LOCAL RIGIDITY OF COVERING CONSTRUCTIONS AND WEIL–PETERSSON SUBVARIETIES OF THE MODULI SPACE OF CURVES

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Abstract. We show that totally geodesic subvarieties of the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points, endowed with the Weil–Petersson metric, are locally rigid. This implies that covering constructions—examples of totally geodesic subvarieties of $\mathcal{M}_{g,n}$ endowed with the Teichmüller metric—are locally rigid. We deduce the local rigidity statement from a more general rigidity result for a class of orbifold maps to $\mathcal{M}_{g,n}$.

1. Introduction

Suppose $3g-3+n>0$ and let $\pi:\mathcal{T}_{g,n}\to\mathcal{M}_{g,n}$ be the projection from the Teichmüller space of a genus g surface with n marked points to the associated moduli space. The Weil–Petersson metric g_{WP} on $\mathcal{T}_{g,n}$ is a negatively curved Kähler metric that descends to a metric on $\mathcal{M}_{g,n}$. The metric g_{WP} induces on $\mathcal{M}_{g,n}$ the structure of a quasi-projective variety [Wol85]. A class of subvarieties[1] of $\mathcal{M}_{g,n}$ naturally associated with g_{WP} is the following:

A complex submanifold $M\subset\mathcal{T}_{g,n}$ is called *Weil–Petersson* if it is totally geodesic with respect to the Weil–Petersson metric g_{WP}. This means that the g_{WP}-geodesic between any two points in M is completely contained in M. Similarly, a subvariety $N\subset\mathcal{M}_{g,n}$ is called *Weil–Petersson* if an irreducible component M of $\pi^{-1}(N)$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$. The complex submanifold M is called a *lift* of N. More generally, we call a subvariety $N\subset\mathcal{M}_{g,n}$ *almost Weil–Petersson* if it has a lift $M\subset\mathcal{T}_{g,n}$ that maps biholomorphically via a forgetful map $\mathcal{T}_{g,n}\to\mathcal{T}_{g,m}$ onto a Weil–Petersson complex submanifold of some $\mathcal{T}_{g,m}$ with $n\geq m$ and $3g-3+m>0$.

We say that a map of complex orbifolds $f:X\to\mathcal{M}_{g,n}$ is an *almost Weil–Petersson immersion* if it is a proper immersion and $f(X)$ is an almost Weil–Petersson subvariety. In particular, this implies that X is a quasi-projective variety. The normalization map [GR84, Chapter 8] of an almost Weil–Petersson subvariety provides the basic example of an of almost Weil–Petersson immersion (see Definition 3.4). In this work we study the following rigidity property of this class of maps.

A proper immersion $f:X\to\mathcal{M}_{g,n}$ from a quasi-projective variety X is *locally rigid* if any holomorphic deformation

$$f_t:X_t\to\mathcal{M}_{g,n}\quad,\quad t\in\Delta$$

with $(X_0,f_0)\cong(X,f:X\to\mathcal{M}_{g,n})$ through proper immersions f_t and quasi-projective varieties X_t is trivial: there is a holomorphic family of biholomorphisms $g_t:X_t\to X$ inducing $(X_t,f_t)\cong(X,f)$. Our maps should be understood in the orbifold sense, i.e. on a level $L\geq 3$ cover of $\mathcal{M}_{g,n}$ (see Definition 5.2 for details). Our main theorem is the following.

Theorem 1.1 (**Local Rigidity**). *Suppose $3g-3+n>0$ and $\dim(X)\geq 1$. Then, any almost Weil–Petersson immersion $f:X\to\mathcal{M}_{g,n}$ is locally rigid.*

Remark 1.2. The assumption on X_t being quasi-projective is only used to get enough maps from finite type curves $C\to X_t$ (cf. Lemma 5.1).

Examples: Covering constructions. A source of examples of almost Weil–Petersson subvarieties is given by *covering constructions* [BS24, MMW17], defined as follows. Let

$$h:\Sigma_{g',n'}\to\Sigma_{g,n}$$

Date: Dec 29 2025. Updated version: rephrasing of the main theorem in terms of maps, corrected typos and improved exposition.

[1]Unless otherwise stated we always consider subvarieties in the algebraic category.

be a finite degree, orientation-preserving, topological branched cover. Assume further that the preimage of the marked points in $\Sigma_{g,n}$ equals the union of the marked points in $\Sigma_{g',n'}$ and the ramification points of h. Pulling back complex structures under h gives a holomorphic map

$$f_h : \mathcal{T}_{g,n} \to \mathcal{T}_{g',\ell}$$

for some $\ell \geq n'$. We call f_h a *totally marked* covering construction (see Section 6 for more details). A *covering construction* is a holomorphic map

$$f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$$

given by the composition of a totally marked covering construction $f_h : \mathcal{T}_{g,n} \to \mathcal{T}_{g',\ell}$ with a forgetful map $\mathcal{T}_{g',\ell} \to \mathcal{T}_{g',n'}$ which forgets *only* ramification points of h. If the map h is a regular branched cover, we call f a *regular covering construction.* The link between covering constructions and almost Weil–Petersson submanifolds of $\mathcal{T}_{g,n}$ is the following.

Proposition 1.3. *Assume $3g-3+n>0$. Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ be a covering construction. The following hold:*

(1) *Assume f is totally marked. Then, the image $f(\mathcal{T}_{g,n})$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g',n'}$.*
(2) *Assume f is regular and not totally marked. If $(g,g') = (0,1)$ assume further that the marked points satisfy $n \geq 5$. Then, the image $f(\mathcal{T}_{g,n})$ is an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g',n'}$.*

In particular, in either case the projection $\pi(f(\mathcal{T}_{g,n})) \subset \mathcal{M}_{g',n'}$ is an almost Weil–Petersson subvariety.

Remark 1.4. To our knowledge, covering constructions provide the only known examples of almost Weil–Petersson subvarieties of $\mathcal{M}_{g,n}$. Thus, we ask the following.

Question 1.5. Does there exist a Weil–Petersson subvariety $N \subset \mathcal{M}_{g,n}$ which is not given by a covering construction?

We now give two applications of Theorem 1.1.

Application 1. Totally geodesic subvarieties for the Teichmüller metric. Replacing the Weil–Petersson metric by the *Teichmüller metric* d_{Teich} in our definition of Weil–Petersson subvarieties gives *Teichmüller* subvarieties $N \subset \mathcal{M}_{g,n}$ and *Teichmüller* complex submanifolds $M \subset \mathcal{T}_{g,n}$ [Wri20, AHW24, BDR24].

1-dimensional Teichmüller subvarieties are called *Teichmüller curves.* McMullen showed [McM09] that the normalization map of a Teichmüller curves is rigid.[2] Arana-Herrera–Wright [AHW24, Question 10.6] ask if the same holds for Teichmüller varieties of higher dimensions. Theorem 1.1 implies a positive answer to [AHW24, Question 10.6] in the case of covering constructions.

Corollary 1.6 (Local rigidity of Covering constructions). *Assume $3g-3+n>0$. Let $f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$ be a totally marked or regular covering construction, and fix the notation and assumptions of Proposition 1.3. The normalization map of $\pi(f(\mathcal{T}_{g,n})) \subset \mathcal{M}_{g',n'}$ is locally rigid.*

Proof. The normalization map of an almost Weil–Petersson subvariety defines an almost Weil–Petersson immersion (see Section 3), and so the claim follows from Theorem 1.1. □

Remark 1.7. Any covering construction arises by applying forgetful maps to a totally marked covering construction. Thus, up to forgetful maps, all covering constructions are locally rigid.

Remark 1.8. Covering constructions encompass all of the known examples of Teichmüller subvarieties of $\mathcal{M}_{g,n}$ of dimension bigger than 1, except for the 2-dimensional examples $\{Y_i\}$ found recently by McMullen, Mukamel, Wright and Eskin [MMW17, EMMW20]. In this regard, the following questions are quite pertinent.

[2]McMullen considered a more general class of maps that are the equivalent (and the inspiration) of our definition of an almost Weil–Petersson immersion for the Teichmüller metric. Furthermore, he considers a more general class of local deformations not assuming that f_t is a proper immersion. For the one-dimensional case, our proof of Theorem 1.1 only requires f_t to be an immersion, and so it gives an alternative proof of McMullen's result for the case of covering constructions. See Remark 5.4 for a discussion on which assumptions can be dropped.

