Modular Curves and Shimura Varieties

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1 Brief Review of Modular Curves

Let $N \geq 1$ be an integer. Recall that we have inclusions of subgroups

$$\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z}).$$

A congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup containing $\Gamma(N)$ for some $N \geq 1$. Let $\mathbb{H}$ denote the complex upper half plane, and note that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ via mobius transformations.

- If Γ is a congruence subgroup, then we set $Y(\Gamma) = \mathbb{H}/\Gamma$. $Y(\Gamma)$ is a complex manifold of dimension 1.
- Let $\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \infty$, and note that the Mobius action naturally extends to $\mathbb{H}^*$. We set $X(\Gamma) = \mathbb{H}^*/\Gamma$, a compact Riemann surface containing $Y(\Gamma)$.
- The complex manifolds $Y_0(N) := Y(\Gamma_0(N))$, $Y_1(N) := Y(\Gamma_1(N))$ admit moduli descriptions, as parameter spaces of complex elliptic curves with extra structure. For example, points of $Y_1(N)$ correspond to isomorphism classes of pairs (E, P) where $E/\mathbb{C}$ is a complex elliptic curve and $P \in E$ is a point of order N. By studying these moduli problems over $\mathbb{Q}$, one can show that these curves all have canonical models as smooth algebraic curves over $\mathbb{Q}$. The same holds for the compactifications $X_0(N)$ and $X_1(N)$ (where one has to use generalized elliptic curves), and thus we get smooth projective algebraic curves over $\mathbb{Q}$ in that case.

  As a remark, the case of $Y(\Gamma(N))$ is a bit more curious: we get $Y(\Gamma(N))$ can be defined over $\mathbb{Q}(\zeta_N)$, so the field of definition grows with N. We will return to this phenomenon in the context of Shimura varieties later.
- Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let f be a meromorphic modular form for Γ of even weight k. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the relation $f(\gamma z) = (cz + d)^k f(z)$ shows that $f(z)dz^{k/2}$ is a meromorphic differentials of degree $k/2$ on $X(\Gamma)$. The most important consequence of this is that we have an isomorphism of $\mathbb{C}$-vector spaces

  $$S_2(\Gamma) \cong H^0(X(\Gamma), \Omega^1_{X(\Gamma)}).$$

  where f is essentially mapped to $f(z)dz$.
- Recall that for each $p \nmid N$, we get a correspondence T_p on $X_0(N)$, defined moduli theoretically as

  $$\begin{tikzcd} & X_0(Np) \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X_0(N) & & X_0(N) \end{tikzcd}$$

  where $g(E, C) = (E/C_p, C/C_p)$ and $f(E, C) = (E, C_N)$. Consequently, we get an action of the algebra $\mathbb{T} = \mathbb{Z}[T_p : p \nmid N]$ on e.g. the singular, étale, and Hodge cohomology of modular curves by using Poincaré duality. Because these correspondences can be defined over $\mathbb{Q}$, any functorial isomorphism of cohomology groups of X is automatically $\mathbb{T}$-equivariant. Under the identification above, it is not hard to see that the induced action of T_p on $H^0(X_0(N), \Omega^1)$ coincides with the usual Hecke operator for modular forms. Moreover it is not hard to see that we could define the same correspondences for e.g. $X_1(N)$.

2 Cohomology of Modular Curves and Galois Representations

Let $X/\mathbb{Q}$ be one of the models for our modular curves as above, and let us use $X(\mathbb{C})$ to denote the corresponding Riemann surface. We consider the singular cohomology group $V := H^1_{\text{sing}}(X(\mathbb{C}), \overline{\mathbb{Q}})$. Note that the Hecke correspondences on the modular curve X induces actions of $\mathbb{T}$ on V.

