LOCAL RIGIDITY OF WEIL–PETERSSON SUBVARIETIES OF THE MODULI SPACE OF CURVES AND COVERING CONSTRUCTIONS

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Abstract. We show that totally geodesic subvarieties of the moduli space $\mathcal{M}_{g,n}$ of genus g curves with n marked points, endowed with the Weil–Petersson metric, are locally rigid. This implies that covering constructions—examples of totally geodesic subvarieties of $\mathcal{M}_{g,n}$ endowed with the Teichmüller metric—are locally rigid. We deduce the local rigidity statement from a more general rigidity result for a class of orbifold maps to $\mathcal{M}_{g,n}$.

1. Introduction

Let $\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the projection from the Teichmüller space of a genus g surface with n marked points to the associated moduli space. The Weil–Petersson metric g_{WP} on $\mathcal{T}_{g,n}$ is a Kähler metric that descends to a metric on $\mathcal{M}_{g,n}$. The metric g_{WP} gives $\mathcal{M}_{g,n}$ the structure of a quasi-projective variety [Wol85].

A complex submanifold $M \subset \mathcal{T}_{g,n}$ is called *Weil–Petersson* if it is totally geodesic with respect to the Weil–Petersson metric g_{WP}. This means that the g_{WP}-geodesic between any two points in M is completely contained in M. Similarly, a subvariety $N \subset \mathcal{M}_{g,n}$ is called *Weil–Petersson* if an irreducible component M of $\pi^{-1}(N)$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$. The complex submanifold M is called a *lift* of N. More generally, we call a subvariety $N \subset \mathcal{M}_{g,n}$ *almost Weil–Petersson* if it has a lift $M \subset \mathcal{T}_{g,n}$ that maps biholomorphically via a forgetful map $\mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ onto a Weil–Petersson complex submanifold of some $\mathcal{T}_{g,m}$.

We say that a subvariety $N \subset \mathcal{M}_{g,n}$ is *locally rigid* if any holomorphic deformation

$$f_t : N_t \to \mathcal{M}_{g,n} \quad , \quad t \in \Delta$$

with $(N_0, f_0) \cong (N, \iota : N \hookrightarrow \mathcal{M}_{g,n})$ through proper immersions f_t and quasi-projective varieties N_t is trivial: there is a holomorphic family of biholomorphisms $g_t : N_t \to N$ inducing $(N_t, f_t) \cong (N, \iota)$. Our maps should be understood in the orbifold sense, i.e. on a level $L \geq 3$ cover of $\mathcal{M}_{g,n}$ (see Definition 5.2 for details). Our main theorem is the following.

Theorem 1.1 (**Local Rigidity**). *Suppose $3g - 3 + n > 0$. Let $N \subset \mathcal{M}_{g,n}$ be an almost Weil–Petersson subvariety of positive dimension. Then N is locally rigid.*

Examples. A source of examples of almost Weil–Petersson subvarieties are given by *covering constructions* [BS24, MMW17], defined as follows. Let

$$h : \Sigma_{g',n'} \to \Sigma_{g,n}$$

be a finite degree, orientation-preserving, topological branched cover. Assume further that the preimage of the marked points in $\Sigma_{g,n}$ equals the union of the marked points in $\Sigma_{g',n'}$ and the ramification points of h. Pulling back complex structures under h gives a holomorphic map

$$f_h : \mathcal{T}_{g,n} \to \mathcal{T}_{g',\ell}$$

for some $\ell \geq n'$. We call f_h a *totally marked* covering construction (see Section 6 for more details). A *covering construction* is a holomorphic map

$$f : \mathcal{T}_{g,n} \to \mathcal{T}_{g',n'}$$

given by the composition of a totally marked covering construction $f_h : \mathcal{T}_{g,n} \to \mathcal{T}_{g',\ell}$ with a forgetful map $\mathcal{T}_{g',\ell} \to \mathcal{T}_{g',n'}$ which forgets *only* ramification points of h. If the map h is a regular branched cover, we call f a *regular covering construction*. The link between covering constructions and almost totally geodesic submanifolds of $\mathcal{T}_{g,n}$ is the following.

Proposition 1.2. *Assume $3g-3+n>0$. Let $f:\mathcal{T}_{g,n}\to\mathcal{T}_{g',n'}$ be a covering construction. The following holds:*

1. *Assume f is totally marked. Then, the image $f(\mathcal{T}_{g,n})$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g',n'}$.*
2. *Assume f is regular and not totally marked. If $(g,g')=(0,1)$ assume further that the marked points satisfy $n\geq 5$. Then, the image $f(\mathcal{T}_{g,n})$ is an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g',n'}$.*

In particular, in either case the projection $\pi(f(\mathcal{T}_{g,n}))\subset\mathcal{M}_{g',n'}$ is an almost Weil–Petersson subvariety.

We now give two applications of Theorem 1.1.

Application 1. Totally geodesic subvarieties for the Teichmüller metric. Replacing the Weil–Petersson metric by the *Teichmüller metric* d_{Teich} in our definition of Weil–Petersson subvarieties gives *Teichmüller* subvarieties $N\subset\mathcal{M}_{g,n}$ and *Teichmüller* complex submanifolds $M\subset\mathcal{T}_{g,n}$ [Wri20, AHW24, BDR24].

1-dimensional Teichmüller subvarieties are called *Teichmüller curves*. McMullen showed [McM09] that Teichmüller curves are rigid.[1] Arana-Herrera–Wright [AHW24, Question 10.6] ask if the same holds for Teichmüller varieties of higher dimensions. The projection of images of covering constructions are Teichmüller subvarieties of $\mathcal{M}_{g,n}$ and so Theorem 1.1 implies:

Corollary 1.3 (**Local rigidity of Covering constructions**). *Assume $3g-3+n>0$. Let $f:\mathcal{T}_{g,n}\to\mathcal{T}_{g',n'}$ be a totally marked or regular covering construction, satisfying the same assumptions of Proposition 1.2. The projection $\pi(f(\mathcal{T}_{g,n}))\subset\mathcal{M}_{g',n'}$ is locally rigid.*

Remark 1.4. Any covering construction arises by applying forgetful maps to a totally marked covering construction. Thus, up to forgetful maps, all covering constructions are locally rigid.

Remark 1.5. The images of covering constructions encompass all of the known examples of Teichmüller complex submanifolds of $\mathcal{T}_{g,n}$ of dimension bigger than 1, except for the family (Y_n) of 2-dimensional examples found by [EMMW20, MMW17]. In this regard, the following questions are quite pertinent.

Question 1.6. Are the Teichmüller surfaces Y_n totally geodesic with respect to the Weil–Petersson metric, i.e. are Y_n also Weil–Petersson surfaces?

Question 1.7. Given a subvariety $N\subset\mathcal{M}_{g,n}$. Suppose that N is totally geodesic with respect to *both* the Teichmüller and the Weil–Petersson metric. Is N the image of a covering construction?

Application 2. Characterization of almost Weil–Petersson submanifolds. The pure mapping class group $\text{PMod}(\Sigma_{g,n}):=\pi_0(\text{Diff}^+(\Sigma_{g,n},\{x_1,\dots,x_n\}))$ acts on $\mathcal{T}_{g,n}$ by biholomorphisms. For an arbitrary analytic subset $W\subset\mathcal{T}_{g,n}$, let Γ_W be the stabilizer of W in $\text{PMod}(\Sigma_{g,n})$. Let M be an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$ of positive dimension. As a corollary of the proof of Theorem 1.1 (see Theorem 3.5), we obtain the following characterization of M.