Question 1.9. Are the Teichmüller surfaces Y_i totally geodesic with respect to the Weil–Petersson metric, i.e. are Y_i also Weil–Petersson surfaces?

Question 1.10. Given a subvariety $N \subset \mathcal{M}_{g,n}$. Suppose that N is totally geodesic with respect to *both* the Teichmüller and the Weil–Petersson metric. Is N given by a covering construction?

Application 2. Characterization of almost Weil–Petersson submanifolds. The pure mapping class group $\mathrm{PMod}(\Sigma_{g,n}) := \pi_0(\mathrm{Diff}^+(\Sigma_{g,n}, \{x_1, \ldots, x_n\}))$ acts on $\mathcal{T}_{g,n}$ by biholomorphisms. For an arbitrary analytic subset $W \subset \mathcal{T}_{g,n}$, let Γ_W be the stabilizer of W in $\mathrm{PMod}(\Sigma_{g,n})$. Let M be an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$ of positive dimension. As a corollary of the proof of Theorem 1.1 (see Theorem 3.6), we obtain the following characterization of M.

Corollary 1.11 (**Maximality**). *Let $3g-3+n>0$. Assume that $\pi(M) \subset \mathcal{M}_{g,n}$ is a subvariety. Then, the submanifold M is maximal among the analytic subsets W of $\mathcal{T}_{g,n}$ such that the following hold:*

(1) $\Gamma_W \subseteq \Gamma_M$.
(2) W/Γ_W *is a quasi-projective variety.*

In particular, the conjugacy class of Γ_M in $\mathrm{PMod}(\Sigma_{g,n})$ is a complete invariant of $\pi(M)$.

Remark 1.12. Corollary 1.11 showcases the following rigidity result for the pair (M, Γ_M). Inclusion at the level of the stabilizers $\Gamma_W < \Gamma_M$ implies the inclusion of *spaces* $W \subset M$. A similar statement holds for families of curves over quasi-projective varieties: if the monodromy factors through Γ_M, then the classifying map to $\mathcal{T}_{g,n}$ factors through M (see Theorem 3.6 for a precise statement).

Paper overview. Let $M \subset \mathcal{T}_{g,n}$ be an almost Weil–Petersson lift of a subvariety of $\mathcal{M}_{g,n}$, and let Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. In Section 2 we quickly review all the necessary material from Teichmüller theory needed for the rest of the paper, focusing on the Weil–Petersson metric. In Section 3 we introduce the concept of a Γ_M-deformation: a class of orbifold maps $\phi : X \to \mathcal{M}_{g,n}$ which includes the classifying maps of families of curves over quasi-projective varieties whose monodromy factors through Γ_M. Theorem 1.1 is a consequence of the stronger rigidity result for Γ_M-deformations given in Theorem 3.6: *any nonconstant Γ_M-deformation factors through M.* Section 3 also shows that Γ_M is the orbifold fundamental group of the *normalization* of $\pi(M)$, and defines the notion an almost Weil–Petersson immersion, which could be roughly stated as an orbifold covering of M/Γ_M. In Section 4 we provide the details of the proof of Theorem 3.6 in the case that X is a finite type curve, starting with an elementary Riemannian geometry lemma and then following the argument appearing in the proof of the Imayoshi–Shiga theorem given in [AAS18, Section 4]. The key observation is that, due to the arguments of [AAS18], *holomorphic maps minimize energy in their homotopy class.*[3] In Section 5 we finish the proof of Theorem 3.6 and show how it implies the rigidity stated in Theorem 1.1. Finally, in Section 6 we prove Proposition 1.3 (see also a remark in page 2 of [KM08] for the totally marked case) thereby concluding the proof of Corollary 1.6.

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[3]For compact C the proof follows from the theory of harmonic maps (cf. [ES64]), but an extra step is needed when C is finite volume.

2. Teichmüller geometry and the Weil–Petersson metric

2.1. **Basic definitions.** Assume $3g-3+n>0$ and let $\Sigma_{g,n}$ be an oriented topological surface of genus g with n-marked points (or n-punctures). A marking is an orientation preserving homeomorphism $\varphi : \Sigma_{g,n} \to X$ from $\Sigma_{g,n}$ to a Riemann surface X. Teichmüller space $\mathcal{T}_{g,n}$ is the space of equivalence classes of markings $[\varphi : \Sigma_{g,n} \to X]$, where two markings are equivalent if they differ up to isotopy by a biholomorphism. Unless we need to stress the marking we will denote $[\varphi : \Sigma_{g,n} \to X]$ simply by X. In case we need to emphasize the subset of points being marked we introduce the following notation: Let Σ be a topological surface and $u \subset \Sigma$ be a finite subset of points. $\mathcal{T}(\Sigma, u)$ denotes the Teichmüller space of Σ with the marked points given by the subset u.

The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$—the group of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ up to isotopy—acts on $\mathcal{T}_{g,n}$ by biholomorphisms. In fact, Royden and Earl–Kra [Roy71, EK74] showed that for $2g+n>4$, the group $\mathrm{Mod}(\Sigma_{g,n})$ agrees with the group of biholomorphisms of $\mathcal{T}_{g,n}$. The pure mapping class group $\mathrm{PMod}(\Sigma_{g,n})$ is the subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ fixing pointwise the marked points. The quotient of $\mathcal{T}_{g,n}$ by $\mathrm{PMod}(\Sigma_{g,n})$ induces the projection

$$\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n} = \mathcal{T}_{g,n} / \mathrm{PMod}(\Sigma_{g,n}).$$

Let $B(X)$ denote the space of *Beltrami forms* on X, i.e. $(-1,1)$-forms μ on X with $\|\mu\|_\infty < \infty$. Denote by $M(X)$ the open unit ball in $B(X)$. There exists a holomorphic submersion

$$\Phi : M(X) \to \mathcal{T}_{g,n} \quad , \quad \mu \to [f^\mu \circ \varphi : \Sigma_{g,n} \to X_\mu]$$

where f^μ is given by solving (locally) the Beltrami equation

$$\partial_{\bar z} f = \mu \partial_z f.$$

In particular, $T_X\mathcal{T}_{g,n}$ is naturally identified with a quotient of $B(X)$.

Let $[\varphi : (\Sigma, u) \to (X, \varphi(u))] \in \mathcal{T}(\Sigma, u)$. Let $Q(X,u)$ denote the space of integrable holomorphic quadratic differentials on X, i.e. meromorphic quadratic differentials q on X with poles of order at most one, and all poles contained in $\varphi(u)$. Equivalently, $Q(X,u)$ is the space of holomorphic quadratic differentials q on $X - \varphi(u)$ so that

$$\int_{X-\varphi(u)} |q| < \infty.$$

When the marked points are not being stressed we denote $Q(X,u)$ simply by $Q(X)$. There is a natural pairing

$$B(X) \times Q(X) \to \mathbb{C} \quad , \quad (\mu, q) \mapsto \int_X \mu q$$

which factors through $T_X\mathcal{T}_{g,n} \times Q(X)$. In particular, $Q(X)$ is naturally identified with $T^*_X\mathcal{T}_{g,n}$.

At the level of the universal cover and working always with punctures, we have the following notation.[4] Let $u \subset \Sigma$ be a finite set of points and $\varphi : (\Sigma, u) \to (X, \varphi(u))$ be a marked Riemann surface. Let $X' = X - \varphi(u)$ and let $\pi : \mathbb{H}^2 \to X'$ be the universal cover of X', with deck group Γ. Then $B(X')$ is given by the Γ-invariant $(-1,1)$-forms on $\mathbb{H}^2$, denoted by $B(\Gamma)$. Similarly, $Q(X')$ is given by $Q(\Gamma)$, the Γ-invariant holomorphic quadratic differentials on $\mathbb{H}^2$ inducing an integrable quadratic differential on X'.

2.2. **Weil–Petersson metric.** The *Weil–Petersson* metric, denoted by g_{WP}, is induced from the hermitian product on $Q(X)$ given by

$$\langle q_1, q_2 \rangle_{\mathrm{WP}} = \int_X q_1 \overline{q_2} (ds^2)^{-1},$$

where ds^2 is the hyperbolic volume form. The hermitian product on $T\mathcal{T}(X)$ is given by

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP}} = \overline{\langle \mu_1^*, \mu_2^* \rangle}_{\mathrm{WP}}$$

where $\mu_i^* \in Q(X)$ is the dual of μ_i. This means that μ_i^* is characterized by

$$\langle q, \mu_i^* \rangle_{\mathrm{WP}} = \int_X \mu_i q, \quad \forall q \in Q(X).$$

[4]A similar description is possible for marked points by using groups with elliptic elements [EK76].