Now let $f \in S_2$ be a *newform* of weight 2, with system of Hecke eigenvalues $\gamma = \{\gamma_p\}_p$. We base change in two different directions. First, we consider

$$H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) = V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

By Hodge theory, we have

$$H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) \cong H^0(X, \Omega^1_X) \oplus H^1(X, \mathcal{O}_X) \text{ with } H^1(X, \mathcal{O}_X) = \overline{H^0(X, \Omega^1_X)}.$$

Thus by what we have seen, we obtain a canonical decomposition

$$H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{C}) \cong S_2(\Gamma) \oplus \overline{S_2(\Gamma)}.$$

From this, we see two things:

1. Since we know that each T_p is diagonalizable on $S_2(\Gamma)$ (for instance by working with the Petersson inner product and using the spectral theorem), it follows that the T_p act diagonalizably on $V_{\mathbb{C}} \cong S_2(\Gamma) \oplus \overline{S_2(\Gamma)}$. Furthermore, the eigenvalues of T_p are all algebraic, and thus the $\mathbb{T}$-action on V itself is diagonalizable. Thus V admits a decomposition into simultaneous eigenspaces for the T_p. We write

$$V = \bigoplus_\lambda V_\lambda$$

where $\lambda = \{\lambda_p\}_p$ is a system of Hecke eigenvalues.

2. Since $f \in S_2(\Gamma)$ is a newform, it is the only element of $S_2(\Gamma)$ with its given system of Hecke eigenvalues by multipicity one. It follows that the γ-eigenspace inside $H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$ is two dimensional, spanned by $\langle f, \overline{f} \rangle$, and thus $V_\gamma \subset V$ is 2-dimensional over $\overline{\mathbb{Q}}$ and satisfies $V_\gamma \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \langle f, \overline{f} \rangle$.

Base changing in another direction, we can consider $H^1_{\text{sing}}(X(\mathbb{C}), \overline{\mathbb{Q}}_\ell)$. By Artin's comparison theorem, we have an isomorphism

$$H^1_{\text{sing}}(X(\mathbb{C}), \overline{\mathbb{Q}}_\ell) \cong H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

Importantly, the left hand side now carries commuting actions of both $\mathbb{T}$ and $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Consequently, the eigenspace $V_\gamma \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}_\ell$ carries an action of $G_{\mathbb{Q}}$, and we obtain the *Galois representation associated to a weight 2 newform*

$$\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}(V_\gamma \otimes \overline{\mathbb{Q}}_\ell) \cong \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

Example 2.1. We provide one example to illustrate the principle that studying the geometry of modular curves can lead to nontrivial facts about Galois representations associated to modular forms.

Let $X/\mathbb{R}$ be a real curve. Then since complex conjugation $\sigma : \mathbb{C} \to \mathbb{C}$ is a continuous automorphism of $\mathbb{C}$ over $\mathbb{R}$, it gives a continuous map $\sigma : X(\mathbb{C}) \to X(\mathbb{C})$, and thus we get an automorphism

$$\sigma : H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}) \to H^1_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}).$$

On the other hand, $H^1_{\text{ét}}(X_{\mathbb{C}}, \overline{\mathbb{Q}}_\ell)$ has a natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$, and the action of complex conjugation here is the same as the one induced via base changing singular cohomology and applying Artin's comparison theorem. But on the complex side, it is clear that σ acts by swapping f and $\overline{f}$; it follows that $\det(\rho_f(\sigma)) = -1$. We deduce that the Galois representation associated to a weight 2 newform form is odd.

3 Shimura Varieties

3.1 Modular Curves as Shimura Varieties

We now want to explain how modular curves can be seen as one instance of the much more general story of Shimura varieties. Let $\mathbb{A}_f$ denote the finite adeles of $\mathbb{Q}$ with the usual restricted product topology.

The first starting point is to observe that we may think of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ in the following alternative way: a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})_+$ is congruence if and only if $\Gamma = K \cap \mathrm{SL}_2(\mathbb{Q})$, where $K \subset \mathrm{SL}_2(\mathbb{A}_f)$ is a compact open subgroup. This is because the $K(N) := \ker(\mathrm{SL}_2(\hat{\mathbb{Z}}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$ form a basic set of compact opens, and $K(N) \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma(N)$. This is good for two reasons:

1. It provides an intrinsic definition of the notion of congruence subgroup which we can then apply to other reductive groups
2. It suggests that modular curves can be described adelically, which would both illuminate the various actions of $\mathrm{GL}_2(\mathbb{Z}_p)$ and give an idea of how to generalize to other groups.