Corollary 1.8 (**Maximality**). *Let $3g-3+n>0$. Assume that $\pi(M)\subset\mathcal{M}_{g,n}$ is a subvariety. Then, the submanifold M is maximal among the analytic subsets W of $\mathcal{T}_{g,n}$ such that the following holds:*

1. *$\Gamma_W\subseteq\Gamma_M$.*
2. *W/Γ_W is a quasi-projective variety.*

Remark 1.9. Corollary 1.8 showcases the following rigidity result for the pair (M,Γ_M). Inclusion at the level of the stabilizers $\Gamma_W<\Gamma_M$ implies the inclusion of *spaces* $W\subset M$. A similar statement holds for families of curves over quasi-projective varieties: if the monodromy factors

[1] McMullen considers a more general class of local deformations, not assuming that f_t is a proper immersion. For the one-dimensional case, our proof of Theorem 1.1 only requires f_t to be an immersion, and so it gives an alternative proof of McMullen's result for the case of covering constructions.

through Γ_M, then the classifying map to $\mathcal{T}_{g,n}$ factors through M (see Theorem 3.5 for a precise statement).

Paper overview. Let $M \subset \mathcal{T}_{g,n}$ be an almost Weil–Petersson lift of a subvariety of $\mathcal{M}_{g,n}$, and let Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. In Section 2 we quickly review all the necessary material from Teichmüller theory needed for the rest of the paper, focusing on the Weil–Petersson metric. In Section 3 we introduce the concept of a Γ_M-deformation: a class of orbifold maps $\phi : X \to \mathcal{M}_{g,n}$ which includes the classifying maps of families of curves over quasi-projective varieties whose monodromy factors through Γ_M. Theorem 1.1 is a consequence of the stronger rigidity result for Γ_M-deformations given in Theorem 3.5: *any nonconstant Γ_M-deformation factors through M*. Section 3 also shows that Γ_M is the orbifold fundamental group of the *normalization* of $\pi(M)$. In Section 4 we provide the details of the proof of Theorem 3.5 for the case of X a finite type curve, starting with an elementary Riemannian geometry lemma and then following the argument appearing in the proof of the Imayoshi–Shiga theorem given in [AAS18, Section 4]. The key observation is that, due to the arguments of [AAS18], *holomorphic maps minimize energy in their homotopy class.*[2] In Section 5 we finish the proof of Theorem 3.5 and show how it implies the rigidity stated in Theorem 1.1 Finally, in Section 6 we prove Proposition 1.2 (see also a remark in page 2 of [KM08] for the totally marked case) thereby concluding the proof of Corollary 1.3.

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2. Teichmüller geometry and the Weil–Petersson metric

2.1. **Basic definitions.** Assume $3g-3+n>0$ and let $\Sigma_{g,n}$ be a topological surface of genus g with n-marked points (or n-punctures). A marking is a homeomorphism $\varphi : \Sigma_{g,n} \to X$ from $\Sigma_{g,n}$ to a Riemann surface X. Teichmüller space $\mathcal{T}_{g,n}$ is the space of equivalence classes of markings $[\varphi : \Sigma_{g,n} \to X]$, where two markings are equivalent if they differ up to isotopy by a biholomorphism of X. Unless we need to stress the marking we will denote $[\varphi : \Sigma_{g,n} \to X]$ simply by X. In case we need to stress the subset of points being marked we introduce the following notation: Let Σ be a topological surface and $u \subset \Sigma$ be a finite subset of points. $\mathcal{T}(\Sigma, u)$ denotes the Teichmüller space of Σ with the marked points given by the subset u.

The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$—the group of orientation-preserving diffeomorphisms of $\Sigma_{g,n}$ up to isotopy—acts on $\mathcal{T}_{g,n}$ by biholomorphisms. In fact, Royden and Earl–Kra [Roy71, EK74] showed that for $2g+n>4$, $\mathrm{Mod}(\Sigma_{g,n})$ agrees with the group of biholomorphisms of $\mathcal{T}_{g,n}$. The pure mapping class group $\mathrm{PMod}(\Sigma_{g,n})$ is the subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ fixing pointwise the marked points. The quotient of $\mathcal{T}_{g,n}$ by $\mathrm{PMod}(\Sigma_{g,n})$ induces the projection

$$\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\,\mathrm{PMod}(\Sigma_{g,n}).$$

Let $B(X)$ denote the space of *Beltrami forms* on X, i.e. $(-1,1)$-forms μ on X with $\|\mu\|_\infty < \infty$. Denote by $M(X)$ the open unit ball in $B(X)$. There exists a holomorphic submersion

$$\Phi : M(X) \to \mathcal{T}_{g,n} \quad , \quad \mu \to [f^\mu \circ \varphi : \Sigma_{g,n} \to X_\mu]$$

where f^μ is given by solving (locally) the Beltrami equation:

$$\partial_{\bar{z}} f = \mu \partial_z f.$$

In particular, $T_X \mathcal{T}_{g,n}$ is naturally identified with a quotient of $B(X)$.

[2] For compact C the proof follows from the theory of harmonic maps (cf. [ES64]), but an extra step is needed when C is finite volume.

Let $[\varphi : (\Sigma, u) \to (X, \varphi(u))] \in \mathcal{T}(\Sigma, u)$. Let $Q(X, u)$ denote the space of integrable holomorphic quadratic differentials on X, i.e. meromorphic quadratic differentials q on X with poles of order at most one, and all poles contained in u. Equivalently, $Q(X, u)$ is the space of holomorphic quadratic differentials q on $X - \varphi(u)$ so that

$$\int_{X-\varphi(u)} |q| < \infty.$$

When the marked points are not being stressed we denote $Q(X, u)$ simply by $Q(X)$. There is a natural pairing

$$B(X) \times Q(X) \to \mathbb{C} \quad , \quad (\mu, q) \mapsto \int_X \mu q$$

which factors through $T_X\mathcal{T}_{g,n} \times Q(X)$. In particular, $Q(X)$ is naturally identified with $T_X^*\mathcal{T}_{g,n}$.

At the level of the universal cover and working always with punctures.[3], we have the following notation. Let $u \subset \Sigma$ be a finite set of points and $\varphi : (\Sigma, u) \to (X, \varphi(u))$ be a marked Riemann surface. Let $X' = X - \varphi(u)$ and let $\pi : \mathbb{H}^2 \to X'$ be the universal cover of X', with deck group Γ. Then $B(X')$ is given by the Γ-invariant $(-1, 1)$-forms on $\mathbb{H}^2$, denoted by $B(\Gamma)$. Similarly, $Q(X')$ is given by $Q(\Gamma)$, the Γ-invariant quadratic differentials on $\mathbb{H}^2$ inducing an integrable quadratic differential on X'.

2.2. **Weil–Petersson metric.** The *Weil–Petersson* metric, denoted by g_{WP}, is induced from the hermitian product on $Q(X)$ given by

$$\langle q_1, q_2 \rangle_{\mathrm{WP}} = \int_X q_1 \overline{q_2} (ds^2)^{-1},$$

where ds^2 is the hyperbolic volume form. The hermitian product on $T\mathcal{T}(X)$ is given by

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP}} = \overline{\langle \mu_1^*, \mu_2^* \rangle}_{\mathrm{WP}}$$

where $\mu_i^* \in Q(X)$ is the dual of μ_i. This means that μ_i^* is characterized by

$$\langle q, \mu_i^* \rangle_{\mathrm{WP}} = \int_X \mu_i q, \quad \forall q \in Q(X).$$

The metric g_{WP} is incomplete, Kähler, negatively curved and geodesically convex [Wol87].

