The Moduli Stack of Displays

Casimir Kothari
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1 Motivation: Dieudonné modules

Let k be a perfect field of characteristic p. Recall that classical Dieudonné theory gives us a classification of p-divisible groups over k in terms of Dieudonné modules, which are finite free $W(k)$-modules M, equipped with σ-linear $F : M \to M$ and σ^{-1}-linear $V : M \to M$ such that $FV = VF = p$.

For a general ring R, σ^{-1} no longer makes sense on $W(R)$. Thus we define a Dieudonné module over R to be a tuple $(M, F^{\#}, V^{\#})$ where M is a finitely generated projective $W(R)$-module, $F^{\#} : M^{(1)} \to M$ and $V^{\#} : M \to M^{(1)}$ are $W(R)$-linear with $F^{\#} \circ V^{\#} = p, V^{\#} \circ F^{\#} = p$.

When R is a perfect ring of characteristic p, it is indeed true that such data classifies p-divisible groups over R, in exact analogy with classical Dieudonné theory. However when R is not perfect, this definition "loses some information"; if $F(x)$ is divisible by p then we can no longer uniquely specify a y such that $py = F(x)$. The problem is thus related to the presence of p-torsion in $W(R)$. We would like to rigidify the situation further.

One way to fix this apparent loss of information is to observe that in the perfect case, on the submodule $V(M) \subset M$, we have that $pV^{-1} = F$; it follows that $V^{-1} : V(M) \to M$ witnesses when F is divisible by p. The idea of Displays is to keep track of $V(M)$ and V^{-1} as part of the data. They were introduced by Zink in [Z].

2 Displays

Let R be a ring in which p is nilpotent. Let $W(R)$ denote the Witt vector ring of R, with Frobenius σ and verschiebung v. Let $I_R \subset W(R)$ be the ideal of elements whose first (ghost) component is zero, i.e. $I_R = v(W(R)) = \ker(W(R) \to R)$. If $g : M \to N$ is a σ-linear map of $W(R)$-modules, we will set $M^{(1)} = W(R) \otimes_{\sigma, W(R)} M$ and let $g^{\#} : M^{(1)} \to M$ the associated $W(R)$-linear map, which is defined by $g^{\#}(s \otimes m) = sg(m)$.

Definition 2.1. A display over R is a quadruple (P, Q, F, F_1), where P is a finitely generated projective $W(R)$-module, $Q \subset P$ is a submodule, and $F : P \to P, F_1 : Q \to P$ are σ-linear maps satisfying

1. $I_R P \subset Q$ and P/Q is a direct summand of $P/I_R P$;
2. P is generated by $\mathrm{im}(F_1)$;
3. $F_1(v(\xi)x) = \xi F(x)$ for $\xi \in W(R), x \in P$.

A morphism of displays $(P, Q, F, F_1) \to (P', Q', F', F_1')$ is a morphism of $W(R)$-modules $P \to P'$ which takes Q to Q' and is compatible with the maps F, F_1, F', F_1'. In this way we get an additive category of displays over R.

The height of a display is $h = \mathrm{rk}_{W(R)} P$, a locally constant function on $\operatorname{Spec} R$. The dimension is $d = \mathrm{rk}_R(P/Q)$. There is a natural notion of base change of displays with respect to ring maps $u : R \to R'$, and in this way, displays of height h and dimension d yield a category fibered in groupoids $\mathrm{Disp}_\infty^{h,d}$ over p-nilpotent rings.

Remark 2.2. 1. Taking $\xi = 1 \in W(R)$, the definition of display gives

$$F(x) = F_1(v(1)x) \text{ for all } x \in P.$$

Thus F is uniquely determined by F_1.

2. Additionally taking $x \in Q$ and using that $\sigma v = p$, we find

$$F(x) = pF_1(x) \text{ for all } x \in Q.$$

It follows that we may think of F_1 as a "partially defined divided Frobenius".

As explained above, one of the main motivations for introducing the notion of display was to obtain a nice generalization of Dieudonné modules in the case of a non-perfect base. For a perfect base, displays behave as expected:

Lemma 2.3 ([L2, Lemma 2.4])**.** If R is a perfect ring of characteristic p, then the category of displays over R is equivalent to the category of Dieudonné modules over R.

Proof. Note first that since R is perfect, σ is an automorphism. Since $\sigma v = p$, it follows that $W(R)$ is p-torsion free and $I_R = (p)$.

If (M, F, V) is a Dieudonné module over R, this implies that $V : M \to M$ is injective, and thus we can consider the natural σ-linear morphism $V^{-1} : V(M) \to M$. Then $(M, V(M), F, V^{-1})$ gives the data of a display over R.

Conversely let (P, Q, F, F_1) be a display over R. We will define a $W(R)$-linear map $V^{\#} : P \to P^{(1)}$ which is the linearization of the appropriate V. Since the image of F_1 generates P, it suffices to specify the values of $V^{\#}$ on $\mathrm{im}(F_1)$, and we set $V^{\#}(F_1(x)) = 1 \otimes x$ for $x \in Q$. (This does in fact exist, since we can write down $V^{\#}$ globally in terms of a normal decomposition of our display, see [L2, Lemma 2.3]).

