ERRATUM TO "ISOMETRIC EMBEDDINGS OF TEICHMÜLLER SPACES ARE COVERING CONSTRUCTIONS"

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The purpose of this erratum is to correct an error in a step of the proof of the following theorem [BS24, Theorem 1.1],

Theorem 1 (**Isometric embeddings are geometric**). *Let $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g',n'}$ be Teichmüller spaces and suppose $2g+n \geq 5$. Then any isometric embedding $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ is a covering construction, up to pre- or post-composition with an orientation-reversing mapping class.*

We emphasize that the results are unchanged.

The mistake. The error appears in Section 5 of Benirschke–Serván, more precisely on [BS24, Remark 5.2]. Let $T : Q(X) \to Q(Y)$ be a linear isometric embedding and $h : Y \to X$ the map given by [GG22, Theorem 1.3]. We claimed that h is given by the diagram

$$\begin{array}{ccc} \mathbb{P}Q(Y)^* & \xrightarrow{T^*} & \mathbb{P}Q(X)^* \\ {\scriptstyle\Psi_Y}\uparrow & & \uparrow{\scriptstyle\Psi_X} \\ Y & \xrightarrow{\quad h \quad} & X. \end{array}$$

The *key issue*, also appearing in the proof of [GG22, Theorem 1.3], is that T^* is only a *rational map*. In particular, it is not defined on the linear subspace $\mathbb{P}\ker T^*$. For the purpose of [GG22, Theorem 1.3] this is not an issue: $T^* \circ \Psi_Y$ can be extended holomorphically over the finite set of points $Z_Y := \Psi_Y^{-1}(\mathbb{P}\ker T^*) \subset Y$.

Let $\mathcal{C}_{g,n}$ and $\mathcal{C}_{g',n'}$ be the universal curves over $\mathcal{T}_{g,n}$ and $\mathcal{T}_{g',n'}$ respectively. Applying [BS24, Theorem 5.1] fiberwise to the *umkehr* map $f_! : Q\mathcal{T}_{g,n} \to f^*Q\mathcal{T}_{g',n'}$ [BS24, Definition 4.1], gives a family of branched coverings $H : f^*\mathcal{C}_{g',n'} \to \mathcal{C}_{g,n}$, fitting into the diagrams

$$\begin{array}{ccc} f^*\mathcal{C}_{g',n'} & \xrightarrow{H} & \mathcal{C}_{g,n} \\ \downarrow & & \downarrow \\ f(\mathcal{T}_{g,n}) & \xleftarrow{f} & \mathcal{T}_{g,n} \end{array}, \qquad \begin{array}{ccc} f^*\mathbb{P}Q\mathcal{T}^*_{g',n'} & \overset{f_!^*}{\dashrightarrow} & \mathbb{P}Q\mathcal{T}^*_{g,n} \\ {\scriptstyle\Phi}\uparrow & & \uparrow{\scriptstyle\Psi} \\ f^*\mathcal{C}_{g',n'} & \xrightarrow{\quad H \quad} & \mathcal{C}_{g,n} \end{array},$$

where $\Phi : f^*\mathcal{C}_{g',n'} \to f^*Q\mathcal{T}^*_{g',n'}$ and $\Psi : \mathcal{C}_{g,n} \to Q\mathcal{T}^*_{g,n}$ are the fiberwise bicanonical maps. In [BS24, Lemma 5.3] we used [BS24, Remark 5.2] to conclude that H is *holomorphic*. The map

$$f_!^* : f^*\mathbb{P}Q\mathcal{T}^*_{g',n'} \dashrightarrow \mathbb{P}Q\mathcal{T}^*_{g,n}$$

is undefined on $\mathbb{P}\ker(f_!)^*$. Thus, the argument on [BS24, Lemma 5.3] only shows the following.

Lemma 2. *Let $2g+n \geq 5$ and $Z = \Phi^{-1}(\mathbb{P}\ker(f_!)^*)$. Then,*

$$H|_{f^*\mathcal{C}_{g',n'}-Z} : f^*\mathcal{C}_{g',n'} - Z \to \mathcal{C}_{g,n}$$

is holomorphic.

The fix. Let $h_X : f(X) \to X$ denote the holomorphic map given by the restriction of H to a fiber. Our proof of Theorem 1 used the holomorphicity of H in a key way to find a connected open dense subset $U \subset \mathcal{T}_{g,n}$ over which the topological profile of h_X is constant.[1] The main observation to salvage our proof is that the argument applies to any holomorphic family of branched coverings between closed Riemann surfaces—we simply used the fact that points of constant profile are generic. Thus, the following is all we need.