Harmonic forms. The map

$$q \mapsto \overline{q} (ds^2)^{-1}$$

sends a quadratic differential to an element of $B(X)$ which is called a *harmonic form*. Let $H(X)$ denote the space of harmonic forms on X. The Weil–Petersson metric has a simple expression on harmonic forms: Let $\mu_i = \overline{q_i}(ds^2)^{-1} \in H(X)$ for $i = 1, 2$. Evidently $\mu_i^* = q_i$, and so

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP}} = \int_X \overline{q_1} q_2 (ds^2)^{-1} = \int_X \mu_1 \overline{\mu_2} ds^2 = \int_X \mu_1 q_2.$$

Definition 2.1 (**Almost Weil–Petersson**)**.** Given a complex submanifold $M \subset \mathcal{T}_{g,n}$, we say that M is *Weil-Petersson* if it is totally geodesic with respect to g_{WP}.[4] More generally, we say that M is *almost Weil–Petersson* if there exists a forgetful map $\mathcal{F} : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m}$ with $n \geq m$ and $3g - 3 + m > 0$. Such that

(1) $\mathcal{F}|_M : M \to \mathcal{F}(M)$ is a biholomorphism
(2) $\mathcal{F}(M)$ is a Weil-Petersson complex submanifold of $\mathcal{T}_{g,m}$.

Let $N \subset \mathcal{M}_{g,n}$ be a subvariety, a *lift* of N is an irreducible component of the preimage of N in $\mathcal{T}_{g,n}$. N is called *(almost) Weil–Petersson* if it has an *(almost) Weil–Petersson* lift.

[3]A similar description is possible for marked points by using groups with elliptic elements [EK76].

[4]It is not hard to see that any totally geodesic analytic subset of $\mathcal{T}_{g,n}$ with respect to g_{WP} must in fact be a submanifold.

3. Γ_M-deformations and the associated orbifold fundamental group

Let $\pi : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ be the projection to moduli space. Let $N \subset \mathcal{M}_{g,n}$ be a subvariety and let $M \subset \mathcal{T}_{g,n}$ be a lift of N. Let Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. In this section we introduce the concept of a Γ_M-deformation which generalizes an important property of classical deformations to the orbifold category. Γ_M-deformations are central to our reformulation of Theorem 1.1, stated at the end of this section (Theorem 3.5). We start by proving some facts about the group $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$.

The associated orbifold fundamental group. The projection π induces a holomorphic map

$$\pi_M : M/\Gamma_M \to N.$$

The group Γ_M is called the *associated orbifold fundamental group* of N(cf. [AHW24]). The following explains the terminology.

Lemma 3.1 (cf. [GDH92, Theorem 1])**.** *Suppose $3g-3+n>0$. Given a subvariety $N \subset \mathcal{M}_{g,n}$ with lift $M \subset \mathcal{T}_{g,n}$, let Γ_M be the stabilizer of M in $\mathrm{PMod}(\Sigma_{g,n})$. The induced holomorphic map*

$$\pi_M : M/\Gamma_M \to N$$

is proper with finite fibers and injective outside a proper analytic subvariety. In particular, it is surjective. Furthermore, if M is smooth π_M gives the normalization of N and M/Γ_M is a quasi-projective variety.

Proof. We need to show that the projection π_M satisfies the following:

(1) has finite fibers,
(2) is closed,
(3) is injective outside a proper analytic subvariety.

The family of closed subsets $\{gM : g \in \mathrm{PMod}(\Sigma_{g,n})\}$ in $\mathcal{T}_{g,n}$ is a subfamily of the decomposition of $\pi^{-1}(N)$ into its irreducible components. Thus, it is a locally finite family of closed subsets of $\mathcal{T}_{g,n}$. This means that for every $x \in \mathcal{T}_{g,n}$, there exists a neighborhood U_x of x intersecting only finitely many g_iM. This automatically gives that π_M has finite fibers, and an elementary argument shows that π_M is closed.

By the Semi-Proper Mapping Theorem [Fis76, Theorem 1.19], the map $M \to M/\Gamma_M$ sends Γ_M-invariant analytic subsets of M to analytic subsets of M/Γ_M. The map π_M is injective on the complement of the projection of the subset

$$W := \bigcup_{g \in \Gamma_M} \{gM \cap M : g(M) \cap M \text{ proper subvariety of } M\} \subset M.$$

W is analytic and Γ_M-invariant. Thus, the projection of W to M/Γ_M is analytic and the third item follows. Since $\pi_M : M/\Gamma_M \to N$ is proper, the proper mapping theorem implies that π_M is surjective.

If M is smooth then M/Γ_M is a normal analytic space by a theorem of Cartan [Car57] and the theorem follows. □

Remark 3.2. As the examples in [GDH92,HPRCR24] show, π_M is not in general a biholomorphism.

Level L-structures. Let $L \geq 3$ and $3g-3+n>0$. Let $\mathrm{PMod}(\Sigma_{g,n})[L]$ be the level L pure mapping class group [FM12, Chapter 6.4.2]. Let

$$\Gamma_M[L] := \Gamma_M \cap \mathrm{PMod}(\Sigma_{g,n})[L].$$

Since $L \geq 3$, $\mathrm{PMod}(\Sigma_{g,n})[L]$ is torsion free. Thus, the induced map

$$M \to M/\Gamma_M[L]$$

is an unramified covering map. Let

$$\pi_L : \mathcal{T}_{g,n} \to \mathcal{M}_{g,n}[L]$$

be the level-L projection to the moduli space of curves with level L structures. The same argument as in lemma 3.1 shows that the induced map

$$\pi_{M[L]} : M/\Gamma_M[L] \to \pi_L(M) \subset \mathcal{M}_{g,n}[L]$$

is a biholomorphism: indeed, $\pi_{M[L]}$ is a local homeomorphism by covering space theory. Since it is also proper it is a covering map, but the argument in Lemma 3.1 shows that it has degree one. The image $N[L] := \pi_L(M)$ is an irreducible component of the preimage of N to $\mathcal{M}_{g,n}[L]$.

Definition 3.3 (**Algebraic Orbifold**)**.** Let $\widetilde{X}$ be a reduced complex space, and Γ_X a group acting properly discontinuously on $\widetilde{X}$ by biholomorphisms, we denote the quotient by $X = \widetilde{X}/\Gamma$. Assume that there exists a finite index subgroup $\Gamma'_X < \Gamma_X$ acting freely on $\widetilde{X}$ and that the induced map $X' := \widetilde{X}/\Gamma'_X \to X$ is a finite ramified covering.

We say that the orbifold $X = \widetilde{X}/\Gamma_X$ with orbifold structure $(\widetilde{X}, \Gamma_X)$ is *algebraic* if either one of the following is satisfied:

(1) X has a finite (proper with finite fibers) ramified covering map to a quasiprojective variety $\overline{X}$.
(2) X' is quasiprojective.

Examples of algebraic orbifolds are given by quasi-projective varieties and by M/Γ_M for M a lift of a subvariety of $\mathcal{M}_{g,n}$.

