_G := G \circ \tilde{F}$ is equivariantly homotopic to $\widetilde{\mathrm{Prym}}$, hence we have a holomorphic map Prym homotopic to an antiholomorphic map F_G. As $\mathcal{R}_g$ contains a smooth closed curve X this is imposible. Indeed let ω be the Kähler form on $\mathcal{A}_{g-1}$, then restricting the maps to X we find $F_G^*(\omega) = f_1 \omega_X$ where $f_1 \leq 0$ and $\mathrm{Prym}^*(\omega) = f_2 \omega_X$ for $f_2 \geq 0$. By Stokes' theorem we then find $f_1 = f_2 = 0$. Hence Prym is constant, which is a contradiction.

3.2.3 ψ-structures

The arguments in Sections 3.1 and 3.2.2 show that the results of Section 2 imply that if $F : \mathcal{R}_g \to \mathcal{A}_{g-1}$ is nonconstant, then it is homotopic to Prym. One could carry the next steps in Farb's

[5] with respect to the action of $\mathrm{Mod}(S_{2g-1}, \sigma)$ on $\mathrm{Fix}(\sigma)$, $f \cdot [(Y, \phi)] \to [(Y, \phi \circ f^{-1})]$, we find equivariance, but with $\mathrm{Sp}(2(g-1), \mathbb{Z})$ acting by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \tau \to (\alpha\tau - \beta)(-\gamma\tau + \delta)^{-1}$

proof [6] under this setting, but the orbifold issues become cumbersome at the last step (existence of rigid curves).

To circumvent these issues we first pass to a finite cover of $\mathcal{R}_g$. Let $[\beta] \in H_1(S_g, \mathbb{Z}/2\mathbb{Z})^*$, and $\rho : S_{2g-1} \to S_g$ be the double cover induced by the map $\pi_1(S_g) \to \mathbb{Z}/2\mathbb{Z}$, given by $\gamma \to \hat{i}_2([\gamma], [\beta])$. Next, let $\hat{S}_L \to S_{2g-1}$ be the cover induced by $\pi_1(S_{2g-1}) \to H_1(S_{2g-1}, \mathbb{Z}/L\mathbb{Z})$ for $L \geq 3$. Then the composite cover is induced by the map

$$\psi : \pi_1(S_g) \to \frac{\pi_1(S_g)}{\langle \pi_1(S_{2g-1})', \pi_1(S_{2g-1})^L \rangle} =: G$$

Where $\pi_1(S_{2g-1})^L := \{\gamma^L : \gamma \in \pi_1(S_{2g-1})\}$. Let $\Gamma_g[\psi]$ be the stabilizer of ψ (as an exterior homomorphism) on $\mathrm{Mod}(S_g)$ and consider the finite cover

$$\hat{\mathcal{R}}_g[\psi] := \mathrm{Teich}(S_g)/\Gamma_g[\psi]$$

of $\hat{\mathcal{R}}_g$, given by attaching to each curve a level ψ structure [1, p.511].

Definition 3.1 (Prym Level-L structures). For any integer $L \geq 0$, we define

$$\mathrm{Mod}(S_{2g-1}, \sigma)[L] = \mathrm{Prym}_*^{-1}(\mathrm{Ker}\{\mathrm{Sp}(2(g-1), \mathbb{Z}) \to \mathrm{Sp}(2(g-1), \mathbb{Z}/L\mathbb{Z})\})$$

Remark 3.1. Unlike $\mathrm{Mod}(S_g)[L]$, the group $\mathrm{Mod}(S_{2g-1}, \sigma)[L]$ contains torsion for $L \geq 3$. This is because the kernel of Prym_* contains torsion: a lift of the hyperelliptic involution from S_g to the cover S_{2g-1} acts under Prym_* in the same way as σ.

Importantly for us $\Gamma_g[\psi]$ satisfy the following properties:

Lemma 3.1.