[1]Constant topological profile means constant degree, ramification *and* branching profile. This last property was omitted in the paper, but it easily follows from our proof (see Section 1).

Lemma 3. *Suppose $2g+n \geq 5$. Then there exists a closed analytic set $Z' \subset f^*\mathcal{C}_{g',n'}$ such that $\dim(Z') < 3g-3+n$ and so that*

$$H|_{f^*\mathcal{C}_{g',n'}-Z'} : f^*\mathcal{C}_{g',n'} - Z' \to \mathcal{C}_{g,n}$$

is holomorphic. In particular, there is a connected open dense set $U \subseteq \mathcal{T}_{g,n}$ such that

$$H|_U : f^*\mathcal{C}_{g',n'}|_{f(U)} \to \mathcal{C}_{g,n}|_U$$

is holomorphic.

Organization. In Section 1, we use Lemma 3 and our original argument to conclude the proof of Theorem 1. In Section 2, we give a proof of Lemma 3 based on a result of Bers [Ber61] and on the fact that a rational map to projective space is defined and holomorphic outside of an analytic subset of codimension at least 2 [GH78, Chapter 4.2, pp. 490-491].

Acknowledgments. We thank Alex Wright for asking us a question related to [GG22, Theorem 1.3] that lead us to discover this mistake. We also thank Benson Farb for comments on an earlier draft that helped us on improving the exposition; and Eduard Looijenga and Sidhanth Raman for helpful discussions that lead us to simplify our proof of Lemma 3.

1. Proof of Theorem 1

In this section we show how to conclude the proof of Theorem 1, replacing our assumption on the holomorphicity of H with Lemma 3. The arguments are essentially the same as the ones appearing in the published version of the paper, the goal being to find a suitable subset $U \subset \mathcal{T}_{g,n}$ over which the family of holomorphic mappings given by $H : f^*\mathcal{C}_{g',n'} \to \mathcal{C}_{g,n}$ has constant topological profile. In the paper, we used a genericity argument to find a subset U over which the degree and ramification profile are constant. The same argument allows us to show more: we can refine U so that the branching profile is constant and the ramification and branch locus interact nicely with the canonical sections given by the marked points.We include the details below.

Proof of Theorem 1. In the following we identify $f(\mathcal{T}_{g,n})$ with $\mathcal{T}_{g,n}$, and successively (and tacitly) apply the proper mapping theorem to the projection maps $\pi' : f^*\mathcal{C}_{g',n'} \to \mathcal{T}_{g,n}$ and $\pi : \mathcal{C}_{g,n} \to \mathcal{T}_{g,n}$. By Lemma 3, there exists a connected open dense subset $U \subseteq \mathcal{T}_{g,n}$ so that

$$H|_U : f^*\mathcal{C}_{g',n'}|_U \to \mathcal{C}_{g,n}$$

is holomorphic. The same arguments appearing in our original proof of Theorem 1 with $\mathcal{T}_{g,n}$ replaced by U *mutatis mutandis* gives us that up to replacing U with a connected dense open subset we can assume that the degree and the *ramification profile* of h_X are constant over U. Moreover, either there is no ramification or the ramification points are given by finitely many smooth, pairwise disjoint, analytic subvarieties $Y_i^r \subset f^*\mathcal{C}_{g',n'}|_U$ satisfying that $\pi' : Y_i^r \to U$ is a covering map. Since $H|_U$ is proper, the set $Y_i^b := H(Y_i^r)$ is an analytic variety. Either Y_i^b is empty or, by removing the singular locus and shrinking U, $\pi : Y_i^b \to U$ is a smooth holomorphic covering map. Removing further proper intersections if needed, we can assume that the family $\{Y_i^b\}$ is pairwise disjoint over U. In particular, the *branching profile* is constant over U. Similarly, we can assume that Y_i^r and Y_i^b are either disjoint from the canonical sections s'_j of $f^*\mathcal{C}_{g',n'}$ and s_j of $\mathcal{C}_{g,n}$, or completely agree with one of them. In particular, by passing to the universal cover $\widehat{U}$ of U, it follows that the ramification and branch locus are given by holomorphic *sections* over $\widehat{U}$.

With these remarks in place, apply *verbatim* the arguments appearing in the last two paragraphs of our original proof. □

2. Proof of Lemma 3

Proof of Lemma 3. In the following we again identify $f(\mathcal{T}_{g,n})$ with $\mathcal{T}_{g,n}$. Since $2g+n \geq 5$, the bicanonical map $\Psi : \mathcal{C}_{g,n} \to \mathbb{P}Q\mathcal{T}^*_{g,n}$ is an embedding so we will replace H with $\Psi \circ H$.