Classical deformations. Let Z, W be two complex manifolds. A local deformation of a holomorphic map $f : Z \to W$ is given by a family of holomorphic maps $f_t : Z_t \to W$ for $t \in \Delta$, with a biholomorphism $\varphi : Z_0 \cong Z$ inducing $(Z_0, f_0) \cong (Z, f)$. If the family (Z_t) is smoothly trivial,[5] it follows that the groups $(f_t)_*(\pi_1(Z_t)) < \pi_1(W)$ are pairwise conjugate. The following is our generalization to the orbifold setting of a deformation of the map $\pi_M : M/\Gamma_M \to N \subset \mathcal{M}_{g,n}$.

Definition 3.4 (Γ**-deformation**)**.** Let $X = \widetilde{X}/\Gamma_X$ be an algebraic orbifold. Let $\phi : X \to \mathcal{M}_{g,n}$ be a holomorphic map of complex orbifolds with associated homomorphism $\phi_* : \Gamma_X \to \mathrm{PMod}(\Sigma_{g,n})$. This means that there is a holomorphic map $\tilde{\phi} : \widetilde{X} \to \mathcal{T}_{g,n}$ equivariant with respect to the homomorphism ϕ_* and fitting into the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{\phi}} & \mathcal{T}_{g,n} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\phi]{} & \mathcal{M}_{g,n} \end{array}$$

Let $\Gamma < \mathrm{PMod}(\Sigma_{g,n})$ be an arbitrary subgroup. We say that ϕ is a Γ-*deformation* if, up to conjugation in $\mathrm{PMod}(\Sigma_{g,n})$, the homomorphism ϕ_* factors through the inclusion $\Gamma \hookrightarrow \mathrm{PMod}(\Sigma_{g,n})$.

Our main rigidity result is the following variant of the Imayoshi-Shiga theorem (see also Theorem 4.10).

Theorem 3.5. *Let $3g - 3 + n > 0$. Let M be an almost Weil–Petersson complex submanifold of $\mathcal{T}_{g,n}$. Let $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$ be the stabilizer of M. Let $X = \widetilde{X}/\Gamma_X$ be an algebraic orbifold and $\phi : X \to \mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. Then, there is a lift $\tilde{\phi} : \widetilde{X} \to \mathcal{T}_{g,n}$ of ϕ that factors through the inclusion $M \hookrightarrow \mathcal{T}_{g,n}$.*

4. The case of a finite type curve C

On this section we prove Theorem 3.5 for the case of X a smooth finite type curve. We start with the following special case.

Proposition 4.1. *Suppose $3g-3+n > 0$. Given a Weil–Petersson complex submanifold $M \subset \mathcal{T}_{g,n}$ with stabilizer $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$. Let C be a smooth finite type curve and $\phi : C \to \mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. Then, there exists a lift of ϕ to $\mathcal{T}_{g,n}$ that factors through the inclusion $M \hookrightarrow \mathcal{T}_{g,n}$.*

[5] note that we are *not assuming* that f is proper.

The first step in the proof of Proposition 4.1 is the following elementary observation, whose proof we include for completeness.

Lemma 4.2 (**Orthogonal Projection**). *Suppose* $3g-3+n>0$. *Let* Z *be a totally geodesic submanifold of* $\mathcal{T}_{g,n}$ *with respect to the Weil–Petersson metric and let* Γ_Z *be the stabilizer of* Z *in* $\mathrm{Mod}(\Sigma_{g,n})$. *There exists a* Γ_Z*-equivariant smooth deformation retraction onto* Z

$$\pi_Z^t : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n} \quad , \quad t \in [0,1]$$

with the following property: for any $w \in T\mathcal{T}_{g,n}$ *the map*

$$t \to \|d\pi_Z^t(w)\|_{\mathrm{WP}}$$

is non-increasing. If $w \notin TZ$ *and* w *is not parallel to a geodesic normal to* Z *then* $\|d\pi_Z^t(w)\|_{\mathrm{WP}}$ *is strictly decreasing.*

Proof. Endow $\mathcal{T}_{g,n}$ with the Weil–Petersson metric g_{WP}. Let $\mathcal{E} \subset T\mathcal{T}_{g,n}$ be the domain of the exponential map. Since g_{WP} is incomplete $\mathcal{E} \neq T\mathcal{T}_{g,n}$. Let $\mathcal{E}_Z := \mathcal{N}Z \cap \mathcal{E}$ for $\mathcal{N}Z$ the normal bundle of Z. The metric g_{WP}) is negatively curved, thus

$$\exp : \mathcal{E}_Z \to \mathcal{T}_{g,n}$$

is injective and a diffeomorphism onto its image. Since g_{WP} is not complete, a priori there is no reason for exp to be surjective. Yet, Wolpert showed that for any $p \in \mathcal{T}_{g,n}$, the map $\exp_p : \mathcal{E}_p \to \mathcal{T}_{g,n}$ is a homeomorphism [Wol87, Corollary 5.4] , and this implies surjectivity of exp.

Let $\phi_t : \mathcal{E}_Z \to \mathcal{E}_Z$ be given by $(p,v) \to (p,tv)$. Define

$$\pi_Z^t : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n} \quad , \quad \pi_Z^t := \exp \circ \phi_{1-t} \circ \exp^{-1}.$$

Since exp and ϕ_t are Γ_Z-equivariant, it follows that π_Z^t is Γ_Z-equivariant. The map π_Z^1 is precisely the orthogonal projection map to Z.

Let $p \in \mathcal{T}_{g,n}$ and $w \in T_p\mathcal{T}_{g,n}$. Observe that $J(t) = d\pi_Z^{1-t}(w)$ is a Jacobi field. A standard computation, using the fact that Z is totally geodesic, shows that $\|J(t)\|_{\mathrm{WP}}$ is non-decreasing. Furthermore, if $p \notin Z$ and w is not parallel to a geodesic normal to Z it follows that $J(t)$ has a non-trivial normal component and so $\|J(t)\|_{\mathrm{WP}}$ is strictly increasing. □

Having Lemma 4.2 at hand, we can prove Proposition 4.1.

Proof of Proposition 4.1. Let $M \subset \mathcal{T}_{g,n}$ be a Weil–Petersson complex submanifold. By Lemma 4.2 there is a smooth homotopy

$$\pi_M^t : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n} \quad , \quad t \in [0,1]$$

between the identity and the orthogonal projection $\pi_M^1 : \mathcal{T}_{g,n} \to M$. Let $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$ be the stabilizer of M. Let C be a smooth finite type curve and $\phi : C \to \mathcal{M}_{g,n}$ be a nonconstant Γ_M-deformation. In particular, the lift $\tilde{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$, which equivariant with respect to the homomorphism $\phi_* : \pi_1(C) \to \mathrm{PMod}(\Sigma_{g,n})$, is *not constant*. Up to post-composing with a mapping class, we can assume that $\phi_* : \pi_1(C) \to \Gamma_M$. Define the map

$$\tilde{\psi} : \mathbb{H}^2 \to \mathcal{T}_{g,n} \quad , \quad \tilde{\psi} = \pi_M^1 \circ \tilde{\phi}.$$

Since π_M^1 is Γ_M-equivariant, it follows that $\tilde{\psi}$ is ϕ_*-equivariant. Proposition 4.1 will be proven if we show that $\tilde{\phi} = \tilde{\psi}$.

Claim 4.3. $\tilde{\phi} = \tilde{\psi}$

Proof of Claim 4.3. The proof follows the argument appearing in the proof of the Imayoshi–Shiga theorem given in Section 4 of [AAS18]. We include the details for completeness, clarifying the equivariance nature of all the maps being used— hence the difference in our notation.