1. *$\Gamma_g[\psi] \subset \mathrm{Mod}(S_g)[L] \cap \pi(\mathrm{Mod}(S_{2g-1}, \sigma)[L])$.*

2. *Let b be a SCC representative of $[\beta]$, and let a be a SCC intersecting b transversely at one point. Let R be the complement of a torus neighborhood of $a \cup b$. Then $\mathrm{Mod}(R)[L] \subset \Gamma_g[\psi]$.*

Proof. Pick a basepoint $x \in S_g - R$ and $\tilde{x}$ the corresponding basepoint for $\pi_1(S_{2g-1})$.

1. We will show that in fact, for any $f \in \Gamma_g[\psi]$, there is a lift $\tilde{f} : S_{2g-1} \to S_{2g-1}$ such that $\tilde{f}$ acts trivially on $H_1(S_{2g-1}, \mathbb{Z}/L\mathbb{Z})$ and $f_* \in \mathrm{Mod}(S_g)[L]$. The latter follows easily as we have a surjection
$$G \twoheadrightarrow H_1(S_g, \mathbb{Z}/L\mathbb{Z}) = \frac{\pi_1(S_g)}{\langle \pi_1(S_g)', \pi_1(S_g)^L \rangle}$$
Such that the projection $\pi_1(S_g) \to H_1(S_g, \mathbb{Z}/L\mathbb{Z})$ factors through ψ. A similar argument shows that $f \in \mathrm{Mod}(S_g, [\beta])$.

   Finally, let $\tilde{\gamma} \in \pi_1(S_{2g-1})$ and γ its image in $\pi_1(S_g)$. Let $f \in \Gamma_g[\psi]$ and pick a representative ϕ fixing x. Then, let $\tilde{\phi}$ be the lift fixing $\tilde{x}$. By assumption there is a loop $\beta \in \pi_1(S_g, x)$, independent of γ, such that $\tilde{\phi}(\gamma).\widetilde{\beta\gamma^{-1}\beta^{-1}} \in \langle \pi_1(S_{2g-1})', \pi_1(S_{2g-1})^L \rangle$. The two possible lifts for $\beta\gamma^{-1}\beta^{-1}$ starting at $\tilde{x}$ are given by $\tilde{\beta}\sigma^i\tilde{\gamma}\tilde{\beta}^{-1}$, for $i = 0, 1$ depending if β lifts to a loop or not. Hence, either $\sigma\tilde{\phi}$ or $\tilde{\phi}$ act trivially on $H_1(S_{2g-1}, \mathbb{Z}/L\mathbb{Z})$ and the result followss.

2. Let $f \in \mathrm{Mod}(R)[L]$, then f has a representative ϕ fixing the complement of R pointwise. In particular $f_*(a) = a$ and $f_*(b) = b$. Furthermore, as R lifts to S_{2g-1}, ϕ has a lift $\tilde{\phi}$ acting trivially on $H_1(S_{2g-1}, \mathbb{Z}/L\mathbb{Z})$ so that $\phi(\gamma).\gamma^{-1} \in \langle \pi_1(S_{2g-1})', \pi_1(S_{2g-1})^L \rangle$. Thus, f_* will fix ψ over $\pi_1(S_{2g-1})$ and a, and so the result follows.

$\square$

Remark 3.2. Note that as $\sigma \notin \mathrm{Mod}(S_{2g-1},\sigma)[L]$ for $L \geq 3$, $\Gamma_g[\psi]$ has a lift $\Lambda_g[\psi] \subset \mathrm{Mod}(S_{2g-1},\sigma)$ and moreover the restriction

$$\mathrm{Prym}_* : \Lambda_g[\psi] \twoheadrightarrow \mathrm{Sp}(2(g-1),\mathbb{Z})[L].$$

Hence, denoting by $\mathcal{R}_g[\psi] = \mathrm{Fix}(\sigma)/\Lambda_g[\psi]$, it follows that $\mathcal{R}_g[\psi] \cong \hat{\mathcal{R}}_g[\psi]$, so to avoid excesive notation we will denote $\hat{\mathcal{R}}_g[\psi]$ by $\mathcal{R}_g[\psi]$. Furthermore, $\Gamma_g[\psi]$ is torsion free and so $\mathcal{R}_g[\psi]$ is a $K(\pi,1)$-manifold, and a non-galois cover of $\mathcal{M}_g$ with fundamental group $\Gamma_g[\psi]$.