Regarding the second point, we have the following:

Lemma 3.1. We have the following isomorphism of manifolds:

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong Y(\Gamma(N)).$$

Proof. First, let's study the set $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_f)/K(N)$. Consider the natural inclusion

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{A}_f)/K(N).$$

Since $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Q}) \cap \mathrm{SL}_2(\hat{\mathbb{Z}})$, this induces an inclusion

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \subset \mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A}_f)/K(N).$$

Moreover since $\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})\,\mathrm{SL}_2(\hat{\mathbb{Z}})$,[1] this inclusion is surjective hence a bijection. But by definition

$$\mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}),$$

so we conclude

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\hat{\mathbb{Z}})/K(N) \cong \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \{1\}.$$

It follows that

$$\mathrm{SL}_2(\mathbb{A}_f) = \mathrm{SL}_2(\mathbb{Q})K(N).$$

Consequently

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K(N) \cong (\mathrm{SL}_2(\mathbb{Q}) \cap K(N))\backslash\mathbb{H} \cong Y(\Gamma(N)),$$

where the first equality follows because every representative (h, g) is equivalent to a representative of the form (h', k), and $(h, k) \sim (h', k')$ if and only if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Q}) \cap K(N)$ such that $\gamma h' = h$. ☐

Of course, one can imagine how to modify this to obtain the quotients $Y_1(N), Y_0(N)$ in this way as well.

For reasons that we will explain later, it ends up being more fruitful in some ways to work with the group $G = \mathrm{GL}_2$ instead of SL_2. Then $G(\mathbb{R})$ acts on $\mathbb{H}^{\pm} = \mathbb{C} \setminus \mathbb{R}$, and $G(\mathbb{R})_+$ act on $\mathbb{H}$. Let's reuse notation and set $K(N) = \ker(\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$. In this setting, the modular curves $Y(\Gamma)$ are just the quotients of $\mathbb{H}$ by a congruence subgroup of $\mathrm{GL}_2(\mathbb{Q})_+$, but they also admit adelic descriptions as before. For instance, arguing as above one can compute that

$$\mathrm{GL}_2(\mathbb{Q})_+\backslash \mathrm{GL}_2(\mathbb{A}_f)/K(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

and consequently

$$G(\mathbb{Q})_+\backslash\mathbb{H} \times G(\mathbb{A}_f)/K(N) = \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(\Gamma(N)).$$

[1] To see this, observe the easier fact that $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\hat{\mathbb{Z}})$ to write any element of $\mathrm{SL}_2(\mathbb{A}_f)$ as a product of matrices of determinant ± 1. Then use $\begin{pmatrix}-1 & 0 & 0 & 1\end{pmatrix}$ to ensure that the determinants are 1.

3.2 Shimura Varieties in General

Now let $G/\mathbb{Q}$ be a reductive group, e.g. $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, O_n$. Let M be a real manifold with a smooth transitive action of $G(\mathbb{R})$. Let M^+ be a connected component of M, and let $G(\mathbb{R})_+$ be the stabilizer of M^+ in $G(\mathbb{R})$.

Definition 3.2 (Informal)**.** Shimura varieties are certain double coset spaces of the form

$$X_K := G(\mathbb{Q})_+ \backslash (M^+ \times G(\mathbb{A}_f))/K.$$

for $K \subset G(\mathbb{A}_f)$ a compact open subgroup.

The above reasoning for modular curves can be generalized to show that X_K is a finite disjoint union of quotients of $\Gamma_i \backslash M^+$, where $\Gamma_i \subset G(\mathbb{Q})_+$ are congruence.

In reality, not just any M will suffice: at the very least we want the X_K's to be the analytification of a complex algebraic variety! So to qualify as a Shimura variety, M must be special, namely it (roughly) has to be a conjugacy class of homomorphisms $\mathbb{C}^\times \to G(\mathbb{R})$.[2] For those familiar with Hodge theory, this suggests some relation between M and a "family of Hodge structures."