For a smooth map $\tilde{f} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ define the *energy density* $E_x(\tilde{f})$ of $\tilde{f}$ at a point $x \in \mathbb{H}^2$ to be

$$E_x(\tilde{f}) := \|d\tilde{f}|_x(v_1)\|_{\mathrm{WP}}^2 + \|d\tilde{f}|_x(v_2)\|_{\mathrm{WP}}^2,$$

for an arbitrary orthonormal basis $\{v_1, v_2\}$ of $T_x\mathbb{H}^2$. Assume further that $\tilde{f}$ is ϕ_*-equivariant, then $E_x(\tilde{f})$ is $\pi_1(C)$-invariant and we can define the *equivariant energy* of $\tilde{f}$ as:

$$E(f) := \int_C E_{\bar{x}}(f)\omega_C$$

where ω_C is the volume form on C and $E_{\bar{x}}(f)$ is given by $E_x(\tilde{f})$ for any lift x of $\bar{x}$.

Since $\tilde{\phi}$ is holomorphic and d_{Teich} agrees with the Kobayashi metric [Roy71], it follows that $\|d\tilde{\phi}(v)\|_{\text{Teich}} \leq \|v\|_{\mathbb{H}^2}$. Recall that the Teichmüller metric dominates the Weil–Petersson metric

$$\|v\|_{\text{WP}} \leq L\|v\|_{\text{Teich}},$$

where $L = |2\pi(2g-2+n)|^{1/2}$ [McM00, Proposition 2.4]. Thus, $\|d\tilde{\phi}(v)\|_{\text{WP}} \leq L\|v\|_{\mathbb{H}^2}$ and we have the following

Lemma 4.4 ([AAS18, Lemma 4.1])**.** *The homotopy*

$$\tilde{F} : \mathbb{H}^2 \times [0,1] \to \mathcal{T}_{g,n}, \quad \tilde{f}_t(x) = \pi_M^t(\tilde{\phi}(x))$$

between $\tilde{\phi}$ and $\tilde{\psi}$ satisfies:

1. *$\tilde{f}_t$ is L-Lipschitz for all t. Thus, the equivariant energy $E(f_t)$ is finite for all t.*
2. *The map $t \to E(f_t)$ is non-increasing along t. If $\tilde{\phi} \neq \tilde{\psi}$, then $E(f_t)$ is not constant.*
3. *For any two points $\bar{x}, \bar{y} \in C$ let $d_C(\bar{x}, \bar{y})$ be their distance in C. Let $x_0 \in \mathbb{H}^2$ and let $\bar{x}_0 \in C$ be its image in C. There are constants $A, B > 0$ such that for all $(x,t) \in \mathbb{H}^2 \times [0,1]$ the operator norm $\|d\tilde{F}_{(x,t)}\|$ satisfies*

$$\|d\tilde{F}_{(x,t)}\|^2 \leq Ad_C(\bar{x}_0, \bar{x})^2 + B.$$

Remark 4.5. Note that since the homotopy π_M^t is norm non-increasing we get stronger results than [AAS18, Lemma 4.1].

The proof of Claim 4.3 is completed by the two following key results. Let ω_{WP} be the Kähler form associated to the Weil–Petersson metric.

Proposition 4.6 (cf. [ES64, Proposition 4.2])**.** *Let $f : \mathbb{H}^2 \to (\mathcal{T}_{g,n}, g_{\text{WP}})$ be a smooth map. Then, for any $x \in \mathbb{H}^2$*

$$E_x(f)\omega_{\mathbb{H}^2} \geq f^*(\omega_{\text{WP}})|_x$$

with equality if f is holomorphic at x.

Remark 4.7. Note that [ES64, Proposition 4.2] is stated for the energy and not the energy density, but their results are given by pointwise estimates for the energy density that imply proposition 4.6. Furthermore, their results give an if and only if condition for equality; this is not necessary for our proof.

For each t, let $f_t^*\omega_{\text{WP}}$ be the 2-form on C induced by the $\pi_1(C)$-invariant form $\tilde{f}_t^*\omega_{\text{WP}}$ on $\mathbb{H}^2$. We have the following,

Proposition 4.8 ([AAS18, Proof of Imayoshi–Shiga])**.** *For all $t \in [0,1]$ we find*

$$\int_C f_t^*(\omega_{\text{WP}}) = \int_C f_0^*(\omega_{\text{WP}}).$$

Proof. Since $\tilde{F}$ satisfies properties (1)-(3) of Lemma 4.4, the same computation as in the proof of the Imayoshi–Shiga theorem in [AAS18] gives the result. This is a generalization of Stoke's theorem by using a nice enough exhaustion of C by compact sets. We remark that the notation of [AAS18] differ from ours as all of their maps should be understood in the orbifold sense, e.g. $\|dF_{(x,t)}\|$ is given in our notation by $\|d\tilde{F}_{(\tilde{x},t)}\|$ for any lift $\tilde{x}$ of x. □

Finishing the proof of $\tilde{\phi} = \tilde{\psi}$. On one hand via proposition 4.6

$$E(f_t) \geq \int_C f_t^*(\omega_{\text{WP}}).$$

On the other hand via proposition 4.8 and the fact that f_0 is holomorphic.

$$\int_C f_t^*(\omega_{\mathrm{WP}}) = \int_C f_0^*(\omega_{\mathrm{WP}}) = E(f_0)$$

Thus $E(f_t) \geq E(f_0)$ and via property (2) in Lemma 4.4 we find $E(f_t) = E(f_0)$ for all t. In particular $\tilde{\phi} = \tilde{\psi}$. □

Having proven Claim 4.3, Proposition 4.1 follows. □

Remark 4.9. Note that the same argument works more generally for M a real totally geodesic submanifold of $\mathcal{T}_{g,n}$ with respect to the Weil–Petersson metric, e.g. an axis of a pseudo-Anosov element of $\mathrm{Mod}(\Sigma_{g,n})$ [DW03]. Furthermore, the same argument yields the following strengthening of the Imayoshi–Shiga theorem.

Theorem 4.10 (**Imayoshi–Shiga**). *Suppose* $3g-3+n > 0$. *Let* $C = \mathbb{H}^2/\pi_1(C)$ *be a smooth finite type curve. Given* $i = 1, 2$ *and smooth maps* $\phi_i : \mathbb{H}^2 \to \mathcal{T}_{g,n}$, *equivariant with respect to the same homomorphism* $\phi_* : \pi_1(C) \to \mathrm{PMod}(\Sigma_{g,n})$. *Assume that* ϕ_1 *is holomorphic and nonconstant. Let* $E(\phi_i)$ *denote the equivariant energy of* ϕ_i. *If either* ϕ_2 *is holomorphic or*

$$E(\phi_2) \leq E(\phi_1)$$

then $\phi_1 = \phi_2$.

Proposition 4.1 implies the following special case of Theorem 3.5 when the algebraic orbifold is a smooth finite type curve.

Proposition 4.11. *Let* $3g-3+n > 0$. *Let* M *be an almost Weil–Petersson complex submanifold of* $\mathcal{T}_{g,n}$ *with stabilizer* $\Gamma_M < \mathrm{PMod}(\Sigma_{g,n})$. *Let* C *be a smooth finite type curve and let* $\phi : C \to \mathcal{M}_{g,n}$ *be a nonconstant* Γ_M*-deformation. Then,* ϕ *has a lift* $\tilde{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ *factoring through* $M \hookrightarrow \mathcal{T}_{g,n}$.