It follows that any nonconstant holomorphic map $F : \mathcal{R}_g \to \mathcal{A}_{g-1}$ induces a holomorphic map $F[\psi] : \mathcal{R}_g[\psi] \to A_{g-1}[L]$, with $F[\psi]_* : \Gamma_g[\psi] \to \mathrm{Sp}(2(g-1),\mathbb{Z})[L]$ equal to $\mathrm{Prym}[\psi]_*$[6]. Consequently, $F[\psi] \sim \mathrm{Prym}[\psi]$ and Steps 2-4 in Farb's proof [6] carry over without modification (In fact, one could have also lifted the period map to $\mathcal{M}_g[L]$ in order to prove its rigidity for $g \geq 3$.). It follows that for a curve $X \subset \mathcal{R}_g[\psi]$ there exists a homotopy F_t between $F[\psi]$ and $\mathrm{Prym}[\psi]$, which is algebraic at each t.

3.2.4 Rigid curves

To conclude the proof of Theorem 1 we just need to show that $\mathcal{A}_{g-1}[L]$-rigid curves exist in our setting. More precisely, we will show that there exists a curve $i : C \hookrightarrow \mathcal{R}_g[\psi]$ so that:

$$\mathrm{Prym}[\psi] \circ i : C \to \mathcal{A}_{g-1}[L]$$

is rigid. As in Farb's case this is done by finding a family satisfying Saito's criterion[13, Thm 8.6].

Let $\overline{\mathcal{M}_g}$ denote the Deligne-Mumford compactification of $\mathcal{M}_g$. Let $\overline{\mathcal{R}_g[\psi]}$ be the compactification of $\mathcal{R}_g[\psi]$, given by the normalization of $\overline{\mathcal{M}_g}$ on the function field of $\mathcal{R}_g[\psi]$, in particular $\overline{\mathcal{R}_g[\psi]} \to \overline{M_g}$ is a finite branched cover and $\overline{\mathcal{R}_g[\psi]}$ is a projective variety. Thus, we can assume that $\mathcal{R}_g[\psi] \subset \overline{\mathcal{R}_g[\psi]} \subset \mathbb{P}^N$ for some N. As $\dim(\mathcal{R}_g[\psi]) = 3g-3$, by Bertini's theorem, the intersection of $\mathcal{R}_g[\psi]$ with $3g-4$ generic hyperplanes is a smooth curve $C \subset \mathcal{R}_g[\psi]$.

By the Lefschetz hyperplane theorem for quasi-projective varieties, the inclusion $C \hookrightarrow \mathcal{R}_g[\psi]$ induces a surjection $\pi_1(C) \twoheadrightarrow \pi_1(\mathcal{R}_g[\psi]) = \Gamma_g[\psi]$.

Let Z be the unique codimension 1-stratum of $\partial\mathcal{M}_g$ containing curves with nodes coming from pinching a unique nonseparating loop, and let $Z[\psi]$ be its preimage on $\partial\mathcal{R}_g[\psi]$.

Let R be as in item 2 of Lemma 3.1, and let $\mathcal{X} \to \Delta$ be the universal family around the nodal curve $\mathcal{X}_0$, where only a nonseparating SCC $\gamma \subset R$ is pinched. In particular, the singular curves are parametrized by $z_1 = 0$. Let

$$U = \{(z,\xi) \in \Delta \times \mathbb{C} : z_1^L = \xi\}.$$

Then $\rho : U \to \Delta$ is an L-cyclic cover, branched along $z_1 = \xi = 0$. Let U^* be the complement of $z_1 = \xi = 0$. The local monodromy for $\rho : \pi_1(\Delta^*) \to \mathrm{Mod}(S_g)$ is generated by T_γ, and $\langle T_\gamma^L \rangle = \rho^{-1}(\Gamma_g[\psi])$. It follows that the pullback family $\rho^*\mathcal{X} \to U$ gives a neighborhood (in $\overline{\mathcal{R}_g[\psi]}$) of a point $y \in Z[\psi]$, and the local monodromy around y is generated by T_γ^L. Let Z_y be the top stratum of the irreducible component of $\partial\mathcal{R}_g[\psi]$ containing y. Then $U \cap \{z_1 = \xi = 0\} \subset Z_y$, so Z_y is of codimension 1 with local monodromy conjugate in $\Gamma_g[\psi]$ to T_γ^L for $\gamma \subset R$. It follows that $\overline{C}$ will intersect Z_y, in particular C is not compact.