The metric g_{WP} is incomplete, Kähler, negatively curved and geodesically convex [Wol87].

Harmonic forms. The map

$$q \mapsto \overline{q}(ds^2)^{-1}$$

sends a quadratic differential to an element of $B(X)$ which is called a *harmonic form*. The space of harmonic forms on X is denoted by $H(X)$. The Weil–Petersson metric has a simple expression on harmonic forms: Let $\mu_i = \overline{q_i}(ds^2)^{-1} \in H(X)$ for $i = 1, 2$. Evidently $\mu_i^* = q_i$, and so

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP}} = \int_X \overline{q_1} q_2 (ds^2)^{-1} = \int_X \mu_1 \overline{\mu_2} ds^2 = \int_X \mu_1 q_2.$$

Definition 2.1 (**Almost Weil–Petersson**). Given a complex submanifold $M \subset \mathcal{T}_{g,n}$, we say that M is *Weil-Petersson* if it is totally geodesic with respect to g_{WP}.[5] More generally, we say that M is *almost Weil–Petersson* if there exists a forgetful map $\mathcal{F} : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ with $n \geq m$ and $3g - 3 + m > 0$. Such that

(1) $\mathcal{F}|_M : M \to \mathcal{F}(M)$ is a biholomorphism
(2) $\mathcal{F}(M)$ is a Weil-Petersson complex submanifold of $\mathcal{T}_{g,m}$.

Let $N \subset \mathcal{M}_{g,n}$ be a subvariety, a *lift* of N is an irreducible component of the preimage of N in $\mathcal{T}_{g,n}$. The subvariety N is called *(almost) Weil–Petersson* if it has an *(almost) Weil–Petersson* lift.

3. Γ_M-deformations, Normalization and the associated orbifold fundamental group

Let $\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the projection to moduli space. Given a subvariety $N \subset \mathcal{M}_{g,n}$ with a lift $M \subset \mathcal{T}_{g,n}$, denote the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$ by Γ_M. In this section we introduce the concept of a Γ_M-deformation which generalizes an important property of classical deformations to the orbifold category. Γ_M-deformations are central to our reformulation of Theorem 1.1, stated at the end of this section (Theorem 3.6). We start by proving some facts about the group $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$.

The associated orbifold fundamental group. The projection π induces a holomorphic map from the analytic quotient[6] M/Γ_M [Car57].

$$\pi_M : M/\Gamma_M \to N.$$

The group Γ_M is called the *associated orbifold fundamental group* of N(cf. [AHW24]). The following explains the terminology.

Lemma 3.1 (cf. [GDH92, Theorem 1]). *Suppose $3g - 3 + n > 0$. Given an analytic subvariety $N \subset \mathcal{M}_{g,n}$ with lift $M \subset \mathcal{T}_{g,n}$, let Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. The following hold:*

(1) *The induced holomorphic map*

$$\pi_M : M/\Gamma_M \to N$$

is proper with finite fibers and injective outside a proper analytic subvariety. In particular, $\pi(M) = N$ and so the conjugacy class of Γ_M in $\mathrm{PMod}(\Sigma_{g,n})$ is an invariant of N.

(2) *If N is quasi-projective, the normalization of M/Γ_M is quasi-projective. If in addition M is normal, M/Γ_M is a quasi-projective variety.*

Proof.
Proof of item (1). We need to show that the projection π_M satisfies the following:

(1) has finite fibers,
(2) is closed,
(3) is injective outside a proper analytic subvariety.

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[5] It is not hard to see that any totally geodesic analytic subset of $\mathcal{T}_{g,n}$ with respect to g_{WP} must in fact be a submanifold. See [Mil68, Remark on p 13].

[6] Note that when forming the quotient we remove the kernel of the action of Γ_M on M.

The family of closed subsets $\{gM : g \in \mathrm{PMod}(\Sigma_{g,n})\}$ in $\mathcal{T}_{g,n}$ is a subfamily of the decomposition of $\pi^{-1}(N)$ into its irreducible components. Thus, it is a locally finite family of closed subsets of $\mathcal{T}_{g,n}$. This means that for every $x \in \mathcal{T}_{g,n}$, there exists a neighborhood U_x of x intersecting only finitely many $g_i M$. This automatically gives that π_M has finite fibers, and an elementary argument shows that π_M is closed.

By the Semi-Proper Mapping Theorem [Fis76, Theorem 1.19], the map $M \to M/\Gamma_M$ sends Γ_M-invariant analytic subsets of M to analytic subsets of M/Γ_M. The map π_M is injective on the complement of the projection of the subset

$$W := \bigcup_{g \in \Gamma_M} \{gM \cap M : g(M) \cap M \text{ proper subvariety of } M\} \subset M.$$

W is analytic and Γ_M-invariant. Thus, the projection of W to M/Γ_M is analytic and the third item follows. Since $\pi_M : M/\Gamma_M \to N$ is proper, the proper mapping theorem implies that π_M is surjective whenever $\dim(M/\Gamma_M) = \dim(N)$ and the last claim follows.

Proof of item (2). If N is quasi-projective its normalization is also quasi-projective as the normalizations in the algebraic and analytic category coincide [Kuh61, Satz 4]. Let $\xi : \hat{M} \to M/\Gamma_M$ be the normalization of M/Γ_M, then $\pi_M \circ \xi : \hat{M} \to N$ is the normalization of N and so $\hat{M}$ is quasi-projective. If M is normal then M/Γ_M is a normal analytic space by a theorem of Cartan [Car57, Theorem 4], and the last claim follows. □

Remark 3.2. As the examples in [GDH92, HPRCR24] show, π_M is not in general a biholomorphism.

Level L-structures and smoothness of the normalization. Suppose $L \geq 3$ and $3g-3+n > 0$, and let $\mathrm{PMod}(\Sigma_{g,n})[L]$ be the level L pure mapping class group [FM12, Chapter 6.4.2]. Define

$$\Gamma_M[L] := \Gamma_M \cap \mathrm{PMod}(\Sigma_{g,n})[L].$$

Since $L \geq 3$, $\mathrm{PMod}(\Sigma_{g,n})[L]$ is torsion free. Thus, the induced map

$$M \to M/\Gamma_M[L]$$

is an unramified covering map. Denote by

$$\pi_L : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}[L]$$

the level-L projection to the moduli space of curves with level L structures, and denote the image $\pi_L(M)$ by $N[L]$. The same argument as in Lemma 3.1 shows that the normalization of $M/\Gamma_M[L]$ is a quasi-projective variety, whenever N is a subvariety of $\mathcal{M}_{g,n}$. In particular, if M is smooth then $M/\Gamma_M[L]$ is a smooth quasi-projective variety and the induced map

$$\pi_{M[L]} : M/\Gamma_M[L] \to N[L] \subset \mathcal{M}_{g,n}[L]$$

defines the normalization of an irreducible component of the preimage of N to $\mathcal{M}_{g,n}[L]$. Observe that if $N[L]$ is normal, then $\pi_{M[L]}$ is a biholomorphism.

Definition 3.3 (**Good and algebraic Orbifolds**)**.** Let $\widetilde{X}$ be a locally compact irreducible analytic space, and Γ_X a group acting properly discontinuously on $\widetilde{X}$ by biholomorphisms, we denote the quotient by $X = \widetilde{X}/\Gamma$. Assume that there exists a finite index normal subgroup $\Gamma'_X \trianglelefteq \Gamma_X$ acting freely on $\widetilde{X}$. We call such structure a *good orbifold* structure on X and define $\pi_1^{\mathrm{orb}}(X) = \Gamma_X$. We say that the orbifold $X = \widetilde{X}/\Gamma_X$ with orbifold structure $(\widetilde{X}, \Gamma_X)$ is *algebraic* if the *normalization* of $X' := \widetilde{X}/\Gamma'_X$ is a quasi-projective variety.

Examples of algebraic orbifolds are given by M/Γ_M for M a lift of a subvariety of $\mathcal{M}_{g,n}$, and by finite type curves C with the orbifold structure $(\widetilde{C}, \pi_1(C))$ for $\widetilde{C}$ the universal cover of C.