Proof. Let $\phi : C \to \mathcal{M}_{g,n}$ be a Γ_M-deformation. Then, there is a lift $\tilde{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ of ϕ, equivariant with respect to a homomorphism

$$\phi_* : \pi_1(C) \to \Gamma_M \hookrightarrow \mathrm{PMod}(\Sigma_{g,n}).$$

Since M is almost Weil–Petersson, there exists a forgetful map

$$\mathcal{F} : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m},$$

with $n \geq m$ and $3g - 3 + m > 0$, so that $M' = \mathcal{F}(M)$ is a Weil–Petersson complex submanifold of $\mathcal{T}_{g,m}$. Let

$$\mathcal{F}_* : \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Mod}(\Sigma_{g,m})$$

be the associated forgetful map. The map $\mathcal{F}$ is $\mathcal{F}_*$-equivariant. It is evident that $\mathcal{F}_*(\Gamma_M) \subset \Gamma_{M'}$. We claim that

$$\mathcal{F}_*|_{\Gamma_M} : \Gamma_M \to \Gamma_{M'}$$

is injective. Indeed, $\mathcal{F}|_M : M \to M'$ is a biholomorphism and so the only way for an element $\gamma \in \ker \mathcal{F}_*$ to be in Γ_M is to fix M *pointwise*. The kernel $\ker \mathcal{F}_*$ is a surface braid group $B_{n-m}(\Sigma_{g,m})$ [FM12, Theorem 9.1]. Surface braid groups for surfaces with negative Euler characteristic are torsion free [FN62, Corollary 2.2]. As $\chi(\Sigma_{g,m}) < 0$ the claim follows.

The map $\tilde{\phi}$ is nonconstant, thus ϕ_* has infinite image. In particular, the group $G := \mathcal{F}_*(\phi_*(\pi_1(C))$ is infinite. It follows that the map $\bar{\phi} := \mathcal{F} \circ \tilde{\phi}$ is not constant: indeed, if not then G fixes a point y, but this is impossible as G is infinite.

Observe that the map $\bar{\phi} : \mathbb{H}^2 \to \mathcal{T}_{g,n}$ is $\mathcal{F}_* \circ \phi_*$-equivariant so via (the proof of) Proposition 4.1 we find that $\bar{\phi}$ factors through $M' \hookrightarrow \mathcal{T}_{g,m}$. To conclude the proof of Proposition 4.11 we show that the holomorphic map

$$\hat{\phi} := \mathcal{F}|_M^{-1} \circ \bar{\phi} : \mathbb{H}^2 \to M \subset \mathcal{T}_{g,n}$$

equals $\tilde{\phi}$. By Theorem 4.10, it is enough to show that $\hat{\phi}$ is ϕ_*-equivariant. Let $x \in M'$ and $\gamma \in \Gamma_M$, then

$$\mathcal{F}|_M^{-1}(\mathcal{F}_*(\gamma) \cdot x) = \mathcal{F}|_M^{-1}(\mathcal{F}_M(\gamma \cdot \mathcal{F}|_M^{-1}(x)) = \gamma \cdot \mathcal{F}|_M^{-1}(x).$$

Thus $\hat{\phi}$ is ϕ_*-equivariant and the claim follows. □

5. Rigidity

In this section we complete the proofs of Theorem 3.5 and Theorem 1.1.

Proof of Theorem 3.5. Let $(\widetilde{X}, \Gamma_X)$ be the orbifold structure of an algebraic orbifold X, with finite index subgroup $\Gamma'_X < \Gamma_X$ and finite branched cover $X' = \widetilde{X}/\Gamma'_X \to X$. Given a nonconstant Γ_M-deformation $\phi : X \to \mathcal{M}_{g,n}$, let $\phi' : X' \to \mathcal{M}_{g,n}$ be the induced orbifold map. By assumption there is a lift $\tilde{\phi} : \widetilde{X} \to \mathcal{T}_{g,n}$ of ϕ that is equivariant with respect to the homomorphism

$$\phi_* : \Gamma_X \to \Gamma_M \hookrightarrow \mathrm{PMod}(\Sigma_{g,n}).$$

Let $C \subset X'$ be a smooth curve so that $\phi'|_C$ is not constant. Via the proof of Proposition 4.11, the induced map $\tilde{\phi}|_{\widetilde{C}}$ on the universal cover $\widetilde{C}$ factors through M. Thus, Theorem 3.5 will be proven if we show that we can cover X' by curves C such that $\phi'|_C$ is not constant. Note that C is not required to be smooth. This is because $\phi'|_C$ will induce a map from its normalization.

Lemma 5.1. *Let $y \in X'$. There exists a curve $C \subset X'$ passing through y so that $\phi'|_C$ is not constant.*

Proof. This is elementary, but we include the details. By assumption there is a finite (proper with finite fibers) ramified covering $\pi : X' \to \overline{X}$ to a quasi-projective variety $\overline{X}$. Let $B \subset \overline{X}$ be the branch locus of π. Given $y \in X'$, let $Y = (\phi')^{-1}(\phi'(y))$. Then, Y is a proper non-empty analytic subset of X'. We will show that there is a curve C passing through y and not contained in Y. By the proper mapping theorem, $Y' = \pi(Y)$ is a proper analytic subset of $\overline{X}$. Pick a point $x \notin Y' \cup B$. Since $\overline{X}$ is quasi-projective, there exists a curve $C' \subset \overline{X}$ passing through $\pi(y)$ and x. Let C_y be an irreducible component of $\pi^{-1}(C')$ containing y, then $x \in \pi(C_y) = C$ and the claim follows. □

Having shown Lemma 5.1, Theorem 3.5 follows. □

Now we show that Theorem 3.5 implies the classical rigidity stated in Theorem 1.1. We consider the following more general class of local deformations,

Definition 5.2. Given an almost Weil–Petersson subvariety $N \subset \mathcal{M}_{g,n}$, let $\iota : N \to \mathcal{M}_{g,n}$ be the inclusion and fix $L \geq 3$. A *proper* local orbifold deformation of ι is a local holomorphic deformation

$$F_t : N_t[L] \to \mathcal{M}_{g,n}[L] \ , \ t \in \Delta$$

of a lift $\iota[L] : N[L] \to \mathcal{M}_{g,n}[L]$ of ι such that the following hold:[6]

(1) Let $\pi : \mathcal{N}[L] \to \Delta$ be the associated deformation of $N[L]$, with $F_0 : N_0[L] \cong N[L]$. Then π is a closed, smooth submersion.
(2) $F_t : N_t[L] \to \mathcal{M}_{g,n}[L]$ is proper.
(3) $\mathcal{N}[L]$ is smoothly isomorphic to $N[L] \times \Delta$.

Theorem 5.3. *Assume $3g - 3 + n > 0$ and $L \geq 3$. Let $N \subset \mathcal{M}_{g,n}$ be an almost Weil–Petersson subvariety of positive dimension. Let*

$$F_t : N_t[L] \to \mathcal{M}_{g,n}[L] \quad , \quad t \in \Delta$$

be a proper local orbifold deformation of $\iota : N \to \mathcal{M}_{g,n}$, such that each element of the family $\pi : \mathcal{N}[L] \to \Delta$ is quasi-projective. Then, up to shrinking Δ, the deformation $\mathcal{F} : \mathcal{N}[L] \to \mathcal{M}_{g,n}[L]$ is trivial, i.e. there exists a holomorphic family $g_t : N_t[L] \to N[L]$ inducing $(N_t[L], F_t) \cong (N[L], \iota[L])$.