Example 3.3. In the modular curve setting, $\mathbb{H}^\pm$ is the conjugacy class of the homomorphism $h : \mathbb{C}^\times \to \mathrm{GL}_2(\mathbb{R})$, $x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Indeed, an easy calculation shows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ centralizes an element in the image of h if and only if $a = d, b = -c$. It follows that the conjugacy class of h is just $\mathrm{GL}_2(\mathbb{R})$ modulo this centralizer. But this centralizer is just the stabilizer of i in the transitive action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}^\pm$, and thus $\mathbb{H}^\pm$ is the isomorphic to the desired conjugacy class. More explicitly, ghg^{-1} will correspond to $g \cdot i \in \mathbb{H}^\pm$.

As one might expect, the Hodge structure corresponding to $\tau \in \mathbb{H}$ is the natural Hodge structure which occurs on the first homology of the elliptic curve $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$.

Interestingly, $\mathbb{H}$ is not realizable as a conjugacy class of homomorphisms $h : \mathbb{C}^\times \to \mathrm{SL}_2(\mathbb{R})$, and consequently our double cosets

$$\mathrm{SL}_2(\mathbb{Q}) \backslash \mathbb{H} \times \mathrm{SL}_2(\mathbb{A}_f)/K$$

from before are not Shimura varieties by this definition. Instead, $\mathbb{H}$ is the conjugacy class of the obvious morphism $h : U_1 \to \mathrm{SL}_2$, making these quotients a slightly different object known as a *connected* Shimura variety, which make up the connected components of Shimura varieties.

Example 3.4. Note that $\mathrm{SL}_2 = \mathrm{Sp}_2$. Replacing the above discussion with $\mathrm{Sp}_{2n} = \{A \in M_{2n} : A^t \Omega A = \Omega\}$, one gets an action on the *Siegel half spaces* $\mathbb{H}_n = \{Z \in \mathrm{Sym}_n(\mathbb{C}) : \mathrm{im}(Z) > 0\}$, and the resulting connected Shimura varieties can be interpreted as moduli spaces of abelian varieties of dimension n with level structure.

Theorem 3.5. Let (G, M) be as in the definition of Shimura variety. Then the entire collection of complex manifolds X_K, $K \subset G(\mathbb{A}_f)$ compact open, is the complex analytification of a canonical collection of smooth quasi-projective varieties over $\mathbb{C}$.

There exists a number field F, depending only on (G, M), such that each X_K admits a canonical model as a smooth quasi-projective variety over F.

The theorem then "explains" the discrepancy between Shimura varieties and connected Shimura varieties we observed earlier: the $Y(\Gamma(N))$'s do not form a collection of Shimura varieties, since their minimal field of definition $\mathbb{Q}(\zeta_N)$ grows with N. However, the disjoint unions $\sqcup_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(\Gamma(N))$ are Shimura varieties, and are in fact all defined over $\mathbb{Q}$.

The theorem also explains the usefulness of Shimura varieties in the Langlands program: the cohomology groups $\varinjlim_K H^*_{\text{ét}}(X_K, \overline{\mathbb{Q}}_\ell)$ will carry actions of both $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and $G(\mathbb{A}_f)$, making it a bit more apparent why they might be useful in realizing Langlands correspondences.

[2] This homomorphism also has to satisfy some additional conditions, but we don't explain them here. It is interesting to note that these conditions prevent the existence of Shimura varieties for GL_n, $n > 2$.

4 Extras

4.1 Hecke Operators, Adelically

Given $g \in G(\mathbb{A}_f)$, we get a morphism $g : X_{gKg^{-1}} \to X_K$, via the left translation action of G on $G(\mathbb{A}_f)$. This leads naturally to an action of $G(\mathbb{A}_f)$ on the inverse system $\{X_K\}_K$, and on cohomology groups $\varinjlim_K H^*(X_K)$, which is known as the *Hecke action.*

The relation to classical Hecke operators in the modular curve case is the following. Given such a g, one gets a correspondence

$$\begin{tikzcd} & X_{K\cap gKg^{-1}} \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X_K & & X_K \end{tikzcd}$$

where f is the natural projection, and g projects to $X_{gKg^{-1}}$ and composes with the Hecke action of g. I think the usual Hecke correspondence T_p occurs in the modular curve case by taking g to be the identity at $\ell \neq p$ and $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ at p.