Definition 3.4 (**Proper and almost Weil–Petersson immersions**)**.** Let X be an orbifold with good orbifold structure $(\widetilde{X}, \Gamma_X)$. Let $f : X \to \mathcal{M}_{g,n}$ be a holomorphic map of complex orbifolds with associated homomorphism $f_* : \Gamma_X \to \mathrm{PMod}(\Sigma_{g,n})$. This means that there is a holomorphic map $\tilde{f} : \widetilde{X} \to \mathcal{T}_{g,n}$ equivariant with respect to the homomorphism f_* and fitting into the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \mathcal{T}_{g,n} \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & \mathcal{M}_{g,n} \end{array}$$

The map f is called a *proper orbifold immersion* if it is proper, $\widetilde{X}$ is a complex manifold and $\tilde{f}$ is an immersion onto an analytic subvariety $M \subset \mathcal{T}_{g,n}$. In particular, the image $N := f(X)$ is an *analytic* subvariety of $\mathcal{M}_{g,n}$, and M is a lift of N. If in addition the image N is an *almost Weil–Petersson* subvariety, f is called an *almost Weil–Petersson immersion*. Note that in this case M is an almost Weil-Petersson submanifold and $\tilde{f}$ is an embedding. As consequences of both definitions we find

(1) X is a normal analytic space.
(2) The homomorphism f_* factors through the stabilizer Γ_M of M.
(3) For any $L \geq 3$, there exists a finite index normal subgroup Γ'_X of Γ_X with manifold quotient $X' = \widetilde{X}/\Gamma'_X$ such that the induced map
$$X' \to M/\Gamma_M[L] \to N[L] \subset \mathcal{M}_{g,n}[L]$$
is a proper immersion.

For an almost Weil–Petersson immersion the following is also satisfied.

(1) X' is a finite unbranched cover of $M/\Gamma_M[L]$, thus X' is a smooth quasi-projective variety and $(\widetilde{X}, \Gamma_X)$ is an algebraic orbifold structure on X. Furthermore, since X is the quotient of X' by a finite group, X also inherits a quasi-projective structure.

Examples of almost Weil–Petersson immersions are given by the orbifold maps $M/\Gamma_M \to \mathcal{M}_{g,n}$ for M a lift of an almost Weil–Petersson subvariety. As Lemma 3.1 shows, this is precisely the normalization map of an almost Weil–Petersson subvariety.

Classical deformations. Let Z, W be two complex manifolds. A local deformation of a holomorphic map $f : Z \to W$ is given by a family of holomorphic maps $f_t : Z_t \to W$ for $t \in \Delta$, with a biholomorphism $\varphi : Z_0 \cong Z$ inducing $(Z_0, f_0) \cong (Z, f)$. If the family (Z_t) is smoothly trivial,[7] it follows that the groups $(f_t)_*(\pi_1(Z_t)) < \pi_1(W)$ are pairwise conjugate. The following is our generalization to the orbifold setting of a deformation of the map $\pi_M : M/\Gamma_M \to N \subset \mathcal{M}_{g,n}$.

Definition 3.5 (**Γ-deformation**)**.** Let $X = \widetilde{X}/\Gamma_X$ be an algebraic orbifold. Let $\phi : X \to \mathcal{M}_{g,n}$ be a holomorphic map of complex orbifolds with associated homomorphism $\phi_* : \Gamma_X \to \mathrm{PMod}(\Sigma_{g,n})$. Given an arbitrary subgroup $\Gamma < \mathrm{PMod}(\Sigma_{g,n})$, we say that ϕ is a Γ-*deformation* if, up to conjugation in $\mathrm{PMod}(\Sigma_{g,n})$, the homomorphism ϕ_* factors through the inclusion $\Gamma \hookrightarrow \mathrm{PMod}(\Sigma_{g,n})$.

Our main rigidity result is the following variant of the Imayoshi-Shiga theorem (see also Theorem 4.10).

Theorem 3.6. *Let $3g - 3 + n > 0$. Let M be an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$, and $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$ be the stabilizer of M. Given an algebraic orbifold $X = \widetilde{X}/\Gamma_X$ and a nonconstan Γ_M-deformation $\phi : X \to \mathcal{M}_{g,n}$. Then, there is a lift $\tilde{\phi} : \widetilde{X} \to \mathcal{T}_{g,n}$ of ϕ that factors through the inclusion $M \hookrightarrow \mathcal{T}_{g,n}$.*

Note that any almost Weil–Petersson immersion is a Γ_M-deformation, Theorem 3.6 says that a Γ_M-deformation closely behaves as an almost Weil–Petersson immersion. In the case that M is a lift of a subvariety, the only missing condition is that the lift is an embedding. This observation is key to our proof of Theorem 1.1.

[7]note that we are *not assuming* that f is proper.

4. The case of a finite type curve C

On this section we prove Theorem 3.6 when X is a smooth finite type curve. We start with the following special case.

Proposition 4.1. *Suppose $3g-3+n>0$. Given a Weil–Petersson complex submanifold $M\subset\mathcal{T}_{g,n}$ with stabilizer $\Gamma_M<\mathrm{PMod}(\Sigma_{g,n})$. Let C be a smooth finite type curve and $\phi: C\to\mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. Then, there exists a lift of ϕ to $\mathcal{T}_{g,n}$ that factors through the inclusion $M\hookrightarrow\mathcal{T}_{g,n}$.*

The first step in the proof of Proposition 4.1 is the following elementary observation, whose proof we include for completeness.

Lemma 4.2 (Orthogonal Projection). *Suppose $3g-3+n>0$. Let Z be a totally geodesic submanifold of $\mathcal{T}_{g,n}$ with respect to the Weil–Petersson metric and let Γ_Z be the stabilizer of Z in $\mathrm{Mod}(\Sigma_{g,n})$. There exists a Γ_Z-equivariant smooth deformation retraction onto Z*

$$\pi_Z^t:\mathcal{T}_{g,n}\to\mathcal{T}_{g,n}\quad,\quad t\in[0,1]$$

with the following property: for any $p\in\mathcal{T}_{g,n}$ and $w\in T_p\mathcal{T}_{g,n}$ the map

$$t\to\|d\pi_Z^t(w)\|_{\mathrm{WP}}$$

is non-increasing. If $p\notin Z$ and w is not parallel to the geodesic joining p and $\pi_Z^1(p)\in Z$ then $\|d\pi_Z^t(w)\|_{\mathrm{WP}}$ is strictly decreasing.

Proof. Endow $\mathcal{T}_{g,n}$ with the Weil–Petersson metric g_{WP}. Let $\mathcal{E}\subset T\mathcal{T}_{g,n}$ be the domain of the exponential map. Since g_{WP} is incomplete $\mathcal{E}\neq T\mathcal{T}_{g,n}$. Let $\mathcal{E}_Z:=\mathcal{N}Z\cap\mathcal{E}$ for $\mathcal{N}Z$ the normal bundle of Z. The metric g_{WP} is negatively curved, thus

$$\exp:\mathcal{E}_Z\to\mathcal{T}_{g,n}$$

is injective and a diffeomorphism onto its image. Since g_{WP} is not complete, a priori there is no reason for exp to be surjective. Yet, Wolpert showed that for any $p\in\mathcal{T}_{g,n}$, the map $\exp_p:\mathcal{E}_p\to\mathcal{T}_{g,n}$ is a homeomorphism [Wol87, Corollary 5.4] , and this implies surjectivity of exp.

Let $\phi_t:\mathcal{E}_Z\to\mathcal{E}_Z$ be given by $(p,v)\to(p,tv)$. Define

$$\pi_Z^t:\mathcal{T}_{g,n}\to\mathcal{T}_{g,n}\quad,\quad \pi_Z^t:=\exp\circ\phi_{1-t}\circ\exp^{-1}.$$

Since exp and ϕ_t are Γ_Z-equivariant, it follows that π_Z^t is Γ_Z-equivariant. The map π_Z^1 is precisely the orthogonal projection map to Z.

Let $p\in\mathcal{T}_{g,n}$ and $w\in T_p\mathcal{T}_{g,n}$. Observe that $J(t)=d\pi_Z^{1-t}(w)$ is a Jacobi field. A standard computation, using the fact that Z is totally geodesic, shows that $\|J(t)\|_{\mathrm{WP}}$ is non-decreasing. Furthermore, if $p\notin Z$ and w is not parallel to the geodesic joining p and $\pi_Z^1(p)\in Z$ it follows that $J(t)$ has a non-trivial normal component and so $\|J(t)\|_{\mathrm{WP}}$ is strictly increasing. □

Having Lemma 4.2 at hand, we can prove Proposition 4.1.