Proof. Let N be an almost Weil–Petersson variety of $\mathcal{M}_{g,n}$ of positive dimension, and let Γ_M be the associated orbifold fundamental group of N. Let

$$F_t : N_t[L] \to \mathcal{M}_{g,n}[L] \quad , \quad t \in \Delta$$

be a proper local orbifold deformation of $\iota : N \hookrightarrow \mathcal{M}_{g,n}$. Let $\pi : \mathcal{N}[L] \to \Delta$ be the associated deformation of $N[L]$. By assumption $N_t[L]$ is quasi-projective for all t. Thus,

[6]Observe that these are automatic for local deformations of compact manifolds.

$$N_t[L] \xrightarrow{F_t} \mathcal{M}_{g,n}[L] \longrightarrow \mathcal{M}_{g,n}$$

is a non-constant Γ_M-deformation. By Theorem 3.5, it follows that $F_t(N_t[L]) \subset N[L] = F_0(N_0[L])$ for all t. The set of points $(x,t) \in \mathcal{N}$ where $dF_t|_x$ is not full rank is closed and does not include the central fiber. Since $\pi : \mathcal{N}[L] \to \Delta$ is closed, by shrinking Δ we can assume that F_t is an immersion for all t. By dimension reasons it follows that F_t is a proper local biholomorphism, i.e. a covering map. Let $\psi_t : N_t[L] \to N[L]$ for $t \in \Delta$ be a smooth trivialization. It follows that $F_t \circ \psi_t^{-1} : N[L] \to N[L]$ is a covering map homotopic to a homeomorphism. In particular, it has degree 1. So F_t is a biholomorphism and $(N_t[L], F_t) \cong (N[L], \iota[L])$ via the family F_t and the claim follows. □

The proof of Theorem 1.1 follows from the proof of Theorem 5.3, skipping the step that shrinks Δ.

6. Covering constructions are Weil–Petersson geodesic

The main goal of this section is the proof of Proposition 1.2 relating covering constructions with almost Weil-Petersson complex submanifolds. Note that by a result of Filip (cf. [Fil16, Remark 1.6]) covering constructions project to subvarieties of $\mathcal{M}_{g,n}$, so we only need to show that the images of covering constructions are almost Weil–Petersson complex submanifolds of $\mathcal{T}_{g,n}$.

We start by recalling the definition of covering constructions. Let

$$h : \Sigma' \to \Sigma$$

be a finite degree, orientation-preserving, topological branched cover between topological surfaces Σ' and Σ. Consider finite subsets $u \subset \Sigma$ and $v \subset \Sigma'$. In the following we will assume that

$$h^{-1}(u) = v \cup R(h) \tag{1}$$

for $R(h)$ the set of ramification points of h. In particular, u contains all the branch points of h.

Definition 6.1 (**Covering constructions**)**.** Pulling back complex structures under the *unbranched covering map*

$$h|_{\Sigma' - h^{-1}(u)} : \Sigma' - h^{-1}(u) \to \Sigma - u$$

induces a holomorphic isometric embedding

$$f_h : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', h^{-1}(u))$$

with respect to the Teichmüller metric. We call f_h a *totally marked covering construction.*
A *covering construction* is a holomorphic map

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

induced by a totally marked covering construction f_h by postcomposition with the forgetful map $\mathcal{F} : \mathcal{T}(\Sigma', h^{-1}(u)) \to \mathcal{T}(\Sigma', v)$. Since $\mathcal{F}$ only forgets ramification points of h, the map f is an isometric embedding [BS24, Proposition A.2]. If the branched cover h is regular we call f a *regular covering construction.*

Remark 6.2. In [BS24, Appendix] it is shown that for $g(\Sigma) \geq 1$ and $\mathcal{T}(\Sigma, u) \neq \mathcal{T}_{1,1}$ the condition given by Equation (1) is necessary and sufficient for h to induce an isometric embedding with respect to the Teichmüller metric.

In the following, we describe in more detail properties of each type of covering construction. Unless otherwise specified $h : (\Sigma', v) \to (\Sigma, u)$ will denote a finite degree, orientation-preserving, topological branched cover. Let $g(\Sigma)$ denote the genus of Σ. Throughout this section we assume that

$$3g(\Sigma) - 3 + |u| > 0.$$

6.1. **Totally marked covering constructions.** Let $v = h^{-1}(u) \subset \Sigma'$ and let

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

be the associated totally marked covering construction. The map f is holomorphic and the derivative is given by pulling back Beltrami forms to the cover. We sketch the details. Let $X = \mathbb{H}^2/\Gamma \in \mathcal{T}(\Sigma, u)$, for X a finite type Riemann surface diffeomorphic to $\Sigma - u$ and $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ a discrete subgroup without elliptic elements. Since h is unbranched over $\Sigma - u$, the image $f(X)$ is given by $f(X) = \mathbb{H}^2/\Gamma'$ for $\Gamma' < \Gamma$ a finite index subgroup. It follows that there exists a pushforward lift of Beltrami forms:

$$f_* : M(\Gamma) \to M(\Gamma')$$

given by the inclusion. Furthermore, f_* induces a holomorphic map

$$\mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$$

which agrees with f.[7] In particular the derivative of f is induced by f_*. Similarly, the pullback of integrable quadratic differentials is given by the inclusion $Q(\Gamma) \to Q(\Gamma')$.

Recall that Harmonic forms $H(X)$ are given by $(z - \bar{z})^2\bar{q}$ for $q \in Q(\Gamma)$. Then, the following corollary is immediate.

Corollary 6.3. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ be a totally marked covering construction. Let $X \in \mathcal{T}(\Sigma, u)$. Then, the derivative df gives an inclusion*

$$df_X : H(X) \hookrightarrow H(f(X)).$$

Furthermore, df_X is induced by the pullback of quadratic differentials and volume forms under h.

Remark 6.4. For Corollary 6.3 it is crucial that the pullback of the complete hyperbolic metric on X is the complete hyperbolic metric on $f(X)$. In particular, it does not apply when we postcompose with forgetful maps.

The Weil–Petersson metric g_{WP} is easily computed for harmonic forms. Thus, we get the following.

Corollary 6.5. *Let $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma, v)$ be a totally marked covering construction induced by a branched cover h. Then,*

$$f^* g_{\mathrm{WP}} = \deg(h) g_{\mathrm{WP}}.$$

Proof. Let $X \in \mathcal{T}(\Sigma, u)$. By Corollary 6.3, $df_X : H(X) \hookrightarrow H(f(X))$ is given by the pullback of quadratic differentials and volume forms under h. Let $\mu_i = \overline{q_i}(ds^2_X)^{-1} \in H(X)$, then:

$$\begin{aligned} \langle df_X(\mu_1), df_X(\mu_2)\rangle &= \int_{f(X)} df_X(\mu_1)\overline{df_X(\mu_2)} ds^2_{f(X)} \\ &= \int_{f(X)} df_X(\mu_1) h^*(q_2) \\ &= \deg(h) \int_X \mu_1 q_2 = \deg(h)\langle \mu_1, \mu_2\rangle. \end{aligned} \tag{2}$$

In the third equality we used the fact that the coderivative of f is given by the trace under h. □

6.2. **Regular H-Covers.** Let Y be a closed Riemann surface, with underlying topological surface Σ'. Let H be a finite group of conformal automorphisms of Y. Let $X = Y/H$ so that H defines a regular branched covering

$$h_X : Y \to X.$$

Let Σ be the underlying topological surface of X. Let $h : \Sigma' \to \Sigma$ be the topological branched cover induced by h_X. As in Equation (1), consider finite subsets $u \subset \Sigma$ and $v \subset \Sigma'$ such that

$$h^{-1}(u) = v \cup R(h).$$

Then, h induces a regular covering construction

$$f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v).$$

[7] Here it is important that h is orientation preserving, so that it agrees with $f(X) \to X$.