Step 1: Local case. First we prove Lemma 3 locally over $\mathcal{T}_{g,n}$. Let $V \subset \mathcal{T}_{g,n}$ be a small enough open subset so that $Q\mathcal{T}_{g,n}$ is holomorphically trivial. Let $\{q^i_X\}$ be a holomorphic frame for $Q\mathcal{T}_{g,n}$

over V. Then, up to shrinking V, there is a holomorphic frame $\{q^i_{f(X)}\}$ for $Q\mathcal{T}_{g',n'}$ over V so that $q^i_{f(X)} = f_!(q^i_X)$ for $i = 1, \ldots, 3g-3+n$. In these coordinates

$$(f_!)^* : f^* Q\mathcal{T}^*_{g',n'} \to Q\mathcal{T}^*_{g,n}$$

is given by (fiberwise) projection onto the first $N+1 = 3g-3+n$ coordinates.

Recall that a model for the universal curve $\mathcal{C}_{g',n'}$ is given by a quotient of the Bers Fiber space $F_{g',n'}$ by a Fuchsian group Γ acting properly discontinuously [EK76, Section 3]. Via a result of Bers, it follows that around any point $(X_0, z_0) \in f^*\mathcal{C}_{g',n'}$ there exist coordinates $W \times \Delta$ so that

$$q^i_{f(X)}(z) = \psi_i(X, z) dz^2,$$

for some meromorphic function ψ_i [Ber61, Theorem II].[2] In particular, $\phi_i(X, z) := \frac{q^i_{f(X)}(z)}{q^0_{f(X)}(z)}$ for $i = 0, \ldots, N$ are *meromorphic* functions on $f^*\mathcal{C}_{g',n'}|_V$. Define the rational map [GH78, Chapter 4.2]

$$F_V : f^*\mathcal{C}_{g',n'}|_V \to \mathbb{P}^N, \qquad (X, z) \mapsto [1 : \phi_1(X, z) : \ldots : \phi_N(X, z)].$$

The map

$$f^*\mathcal{C}_{g',n'}|_V \to \mathbb{P}Q\mathcal{T}^*_{g,n}|_V \cong V \times \mathbb{P}^N, \qquad (X, z) \mapsto (X, F_V(X, z))$$

agrees with H wherever it is defined. Since F_V is rational, there exists a closed analytic subset Z_V, of codimension at least 2, so that F_V is holomorphic on $f^*\mathcal{C}_{g',n'}|_V - Z_V$. The result follows.

Step 2: Globalizing. Let $\{V_\alpha\}$ be a locally finite cover of $\mathcal{T}_{g,n}$ so that for each α there is a holomorphic frame for $Q\mathcal{T}_{g,n}|_{V_\alpha}$ that can be completed to a frame for $f^*Q\mathcal{T}_{g',n'}|_{V_\alpha}$. Step 1 implies that H is holomorphic on the open set $W := \cup_\alpha f^*\mathcal{C}_{g',n'}|_{V_\alpha} - Z_\alpha$, where Z_α is a closed analytic subset of $f^*\mathcal{C}_{g',n'}|_{V_\alpha}$, of codimension at least two. Let $Z' = f^*\mathcal{C}_{g',n'} - W$, then Z' is closed and as the cover is locally finite, Z' is also analytic and of codimension at least two. Since $\dim(f^*\mathcal{C}_{g',n'}) = 3g-2+n$, it follows that $\dim(Z') < 3g-3+n$ as needed.

The second claim of the Lemma follows by taking $U = \mathcal{T}_{g,n} - \pi'(Z')$ and applying the proper mapping theorem to the proper map $\pi' : f^*\mathcal{C}_{g',n'} \to \mathcal{T}_{g,n}$ (see e.g. [Fis06, Theorem 1.19]). □

Some minor extra correction. In our definition of totally geodesic submanifold, we claimed that a totally geodesic submanifold $M \subseteq \mathcal{T}_{g,n}$ is a complex submanifold. This is false, as one can see by considering a real geodesic. We solely worked in the complex category, so this does not affect our results..

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[2]Note that, as used in [Ear77, Section 1], Bers' theorem also applies to the finite type case by considering Fuchsian groups with elliptic elements. Bers' theorem gives holomorphic representatives of the quadratic differentials when defined on the Bers Fiber space $F_{g',n'}$, the meromorphic representatives on the universal curve $\mathcal{C}_{g',n'}$ appear precisely because we consider groups with elliptic elements. Further, all ψ_i have at worst simple poles and this implies the holomorphicity of the bicanonical map (see also [GG22, Prop 3.2]).