Proof of Proposition 4.1. Let $M\subset\mathcal{T}_{g,n}$ be a Weil–Petersson complex submanifold, with stabilizer $\Gamma_M<\mathrm{PMod}(\Sigma_{g,n})$. By Lemma 4.2 there is a smooth Γ_M-equivariant homotopy

$$\pi_M^t:\mathcal{T}_{g,n}\to\mathcal{T}_{g,n}\quad,\quad t\in[0,1]$$

between the identity and the orthogonal projection $\pi_M^1:\mathcal{T}_{g,n}\to M$. Let C be a smooth finite type curve and $\phi: C\to\mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. In particular, C is of hyperbolic type and the lift $\tilde\phi:\mathbb{H}^2\to\mathcal{T}_{g,n}$, which is equivariant with respect to the homomorphism $\phi_*:\pi_1(C)\to\mathrm{PMod}(\Sigma_{g,n})$, is *not constant.* Up to post-composing with a mapping class, we can assume that $\phi_*:\pi_1(C)\to\Gamma_M$. Define the map

$$\tilde\psi:\mathbb{H}^2\to\mathcal{T}_{g,n}\quad,\quad \tilde\psi=\pi_M^1\circ\tilde\phi.$$

Since π_M^1 is Γ_M-equivariant, it follows that $\tilde\psi$ is ϕ_*-equivariant. Proposition 4.1 will be proven if we show that $\tilde\phi=\tilde\psi$.

Claim 4.3. $\tilde\phi=\tilde\psi$

Proof of Claim 4.3. The proof follows the argument appearing in the proof of the Imayoshi–Shiga theorem given in Section 4 of [AAS18]. We include the details for completeness, clarifying the equivariance nature of all the maps being used— hence the difference in our notation.

For a smooth map $\tilde{f} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ define the *energy density* $E_x(\tilde{f})$ of $\tilde{f}$ at a point $x \in \mathbb{H}^2$ to be

$$E_x(\tilde{f}) := \|d\tilde{f}|_x(v_1)\|^2_{\mathrm{WP}} + \|d\tilde{f}|_x(v_2)\|^2_{\mathrm{WP}},$$

for an arbitrary orthonormal basis $\{v_1, v_2\}$ of $T_x\mathbb{H}^2$. Assume further that $\tilde{f}$ is ϕ_*-equivariant, then $E_x(\tilde{f})$ is $\pi_1(C)$-invariant and we can define the *equivariant energy* of $\tilde{f}$ as:

$$E(f) := \int_C E_{\bar{x}}(f)\omega_C$$

where ω_C is the volume form on C and $E_{\bar{x}}(f)$ is given by $E_x(\tilde{f})$ for any lift x of $\bar{x}$.

Since $\tilde{\phi}$ is holomorphic and d_{Teich} agrees with the Kobayashi metric [Roy71], it follows that $\|d\tilde{\phi}(v)\|_{\mathrm{Teich}} \leq \|v\|_{\mathbb{H}^2}$. Recall that the Teichmüller metric dominates the Weil–Petersson metric

$$\|v\|_{\mathrm{WP}} \leq L\|v\|_{\mathrm{Teich}},$$

where $L = |2\pi(2g-2+n)|^{1/2}$ [McM00, Proposition 2.4]. Thus, $\|d\tilde{\phi}(v)\|_{\mathrm{WP}} \leq L\|v\|_{\mathbb{H}^2}$ and we have the following

Lemma 4.4 ([AAS18, Lemma 4.1])**.** *The homotopy*

$$\tilde{F} : \mathbb{H}^2 \times [0,1] \to \mathcal{T}_{g,n}, \quad \tilde{f}_t(x) = \pi^t_M(\tilde{\phi}(x))$$

between $\tilde{\phi}$ and $\tilde{\psi}$ satisfies:

(1) *$\tilde{f}_t$ is L-Lipschitz for all t. Thus, the equivariant energy $E(f_t)$ is finite for all t.*
(2) *The map $t \to E(f_t)$ is non-increasing along t. If $\tilde{\phi} \neq \tilde{\psi}$, then $E(f_t)$ is not constant.*
(3) *For any two points $\bar{x}, \bar{y} \in C$ let $d_C(\bar{x}, \bar{y})$ be their distance in C. Let $x_0 \in \mathbb{H}^2$ and let $\bar{x}_0 \in C$ be its image in C. There are constants $A, B > 0$ such that for all $(x,t) \in \mathbb{H}^2 \times [0,1]$ the operator norm $\|d\tilde{F}_{(x,t)}\|$ satisfies*
$$\|d\tilde{F}_{(x,t)}\|^2 \leq A d_C(\bar{x}_0, \bar{x})^2 + B.$$

Remark 4.5. Note that since the homotopy π^t_M is norm non-increasing we get stronger results than [AAS18, Lemma 4.1].

The proof of Claim 4.3 is completed by the two following key results. Let ω_{WP} be the Kähler form associated to the Weil–Petersson metric.

Proposition 4.6 (cf. [ES64, Proposition 4.2])**.** *Let $f : \mathbb{H}^2 \to (\mathcal{T}_{g,n}, g_{\mathrm{WP}})$ be a smooth map. Then, for any $x \in \mathbb{H}^2$*

$$E_x(f)\omega_{\mathbb{H}^2} \geq f^*(\omega_{\mathrm{WP}})|_x$$

with equality if f is holomorphic at x.

Remark 4.7. Note that [ES64, Proposition 4.2] is stated for the energy and not the energy density, but their results are given by pointwise estimates for the energy density that imply proposition 4.6. Furthermore, their results give an if and only if condition for equality; this is not necessary for our proof.

For each t, let $f_t^*\omega_{\mathrm{WP}}$ be the 2-form on C induced by the $\pi_1(C)$-invariant form $\tilde{f}_t^*\omega_{\mathrm{WP}}$ on $\mathbb{H}^2$. We have the following,

Proposition 4.8 ([AAS18, Proof of Imayoshi–Shiga])**.** *For all $t \in [0,1]$ we find*

$$\int_C f_t^*(\omega_{\mathrm{WP}}) = \int_C f_0^*(\omega_{\mathrm{WP}}).$$

Proof. Since $\tilde{F}$ satisfies properties (1)-(3) of Lemma 4.4, the same computation as in the proof of the Imayoshi–Shiga theorem in [AAS18] gives the result. This is a generalization of Stoke's theorem by using a nice enough exhaustion of C by compact sets. We remark that the notation of [AAS18] differ from ours as all of their maps should be understood in the orbifold sense, e.g. $\|dF_{(x,t)}\|$ is given in our notation by $\|d\tilde{F}_{(\tilde{x},t)}\|$ for any lift $\tilde{x}$ of x. □

Finishing the proof of $\tilde{\phi} = \tilde{\psi}$. On one hand via proposition 4.6

$$E(f_t) \geq \int_C f_t^*(\omega_{\mathrm{WP}}).$$

On the other hand via proposition 4.8 and the fact that f_0 is holomorphic.

$$\int_C f_t^*(\omega_{\mathrm{WP}}) = \int_C f_0^*(\omega_{\mathrm{WP}}) = E(f_0)$$

Thus $E(f_t) \geq E(f_0)$ and via property (2) in Lemma 4.4 we find $E(f_t) = E(f_0)$ for all t. In particular $\tilde{\phi} = \tilde{\psi}$. □

Having proven Claim 4.3, Proposition 4.1 follows. □

Remark 4.9. Note that the same argument works more generally for M a real totally geodesic submanifold of $\mathcal{T}_{g,n}$ with respect to the Weil–Petersson metric, e.g. an axis of a pseudo-Anosov element of $\mathrm{Mod}(\Sigma_{g,n})$ [DW03]. Furthermore, the same argument yields the following strengthening of the Imayoshi–Shiga theorem.

Theorem 4.10 (**Imayoshi–Shiga**)**.** *Suppose $3g-3+n > 0$. Let $C = \mathbb{H}^2/\pi_1(C)$ be a smooth finite type curve. Given $i = 1, 2$ and smooth maps $\phi_i : \mathbb{H}^2 \to \mathcal{T}_{g,n}$, equivariant with respect to the same homomorphism $\phi_* : \pi_1(C) \to \mathrm{PMod}(\Sigma_{g,n})$. Assume that ϕ_1 is holomorphic and nonconstant. Denote the equivariant energy of ϕ_i by $E(\phi_i)$. If either ϕ_2 is holomorphic or*

$$E(\phi_2) \leq E(\phi_1)$$

then $\phi_1 = \phi_2$.