Suppose that v is H-invariant. Then H is a group of conformal automorphisms of $Y' := Y - v$. Recall that H acts on $\mathcal{T}(\Sigma', v)$ by biholomorphisms. We claim the following,

Lemma 6.6.

$$f(\mathcal{T}(\Sigma, u)) = \mathrm{Fix}(H)$$

for $\mathrm{Fix}(H)$ *the fixed set of* H *in* $\mathcal{T}(\Sigma', v)$.

This is already known (e.g. [Kra59]), but we provide a proof using the terminology introduced in [BS24].

Proof. Let $M := f(\mathcal{T}(\Sigma, u))$. Note that H is a finite group of biholomorphic isometries of the Teichmüller metric. Thus, $\mathrm{Fix}(H) \subset \mathcal{T}(\Sigma', v)$ is a totally geodesic complex submanifold with respect to the Teichmüller metric. The inclusion $M \subseteq \mathrm{Fix}(H)$ is evident, as H can be represented by biholomorphisms of any $f(Z)$ for $Z \in \mathcal{T}(\Sigma, u)$. To prove the converse, we will consider the totally geodesic bundles QM and $Q\,\mathrm{Fix}(H)$ [BS24, Section 3]. Let $f(X) \in M \subseteq \mathrm{Fix}(H)$ and identify H with a subgroup of biholomorphisms of $f(X)$. Recall that $Q_{f(X)}M \subset Q(f(X))$ equals the H-invariant differentials on $Q(f(X))$. We claim that $Q_{f(X)}\,\mathrm{Fix}(H)$ admits the same description. In particular, $Q_{f(X)}\,\mathrm{Fix}(H) = Q_{f(X)}M$ and we are done.

To prove the claim, let $t \in H$ and denote by $\mathfrak{t} : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma', v)$ to the biholomorphism induced by t. Since $\mathfrak{t}(f(X)) = f(X)$, the umkehr map $\mathfrak{t}_!$ [BS24, Section 4] associated to $\mathfrak{t}$ acts on $Q(f(X))$ by:

$$\mathfrak{t}_! : Q(f(X)) \to Q(f(X)) \quad , \quad q \to t^*(q) \in Q(f(X)).$$

Any geodesic in $\mathrm{Fix}(H)$ is fixed by $\mathfrak{t}$. Thus, $q = t^*(q)$ for any $q \in Q_{f(X)}\,\mathrm{Fix}(H)$. The claim follows. □

H acts on $\mathcal{T}(\Sigma', v)$ by Weil–Petersson isometries, thus the following is immediate.

Corollary 6.7. *Let* $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ *be a regular covering construction induced by a regular covering with deck group* H. *Assume that* v *is* H*-invariant. Then* $f(\mathcal{T}(\Sigma, u))$ *is totally geodesic with respect to the Weil–Petersson metric, i.e. it is Weil–Petersson.*

In general, we have a weaker statement.

Corollary 6.8. *Let* $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma', v)$ *be a regular covering construction, induced by a regular cover* $h : \Sigma' \to \Sigma$. *If* $(g(\Sigma), g(\Sigma')) = (0, 1)$ *assume further that* $|u| \geq 5$. *Then* $f(\mathcal{T}(\Sigma, u))$ *is almost Weil–Petersson.*

Proof. Let $v' = v \setminus R(h) = h^{-1}(u \setminus B(h))$, for $B(h)$ the branch locus of h. Then, v' is H-invariant. We claim that $g(\Sigma') - 3 + |v'| > 0$. This is immediate if $g(\Sigma') > 1$ so assume otherwise.

We proceed by cases. If $g(\Sigma) = 1$ the map h is actually unramified and $|v'| = |u| \geq 1$. For the remaining cases we use the following facts from [Mir95, Lemma 3.8 and discussion thereafter]:

(1) The map h can have at most 4 branch points and less than 4 if $g(\Sigma') = 0$.
(2) If $g(\Sigma') = 0$ and h has 3 branch points, then $\deg(h) \geq 4$.

The claim follows. The forgetful map

$$\mathcal{F} : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma', v')$$

satisfies that $M' = \mathcal{F}(f(\mathcal{T}(\Sigma, u))$ is Weil–Petersson (Corollary 6.7). Moreover, as $\mathcal{F}$ only forgets ramification points, $\mathcal{F}$ restricts to a biholomorphism between $f(\mathcal{T}(\Sigma, u))$ and M'. The result follows. □

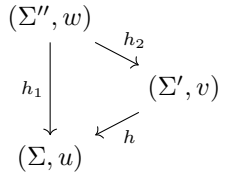
Combining Corollaries 6.5 and 6.7, we complete the proof of Proposition 1.2 by showing the following.

Proposition 6.9. *Let* $f : \mathcal{T}(\Sigma, u) \to \mathcal{T}(\Sigma, v)$ *be a totally marked covering construction induced by a branched cover* h. *Then,* $f(\mathcal{T}(\Sigma, u))$ *is a Weil–Petersson complex submanifold.*

Proof. The case of totally marked regular coverings follows from Corollary 6.7. So assume that h is not regular. Let

$$h_2 : (\Sigma'', w) \to (\Sigma', v)$$

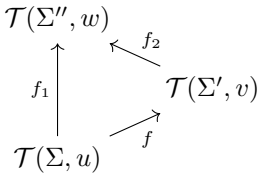
be the *normal closure* of h, i.e. the regular cover induced by taking a finite index subgroup $\Gamma < h_*(\pi_1(\Sigma' - v)) < \pi_1(\Sigma - u))$ such that $\Gamma \trianglelefteq \pi_1(\Sigma - u)$. This means that have a commutative diagram:



where h_i for $i = 1, 2$ are regular covers and the marked points satisfy

$$h_1^{-1}(u) = h_2^{-1}(v) = w.$$

By Corollary 6.7, the induced maps



satisfy that $f_1(\mathcal{T}(\Sigma, u))$ and $f_2(\mathcal{T}(\Sigma', v))$ are Weil–Petersson submanifolds of $\mathcal{T}(\Sigma'', w)$. Since $f_1 = f_2 \circ f$, it follows that $f_1(\mathcal{T}(\Sigma, u)) \subset f_2(\mathcal{T}(\Sigma', v))$ and

$$f(\mathcal{T}(\Sigma, u)) = f_2^{-1}(f_1(\mathcal{T}(\Sigma, u))).$$

In particular, $f_1(\mathcal{T}(\Sigma, u))$ is a totally geodesic submanifold of $f_2(\mathcal{T}(\Sigma', v))$ with respect to g_{WP}. The map $f_2 : \mathcal{T}(\Sigma', v) \to \mathcal{T}(\Sigma'', w)$ is a totally marked covering construction. Thus, by Corollary 6.5 it is (up to a constant) a Riemannian isometric embedding for g_{WP}. The claim follows. □

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