This is enough to conclude that $\mathrm{Prym}[\psi](C)$ is rigid: Let $\mathcal{X}[L] \to \mathcal{A}_{g-1}[L]$ be the universal family of PPAVs with level L structure. Then, let $E[L] \to C$ be the pullback of $\mathcal{X}[L]$ under

[6]Our original F_* would induce a map to $\mathrm{PSp}(2(g-1),\mathbb{Z})[L]$ which we lift to $\mathrm{Sp}(2(g-1),\mathbb{Z})[L]$.

$\mathrm{Prym} \circ i : C \to \mathcal{A}_{g-1}[L]$. Forgetting the level L structure, exists a family $\rho : E \to C$ of PPAVs over C. As $\mathcal{A}_{g-1}[L] \to \mathcal{A}_{g-1}$ is a finite branched cover, it is enough to show that E is rigid.

Since $i_* : \pi_1(C) \to \pi_1(\mathcal{R}_g[\psi])$ is surjective and $\mathrm{Prym}_*(\Gamma_g[\psi]) = \mathrm{Sp}(2(g-1), \mathbb{Z})[L]$, it follows that the monodromy representation $\rho_* : \pi_1(C) \to \mathrm{Sp}(2(g-1), \mathbb{Z})$ is irreducible.

Finally, there is a point $y' \in \overline{C} \cap Z_y$ so that the local monodromy of E around y is conjugate to $\mathrm{Prym}_*(T_\gamma^L)$ for some $T_\gamma \in \mathrm{Mod}(R)[L]$ along a nonseparating SCC γ. As T_γ maps to a transvection under Prym_* the local monodromy has infinite order and the claim follows by applying [13, Thm 8.6].

3.2.5 Finishing the proof

By the previous steps the homotopy $F_t : C \to \mathcal{A}_{g-1}[L]$ satisfies $F_t = \mathrm{Prym}[\psi]$ at each t, hence $\mathrm{Prym}[\psi]$ and $F[\psi]$ agree over C. As $\mathcal{R}_g[\psi]$ is a quasiprojective variety and $\mathfrak{h}_{g-1}$ is a bounded domain, by the criterion of Borel-Narasimhan[3, Thm 3.6], it follows that $F[\psi] = \mathrm{Prym}[\psi]$, hence also $\mathrm{Prym} = F$ and Theorem 1 is proven.

4 Appendix

Let S_g be a closed surface of genus $g \geq 1$, and $[\beta] \in H_1(S, \mathbb{Z}/2\mathbb{Z})^*$. Then, there is a (unique up to isomorphism) double cover

$$p : S_{2g-1} \to S_g,$$

with deck transform σ, and monodromy given by intersection with $[\beta]$. Define

$$\mathrm{Mod}(S_g, [\beta]) := \mathrm{Mod}(S_g, [\beta]), \text{ and } \mathrm{Mod}(S_{2g-1}, \sigma) := C_{\mathrm{Mod}(S_{2g-1})}(\sigma)$$

In this section we provide a short proof of the following

Theorem 13. *Let $g \geq 4$, then $\mathrm{Mod}(S_g, [\beta])^{\mathrm{Ab}}$ is cyclic of order at most 4 and is $\mathbb{Z}/2\mathbb{Z}$ for g even*

Proof. Let b be a nonseparating SCC representing $[\beta]$, and a be a SCC intersecting b once transversely. Let R be the complement of an open annulus neighborhood of b. Then $R \cong S_{g-1}^2$ and there is an inclusion $j : \mathrm{Mod}(R) \to \mathrm{Mod}(S_g, [\beta])$, with kernel generated by $T_{b'} T_{b''}^{-1}$ corresponding to twists along the boundary components of R. As $\mathrm{Mod}(R)^{\mathrm{Ab}} = 0$, it follows that $[T_c] = 0 \in \mathrm{Mod}(S_g, [\beta])^{\mathrm{Ab}}$ for any SCC c disjoint from b. In particular, by Lemma 2.2, $\mathrm{Mod}(S_g, [\beta])^{\mathrm{Ab}} = \langle [T_a^2] \rangle$.