Proposition 4.1 implies the following special case of Theorem 3.6 when the algebraic orbifold is a smooth finite type curve.

Proposition 4.11. *Let $3g - 3 + n > 0$. Given an almost Weil–Petersson complex submanifold $M \subset \mathcal{T}_{g,n}$ with stabilizer $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$. Let C be a smooth finite type curve and $\phi : C \to \mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. Then, ϕ has a lift $\tilde{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ factoring through $M \hookrightarrow \mathcal{T}_{g,n}$.*

Proof. Let $\phi : C \to \mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. Then, there is a lift $\tilde{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ of ϕ, equivariant with respect to a homomorphism

$$\phi_* : \pi_1(C) \to \Gamma_M \hookrightarrow \mathrm{PMod}(\Sigma_{g,n}).$$

Since M is almost Weil–Petersson, there exists a forgetful map

$$\mathcal{F} : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m},$$

with $n \geq m$ and $3g - 3 + m > 0$, so that $M' = \mathcal{F}(M)$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g,m}$. Let

$$\mathcal{F}_* : \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Mod}(\Sigma_{g,m})$$

be the associated forgetful map. The map $\mathcal{F}$ is $\mathcal{F}_*$-equivariant. It is evident that $\mathcal{F}_*(\Gamma_M) \subset \Gamma_{M'}$. We claim that

$$\mathcal{F}_*|_{\Gamma_M} : \Gamma_M \to \Gamma_{M'}$$

is injective. Indeed, $\mathcal{F}|_M : M \to M'$ is a biholomorphism and so the only way for an element $\gamma \in \ker \mathcal{F}_*$ to be in Γ_M is to fix M *pointwise*. The kernel $\ker \mathcal{F}_*$ is a surface braid group $B_{n-m}(\Sigma_{g,m})$ [FM12, Theorem 9.1]. Surface braid groups for surfaces with negative Euler characteristic are torsion free [FN62, Corollary 2.2]. As $\chi(\Sigma_{g,m}) < 0$ the claim follows.

The map $\tilde{\phi}$ is nonconstant, thus ϕ_* has infinite image. In particular, the group $G := \mathcal{F}_*(\phi_*(\pi_1(C)))$ is infinite. It follows that the map $\bar{\phi} := \mathcal{F} \circ \tilde{\phi}$ is not constant: indeed, if not then G fixes a point y, but this is impossible as G is infinite.

Observe that the map $\bar{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ is $\mathcal{F}_* \circ \phi_*$-equivariant so via (the proof of) Proposition 4.1 we find that $\bar{\phi}$ factors through $M' \hookrightarrow \mathcal{T}_{g,m}$. To conclude the proof of Proposition 4.11 we show that the holomorphic map

$$\hat{\phi} := \mathcal{F}|_M^{-1} \circ \bar{\phi} : \mathbb{H}^2 \to M \subset \mathcal{T}_{g,n}$$

equals $\tilde{\phi}$. By Theorem 4.10, it is enough to show that $\hat{\phi}$ is ϕ_*-equivariant. Let $x \in M'$ and $\gamma \in \Gamma_M$, then

$$\mathcal{F}|_M^{-1}(\mathcal{F}_*(\gamma)\cdot x) = \mathcal{F}|_M^{-1}(\mathcal{F}_M(\gamma\cdot\mathcal{F}|_M^{-1}(x)) = \gamma\cdot\mathcal{F}|_M^{-1}(x).$$

Thus $\hat{\phi}$ is ϕ_*-equivariant and the claim follows. □

5. Rigidity

In this section we complete the proofs of Theorem 3.6 and Theorem 1.1.

Proof of Theorem 3.6. Let $(\widetilde{X}, \Gamma_X)$ be the orbifold structure of an algebraic orbifold X, with finite index subgroup $\Gamma'_X < \Gamma_X$ and $X' = \widetilde{X}/\Gamma'_X$ having a quasi-projective normalization $\hat{X}$. Given a nonconstant Γ_M-deformation $\phi : X \to \mathcal{M}_{g,n}$, let $\phi' : X' \to \mathcal{M}_{g,n}$ be the induced orbifold map. By assumption there is a lift $\tilde{\phi} : \widetilde{X} \to \mathcal{T}_{g,n}$ of ϕ that is equivariant with respect to the homomorphism

$$\phi_* : \Gamma_X \to \Gamma_M \hookrightarrow \mathrm{PMod}(\Sigma_{g,n}).$$

Let $C \subset X'$ be a smooth finite type curve so that $\phi'|_C$ is not constant. Via the proof of Proposition 4.11, the induced map $\tilde{\phi}|_{\widetilde{C}}$ on the universal cover $\widetilde{C}$ factors through M. Thus, Theorem 3.6 will be proven if we show that we can cover X' by finite type curves C such that $\phi'|_C$ is not constant. Note that C is not required to be smooth. This is because $\phi'|_C$ will induce a map from its normalization.

Lemma 5.1. *Let $y \in X'$. There exists a finite type curve $C \subset X'$ passing through y so that $\phi'|_C$ is not constant.*

Proof. This is elementary, but we include the details. Given $y \in X'$, let $Y = (\phi')^{-1}(\phi'(y))$. Then, Y is a proper non-empty analytic subset of X'. Let $\xi : \hat{X} \to X'$ be the normalization of X' with ramification locus R. Since $\hat{X}$ is quasi-projective there exists a finite type curve C (not necessarily smooth) passing through any two of its points. Pick a point $x \in \hat{X} \setminus \xi^{-1}(Y) \cup R$, and let C be a finite type curve passing through x and an arbitrary preimage y' of y. Then $C_y := \xi(C)$ is a finite type curve passing through y. Indeed, the last claim follows since C and C_y have the same normalization. □

Having shown Lemma 5.1, Theorem 3.6 follows. □

Now we show that Theorem 3.6 implies the rigidity stated in Theorem 1.1. We consider the following more general class of local deformations,

Definition 5.2. Let $f : X \to \mathcal{M}_{g,n}$ be a proper orbifold immersion (see Definition 3.4) from a good orbifold $X = \widetilde{X}/\Gamma_X$ and $L \geq 3$. Let $\Gamma_X[L]$ be any finite index subgroup of Γ_X acting freely on $\widetilde{X}$ and so that $f_*(\Gamma_X[L]) < \mathrm{PMod}(\Sigma_{g,n})[L]$. A *proper* local orbifold deformation of f is a classical local holomorphic deformation

$$F_t : X_t[L] \to \mathcal{M}_{g,n}[L]\ ,\ t \in \Delta$$

of the lift $f[L] : X[L] := \widetilde{X}/\Gamma_X[L] \to \mathcal{M}_{g,n}[L]$ of f. In addition, we require the following to hold:[8]

(1) Let $\pi : \mathcal{X}[L] \to \Delta$ be the associated deformation of $X[L]$, with total space the complex manifold $\mathcal{X}[L]$. Then π is a closed, smooth submersion.
(2) $F_t : X_t[L] \to \mathcal{M}_{g,n}[L]$ is proper for each $t \in \Delta$.
(3) $\mathcal{X}[L]$ is smoothly isomorphic to $X[L] \times \Delta$.

Theorem 5.3. *Assume $3g-3+n > 0$, $L \geq 3$ and $\dim(X) \geq 1$. Be given an almost Weil–Petersson immersion $f : X \to \mathcal{M}_{g,n}$. Let*

$$F_t : X_t[L] \to \mathcal{M}_{g,n}[L] \quad , \quad t \in \Delta$$

be a proper local orbifold deformation of f such that each element of the family $\pi : \mathcal{X}[L] \to \Delta$ is quasi-projective. Then, up to shrinking Δ, the deformation is trivial, i.e. there exists a holomorphic family of biholomorphisms $g_t : X_t[L] \to X[L]$ inducing $(X_t[L], F_t) \cong (X[L], f[L])$.

[8]Observe that these are automatic for local deformations of compact manifolds.