This $G(\mathbb{A}_f)$ action is crucial in the Langlands program, as one ultimately wants to relate the $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ representations coming from $\varinjlim_K H^*_{\text{ét}}(X_K, \overline{\mathbb{Q}}_\ell)$ (leading to the statements of *local-global compatibility).*

4.2 A Double Coset Calculation

We prove in detail a calculation which was just stated earlier.

Proposition 4.1. We have the following isomorphism of manifolds

$$G(\mathbb{Q})_+ \backslash \mathbb{H} \times G(\mathbb{A}_f)/K(N) = \bigsqcup_{(\mathbb{Z}/N\mathbb{Z})^\times} Y(\Gamma(N))$$

Proof. For simplicity let us assume $N = p$ is a prime. First we study the double coset $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K(p)$.

Consider the natural inclusion

$$\mathrm{GL}_2(\hat{\mathbb{Z}})/K(p) \subset \mathrm{GL}_2(\mathbb{A}_f)/K(p).$$

Since $\mathrm{GL}_2(\mathbb{Z})_+ = \mathrm{GL}_2(\mathbb{Q})_+ \cap \mathrm{GL}_2(\hat{\mathbb{Z}})$, this induces an inclusion

$$\mathrm{GL}_2(\mathbb{Z})_+ \backslash \mathrm{GL}_2(\hat{\mathbb{Z}})/K(p) \subset \mathrm{GL}_2(\mathbb{Q})_+ \backslash \mathrm{GL}_2(\mathbb{A}_f)/K(p).$$

Moreover, since $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q})_+ \mathrm{GL}_2(\hat{\mathbb{Z}})$, this inclusion is surjective, hence a bijection. But by definition,

$$\mathrm{GL}_2(\hat{\mathbb{Z}})/K(p) \cong \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}),$$

so we conclude

$$\mathrm{GL}_2(\mathbb{Z})_+ \backslash \mathrm{GL}_2(\hat{\mathbb{Z}})/K(p) \cong \mathrm{GL}_2(\mathbb{Z})_+ \backslash \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times,$$

where the last map is given by the determinant. It follows that

$$G(\mathbb{A}_f) = \bigsqcup_{i \in (\mathbb{Z}/p\mathbb{Z})^\times} G(\mathbb{Q})_+ g_i K(p),$$

where $g_i \in G(\mathbb{A}_f)$ is any element which is integral at p and has determinant i modulo p. For instance, we can take g_i to be the identity at all $\ell \neq p$ and $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ at p. Then we have

$$G(\mathbb{Q})_+ \backslash \mathbb{H} \times G(\mathbb{A}_f)/K(p) \cong \bigsqcup_{i \in (\mathbb{Z}/p\mathbb{Z})^\times} G(\mathbb{Q})_+ \backslash \mathbb{H} \times G(\mathbb{Q})_+ g_i K(p)/K(p) \cong$$

Note that $(h, \gamma g_i k) \sim (\gamma^{-1}h, g_i k)$, and $(h, g_i k) \sim (h', g_i k')$ if and only if $h' = g_i u g_i^{-1} g$ for $u \in K(p)$. It follows that if we set $\Gamma_i = \mathbb{G}(\mathbb{Q})_+ \cap g_i K(p) g_i^{-1}$, then we have

$$\bigsqcup_{i \in (\mathbb{Z}/p\mathbb{Z})^\times} \Gamma_i \backslash \mathbb{H},$$

But it is easy to check from our explicit choice of g_i that Γ_i is just $\Gamma(p) \subset \mathrm{GL}_2(\mathbb{Z})_+$ for all i; it follows that $\Gamma_i \backslash \mathbb{H} \cong Y(\Gamma(p))$, as desired. □

5 References

For basics on modular curves, modular forms, and Hecke operators:

- Diamond, Shurman. A First Course in Modular Forms.
- Diamond, Im. Modular Forms and Modular Curves.
- Andrew Snowden. Online course on Mazur's theorem. link.
- Gal Porat. Attaching Galois Representations to Modular Forms. link.

For basics on Shimura Varieties:

- Kai Wen Lan. An example based introduction to Shimura varieties.
- J.S. Milne. Introduction to Shimura Varieties.