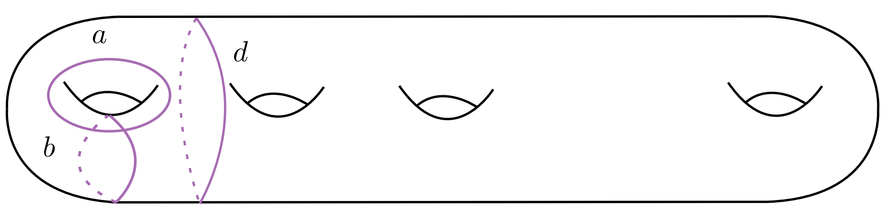

Figure 5: 2-chain relation.

By the 2-chain relation $(T_a^2 T_b)^4 = T_d$. As d is disjoint from b, $[T_a^2]^4 = 0$. Similarly, using the k-relation depicted in Figure 3

$$(T_a^2 T_{c_1} T_{c_2})^3 = (T_{a'}^2 T_{c_3} \dots T_{c_{2g-2}})^{2g-3}.$$

Thus, for g even $[T_a^2]^2 = 0$.

The nontriviality of $\mathrm{Mod}(S_g, [\beta])^{\mathrm{Ab}}$ follows from.

Lemma 4.1. *Let $p \in \mathbb{N}$ and $\Lambda_g[p] = \{A \in \mathrm{Sp}(2g;\mathbb{Z}); Ae_1 = e_1 + pa_1\}$ and denote by $\wedge$ the symplectic pairing. The map $\varphi : \Lambda_g[p] \to \mathbb{Z}_p$ defined by*

$$A \mapsto \frac{1}{p}(Ae_1 \wedge e_1) \mod p$$

is a surjective homomorphism. In particular $H_1(\Lambda_g[p];\mathbb{Z})$ is of order at least p.

Proof. Let $A, B \in \Lambda_g[p]$, then:

$$(ABe_1) = A(pb_1 + e_1) = pAb_1 + pa_1 + e_1$$

Now as A preserves the symplectic pairing $\wedge$, we have:

$$Ab_1 \wedge Ae_1 = Ab_1 \wedge (pa_1 + e_1) = b_1 \wedge e_1$$

And so we get $Ab_1 \wedge e_1 = b_1 \wedge e_1 - pAb_1 \wedge a_1$. Hence:

$$(AB)e_1 \wedge e_1 = p(a_1 \wedge e_1 + b_1 \wedge e_1 - pAb_1 \wedge a_1)$$

and so φ is a group homomorphism. To see that it is surjective just take the p-th powers of the transvection given by $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$.

□

□

The results of Section 2.4 imply the following.

Corollary 14. *Let $g \geq 4$, then $\mathrm{Mod}(S_{2g-1}, \sigma)$ is cyclic of order at most 4. Furthermore, it is generated by the class of $T_{\tilde{a}}$, where a is a nonseparating SCC with $\hat{\imath}_2([a],[\beta]) = 1$. For g even, $[\sigma] = [T_{\tilde{a}}]^2$ and $[\sigma] = 0$ for g odd.*

Remark 4.1. Sato [14, Theorem 0.2] has shown that $\mathrm{Mod}(S_g, [\beta])^{\mathrm{Ab}} = \mathbb{Z}/4\mathbb{Z}$ for g odd, and $\mathrm{Mod}(S_{2g-1}, \sigma)^{\mathrm{Ab}} = \mathbb{Z}/4\mathbb{Z}$.

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