Proof. Let M be a lift of the image $N := f(X)$ and Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. Consider the lift $f[L] : X[L] \to N[L]$, the deformation $\pi : \mathcal{X}[L] \to \Delta$ of $X[L]$, and the holomorphic map $\mathcal{F} : \mathcal{X}[L] \to \mathcal{M}_{g,n}[L]$ associated to a proper local orbifold deformation of f

$$F_t : X_t[L] \to \mathcal{M}_{g,n}[L] \quad , \quad t \in \Delta.$$

The set of points $(x, t) \in \mathcal{X}[L]$ where $dF_t|_x$ is not full rank is closed and does not include the central fiber. Since $\pi : \mathcal{X}[L] \to \Delta$ is closed, by shrinking Δ, we can assume that F_t is an immersion for all t. By assumption $X_t[L]$ is quasi-projective for all t, thus

$$X_t[L] \xrightarrow{F_t} \mathcal{M}_{g,n}[L] \longrightarrow \mathcal{M}_{g,n}$$

is a non-constant Γ_M-deformation. Since M is almost Weil-Petersson, Theorem 3.6 implies that $F_t(X_t[L]) \subset N[L] = F_0(X_0[L])$ for all t. In particular, the map $\mathcal{F} : \mathcal{X}[L] \to \mathcal{M}_{g,n}$ factors through $N[L]$. Recall that $\pi_{M[L]} : M/\Gamma_M[L] \to N[L]$ gives the normalization of $N[L]$. Hence, since $\mathcal{F}$ is surjective and $\mathcal{X}[L]$ is normal and irreducible, there is a unique holomorphic map $\mathcal{G} : \mathcal{X}[L] \to M/\Gamma_M[L]$ making the following diagram commute [GR84, Section 8.4.3]

$$\begin{tikzcd} & M/\Gamma_M[L] \arrow[d, "\pi_{M[L]}"] \\ \mathcal{X}[L] \arrow[ur, "\mathcal{G}"] \arrow[r, "\mathcal{F}"'] & N[L]. \end{tikzcd}$$

Let $\phi : X[L] \to X_0[L] \subset \mathcal{X}[L]$ be the biholomorphism so that $F_0 \circ \phi = f[L] : X[L] \to N[L]$. Observe that the surjective map $f[L] : X[L] \to N[L]$ factors through a finite holomorphic covering map

$$\hat{f}[L] : X[L] \to M/\Gamma_M[L].$$

By uniqueness of lifts to the normalization, it follows that $\mathcal{G} \circ \phi = \hat{f}[L]$. In particular, $\mathcal{G}_*(\pi_1(\mathcal{X}[L])) = \hat{f}[L]_*(\pi_1(X[L]))$ and so we have a further lift

$$\begin{tikzcd} & X[L] \arrow[d, "\hat{f}[L]"] \\ \mathcal{X}[L] \arrow[ur, "\Psi"] \arrow[r, "\mathcal{G}"'] & M/\Gamma_M[L]. \end{tikzcd}$$

Denote by Ψ_t the restriction of Ψ to each fiber $X_t[L]$. Choosing Ψ such that $\Psi \circ \phi : X[L] \to X[L]$ fixes a point, it follows that $\Psi \circ \phi$ is the identity, and so $\Psi_0 = \phi^{-1}$ is a biholomorphism onto $X[L]$. Furthermore, by dimension reasons it follows that Ψ_t is a proper local biholomorphism, i.e. a finite holomorphic covering map. Let $\varphi_t : X_t[L] \to X[L]$ for $t \in \Delta$, be a smooth trivialization. It follows that for each t, $\Psi_t \circ \varphi_t^{-1} : X[L] \to X[L]$ is a finite covering map homotopic to a homeomorphism. In particular, it has degree 1. So Ψ_t is a biholomorphism and $(X_t[L], F_t) \cong (X[L], f[L])$ via the family Ψ_t and the claim follows. □

The proof of Theorem 1.1 follows from the proof of Theorem 5.3, skipping the step that shrinks Δ.

Remark 5.4 (One dimensional case)**.** The following simplifications can be made if $\dim(X) = 1$.

(1) Any non-constant holomorphic map between finite type curves of the same type is proper, so the assumption that F_t is proper can be dropped.
(2) If in addition $X[L]$ is a finite type curve of genus $g > 0$, all assumptions on F_t can be dropped. This is due to the fact that a non-constant holomorphic map between finite type curves of fixed genus at least 1 is either of degree one or unramified.

6. Covering constructions are Weil–Petersson geodesic

The main goal of this section is the proof of Proposition 1.3 relating covering constructions with almost Weil-Petersson complex submanifolds. Note that by a result of Filip (cf. [Fil16, Remark 1.6]) covering constructions project to subvarieties of $\mathcal{M}_{g,n}$, so we only need to show that the images of covering constructions are almost Weil–Petersson complex submanifolds of $\mathcal{T}_{g,n}$.

We start by recalling the definition of covering constructions. Let

$$h : \Sigma' \to \Sigma$$

be a finite degree, orientation-preserving, topological branched cover between topological surfaces Σ' and Σ. Consider finite subsets $u \subset \Sigma$ and $v \subset \Sigma'$. In the following we will assume that

$$h^{-1}(u) = v \cup R(h) \tag{1}$$

for $R(h)$ the set of ramification points of h. In particular, u contains all the branch points of h.

Definition 6.1 (Covering constructions). Pulling back complex structures under the *unbranched covering map*

$$h|_{\Sigma'-h^{-1}(u)} : \Sigma' - h^{-1}(u) \to \Sigma - u$$

induces a holomorphic isometric embedding

$$f_h : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', h^{-1}(u))$$

with respect to the Teichmüller metric. We call f_h a *totally marked covering construction*.
A *covering construction* is a holomorphic map

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

induced by a totally marked covering construction f_h by postcomposition with the forgetful map $\mathcal{F} : \mathcal{T}(\Sigma', h^{-1}(u)) \to \mathcal{T}(\Sigma', v)$. Since $\mathcal{F}$ only forgets ramification points of h, the map f is an isometric embedding [BS24, Proposition A.2]. If the branched cover h is regular we call f a *regular covering construction*.

Remark 6.2. In [BS24, Appendix] it is shown that for $g(\Sigma) \geq 1$ and $\mathcal{T}(\Sigma, u) \neq \mathcal{T}_{1,1}$ the condition given by Equation (1) is necessary and sufficient for h to induce an isometric embedding with respect to the Teichmüller metric.

In the following, we describe in more detail properties of each type of covering construction. Unless otherwise specified $h : (\Sigma', v) \to (\Sigma, u)$ will denote a finite degree, orientation-preserving, topological branched cover. Let $g(\Sigma)$ denote the genus of Σ. Throughout this section we assume that

$$3g(\Sigma) - 3 + |u| > 0.$$

6.1. Totally marked covering constructions.

Let $v = h^{-1}(u) \subset \Sigma'$ and let

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

be the associated totally marked covering construction. The map f is holomorphic and the derivative is given by pulling back Beltrami forms to the cover. We sketch the details. Let $X = \mathbb{H}^2/\Gamma \in \mathcal{T}(\Sigma, u)$, for X a finite type Riemann surface diffeomorphic to $\Sigma - u$ and $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ a discrete subgroup without elliptic elements. Since h is unbranched over $\Sigma - u$, the image $f(X)$ is given by $f(X) = \mathbb{H}^2/\Gamma'$ for $\Gamma' < \Gamma$ a finite index subgroup. It follows that there exists a pushforward lift of Beltrami forms:

$$f_* : M(\Gamma) \to M(\Gamma')$$

given by the inclusion. Furthermore, f_* induces a holomorphic map

$$\mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

which agrees with f.[9] In particular the derivative of f is induced by f_*. Similarly, the pullback of integrable quadratic differentials is given by the inclusion $Q(\Gamma) \to Q(\Gamma')$.

[9]Here it is important that h is orientation preserving, so that it agrees with $f(X) \to X$.

Recall that Harmonic forms $H(X)$ are given by $(z-\bar{z})^2\bar{q}$ for $q \in Q(\Gamma)$. Then, the following corollary is immediate.

Corollary 6.3. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ be a totally marked covering construction. Let $X \in \mathcal{T}(\Sigma, u)$. Then, the derivative df gives an inclusion*

$$df_X : H(X) \hookrightarrow H(f(X)).$$

Furthermore, df_X is induced by the pullback of quadratic differentials and volume forms under h.

Remark 6.4. For Corollary 6.3 it is crucial that the pullback of the complete hyperbolic metric on X is the complete hyperbolic metric on $f(X)$. In particular, it does not apply when we postcompose with forgetful maps.

The Weil–Petersson metric g_{WP} is easily computed for harmonic forms. Thus, we get the following.

Corollary 6.5. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma, v)$ be a totally marked covering construction induced by a branched cover h. Then,*

$$f^* g_{\mathrm{WP}} = \deg(h) g_{\mathrm{WP}}.$$

Proof. Let $X \in \mathcal{T}(\Sigma, u)$. By Corollary 6.3, $df_X : H(X) \hookrightarrow H(f(X))$ is given by the pullback of quadratic differentials and volume forms under h. Let $\mu_i = \overline{q_i}(ds_X^2)^{-1} \in H(X)$, then:

$$\begin{aligned} \langle df_X(\mu_1), df_X(\mu_2)\rangle &= \int_{f(X)} df_X(\mu_1)\overline{df_X(\mu_2)} ds^2_{f(X)} \\ &= \int_{f(X)} df_X(\mu_1) h^*(q_2) \qquad (2) \\ &= \deg(h) \int_X \mu_1 q_2 = \deg(h)\langle \mu_1, \mu_2\rangle. \end{aligned}$$

The third equality follows from the fact that $f^*|_X : Q(f(X)) \to Q(X)$ is given by the trace under h. ☐

6.2. **Regular H-Covers.** Let Y be a closed Riemann surface, with underlying topological surface Σ'. Let H be a finite group of conformal automorphisms of Y. Let $X = Y/H$ so that H defines a regular branched covering

$$h_X : Y \to X.$$

Let Σ be the underlying topological surface of X. Let $h : \Sigma' \to \Sigma$ be the topological branched cover induced by h_X. As in Equation (1), consider finite subsets $u \subset \Sigma$ and $v \subset \Sigma'$ such that

$$h^{-1}(u) = v \cup R(h).$$

Then, h induces a regular covering construction

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v).$$

Suppose that v is H-invariant. Then H is a group of conformal automorphisms of $Y' := Y - v$. Recall that H acts on $\mathcal{T}(\Sigma', v)$ by biholomorphisms. We claim the following,

Lemma 6.6.

$$f(\mathcal{T}(\Sigma, u)) = \mathrm{Fix}(H)$$

for $\mathrm{Fix}(H)$ *the fixed set of H in $\mathcal{T}(\Sigma', v)$.*

This is already known (e.g. [Kra59]), but we provide a proof using the terminology introduced in [BS24].

Proof. Let $M := f(\mathcal{T}(\Sigma, u))$. Note that H is a finite group of biholomorphic isometries of the Teichmüller metric. Thus, $\mathrm{Fix}(H) \subset \mathcal{T}(\Sigma', v)$ is a totally geodesic complex submanifold with respect to the Teichmüller metric. The inclusion $M \subseteq \mathrm{Fix}(H)$ is evident, as H can be represented by biholomorphisms of any $f(Z)$ for $Z \in \mathcal{T}(\Sigma, u)$. To prove the converse, we will consider the totally geodesic bundles QM and $Q\,\mathrm{Fix}(H)$ [BS24, Section 3]. Let $f(X) \in M \subseteq \mathrm{Fix}(H)$ and identify

H with a subgroup of biholomorphisms of $f(X)$. Recall that $Q_{f(X)}M \subset Q(f(X))$ equals the H-invariant differentials on $Q(f(X))$. We claim that $Q_{f(X)}\operatorname{Fix}(H)$ admits the same description. In particular, $Q_{f(X)}\operatorname{Fix}(H) = Q_{f(X)}M$ and we are done.

To prove the claim, let $t \in H$ and denote by $\mathfrak{t} : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma', v)$ the biholomorphism induced by changing the marking by t. Since $\mathfrak{t}(f(X)) = f(X)$, the umkehr map $\mathfrak{t}_!$ [BS24, Section 4] associated to $\mathfrak{t}$ acts on $Q(f(X))$ by:

$$\mathfrak{t}_! : Q(f(X)) \to Q(f(X)) \quad , \quad q \to t^*(q) \in Q(f(X)).$$

Any geodesic in $\operatorname{Fix}(H)$ is fixed by $\mathfrak{t}$. Thus, $q = t^*(q)$ for any $q \in Q_{f(X)}\operatorname{Fix}(H)$. The claim follows. □

H acts on $\mathcal{T}(\Sigma', v)$ by Weil–Petersson isometries, thus the following is immediate.

Corollary 6.7. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ be a regular covering construction induced by a regular covering with deck group H. Assume that v is H-invariant. Then $f(\mathcal{T}(\Sigma, u))$ is totally geodesic with respect to the Weil–Petersson metric, i.e. it is Weil–Petersson.*

In general, we have a weaker statement.

Corollary 6.8. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ be a regular covering construction, induced by a regular cover $h : \Sigma' \to \Sigma$. If $(g(\Sigma), g(\Sigma')) = (0, 1)$ assume further that $|u| \geq 5$. Then $f(\mathcal{T}(\Sigma, u))$ is almost Weil–Petersson.*

Proof. Let $v' = v \setminus R(h) = h^{-1}(u \setminus B(h))$, for $B(h)$ the branch locus of h. Then, v' is H-invariant. We claim that $g(\Sigma') - 3 + |v'| > 0$. This is immediate if $g(\Sigma') > 1$ so assume otherwise.

We proceed by cases. If $g(\Sigma) = 1$ the map h is actually unramified and $|v'| = |u| \geq 1$. For the remaining cases we use the following facts from [Mir95, Lemma 3.8 and discussion thereafter]:

(1) The map h can have at most 4 branch points and less than 4 if $g(\Sigma') = 0$.
(2) If $g(\Sigma') = 0$ and h has 3 branch points, then $\deg(h) \geq 4$.

The claim follows. The forgetful map

$$\mathcal{F} : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma', v')$$

satisfies that $M' = \mathcal{F}(f(\mathcal{T}(\Sigma, u))$ is Weil–Petersson (Corollary 6.7). Moreover, as $\mathcal{F}$ only forgets ramification points, $\mathcal{F}$ restricts to a biholomorphism between $f(\mathcal{T}(\Sigma, u))$ and M'. The result follows. □

Combining Corollaries 6.5 and 6.7, we complete the proof of Proposition 1.3 by showing the following.

Proposition 6.9. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma, v)$ be a totally marked covering construction induced by a branched cover h. Then, $f(\mathcal{T}(\Sigma, u))$ is a Weil–Petersson complex submanifold.*

Proof. The case of totally marked regular coverings follows from Corollary 6.7. So assume that h is not regular. Let

$$h_2 : (\Sigma'', w) \to (\Sigma', v)$$

be the *normal closure* of h, i.e. the regular cover induced by taking a finite index subgroup $\Gamma < h_*(\pi_1(\Sigma' - v)) < \pi_1(\Sigma - u))$ such that $\Gamma \trianglelefteq \pi_1(\Sigma - u)$. This means that there is a commutative diagram:

$$\begin{tikzcd}
(\Sigma'', w) \arrow[dd, "h_1"'] \arrow[dr, "h_2"] & \\
& (\Sigma', v) \arrow[dl, "h"] \\
(\Sigma, u) &
\end{tikzcd}$$

where h_i for $i = 1, 2$ are regular covers and the marked points satisfy

$$h_1^{-1}(u) = h_2^{-1}(v) = w.$$

By Corollary 6.7, the induced maps

$$\begin{tikzcd} \mathcal{T}(\Sigma'', w) & \\ & \mathcal{T}(\Sigma', v) \arrow[ul, "f_2"'] \\ \mathcal{T}(\Sigma, u) \arrow[uu, "f_1"] \arrow[ur, "f"'] & \end{tikzcd}$$

satisfy that $f_1(\mathcal{T}(\Sigma, u))$ and $f_2(\mathcal{T}(\Sigma', v))$ are Weil–Petersson submanifolds of $\mathcal{T}(\Sigma'', w)$. Since $f_1 = f_2 \circ f$, it follows that $f_1(\mathcal{T}(\Sigma, u)) \subset f_2(\mathcal{T}(\Sigma', v))$ and

$$f(\mathcal{T}(\Sigma, u)) = f_2^{-1}(f_1(\mathcal{T}(\Sigma, u))).$$

In particular, $f_1(\mathcal{T}(\Sigma, u))$ is a totally geodesic submanifold of $f_2(\mathcal{T}(\Sigma', v))$ with respect to g_{WP}. The map $f_2 : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma'', w)$ is a totally marked covering construction. Thus, by Corollary 6.5 it is (up to a constant) a Riemannian isometric embedding for g_{WP}. The claim follows. $\square$

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