This yields a tuple (P, F, V). The checks that $F^{\#} \circ V^{\#} = p, V^{\#} \circ F^{\#} = p$ essentially boil down to the definitions and the facts that $pF_1 = F$ on Q and $\sigma \circ v = p$ on $W(R)$.

□

Remark 2.4. As the proof shows, there is always a natural functor from displays to Dieudonne modules, even for non-perfect rings. However, in the presence of p-torsion in $W(R)$, this functor need not be fully faithful. The kind of thing that can happen is we could have two displays $(P, Q, F, F_1), (P, Q, F, F_1')$ producing the same Dieudonné module because $pF_1 = pF_1'$ but $F_1 \neq F_1'$.

An important tool for studying displays is that of normal decompositions.

Definition 2.5. A normal decomposition of a display $\mathscr{P}$ over R is direct sum decomposition $P = T \oplus L$ such that $Q = I_R T \oplus L$.

Any display has a normal decomposition. Indeed, since P/Q is a direct summand of $P/I_R P$, it is a finitely generated projective R-module. Since $R = W(R)/I_R$ and $W(R)$ is I_R-adically complete, we may choose a finitely generated projective $W(R)$-module T with $T/I_R T = P/Q$. Choose a surjection $P \to T$ (by lifting $P/IP \to P/Q$), and let L be the kernel, so that $P = L \oplus T$. We have that $Q = \ker(P \to P/Q) = \ker(P \to T) \oplus \ker(T \to P/Q) = L \oplus I_R T$ as desired.

The following lemma is the key which allows us to understand displays explicitly via matrices.

Lemma 2.6. The display $\mathscr{P}$ is determined by a normal decomposition together with $F|_T$ and $F_1|_L$. Moreover, the map $F \oplus F_1 : T \oplus L \to P$ is an f-linear isomorphism.

Proof. By what was established above, we have $F|_L = pF_1|_L$, so $F_1|_L$ determines $F|_L$. Similarly, $\xi F(t) = F_1(v(\xi)t)$ for all $\xi \in W(R)$, so $F|_T$ determines $F_1|_{I_R T}$. Thus the given data allow us to reconstruct $F : P \to P$ and $F_1 : Q \to P$.

The linearization $(F \oplus F_1)^{\#} : (T \oplus L)^{(1)} \to P$ is a map of projective $W(R)$-modules of the same rank, so it suffices to show it is surjective. By hypothesis, P is generated by $\mathrm{im}(F_1)$, i.e. by elements of the form $F_1(v(\xi)t + l)$ with $v(\xi) \in I_R, t \in T, l \in L$. But note that $F_1(v(\xi)t + l) = \xi F(t) + F_1(l)$, and thus $F \oplus F_1$ is surjective, as desired. □

3 Describing Displays Via Matrices

We will explain, following Bueltel and Pappas [BP], how to think of displays as described by matrices. This will allow us to describe the moduli stack of displays very explicitly. It will also make it easier to formulate the notions of truncated displays (which are associated to truncated Barsotti-Tate groups).

Before doing so, we make a couple notational remarks.

1. We let $\mathrm{GL}_h(W)$ denote the functor $R \mapsto \mathrm{GL}_h(W(R))$ on p-nilpotent rings R. It is represented by a formally smooth affine group scheme over $\mathbb{Z}_p$, which follows from the existence of the Witt polynomials for addition and multiplication.

2. We let $\mathrm{GL}_h(W_n)$ denote the functor $R \mapsto \mathrm{GL}_h(W_n(R))$ on $\mathbb{F}_p$-algebras R. It is representable by an affine group scheme over $\mathbb{F}_p$; the natural truncation morphisms $\mathrm{GL}_h(W_n) \to \mathrm{GL}_h(W_{n-1})$ are smooth and $\mathrm{GL}_h(W_n)$ is smooth as well.

3.1 Displays

Let R be a p-nilpotent ring, and let $\mathscr{P} = (P, Q, F, F_1)$ be a display over R. Let $P = T \oplus L$ be a normal decomposition. Let us view $F \oplus F_1$ as a map in blocks

$$F \oplus F_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $A : T^{(1)} \to T, B : L^{(1)} \to T, C : T^{(1)} \to L$, and $D : L^{(1)} \to L$ are linear maps.

Working Zariski-locally on R, we may assume that T and L are free of rank d and $h - d$, respectively. Choosing a basis, write $T = \langle e_1, \ldots, e_d \rangle$ and $L = \langle e_{d+1}, \ldots, e_h \rangle$. By the lemma, $\mathscr{P}$ is determined by the block matrix

$$F \oplus F_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_h(W(R)),$$

where A has size $d \times d$ and D has size $(h - d) \times (h - d)$.

Using this data, a morphism of displays $\mathscr{P} \to \mathscr{P}'$ is given by an element $H \in M_h(W(R))$, but the condition that it sends $Q = I_R T \oplus L \subset P$ to $Q' \subset P'$ means that H can be written as a block matrix of the form

$$\begin{pmatrix} X & v(Y) \\ Z & T \end{pmatrix}.$$

where $v(Y)$ means apply v entry-wise to Y. Furthermore, the manifestation of the compatibility of H with F, F_1, F', F_1' is given by the matrix equation

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \sigma(X) & Y \\ p\sigma(Z) & \sigma(T) \end{pmatrix} = \begin{pmatrix} X & v(Y) \\ Z & T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The applications of σ come from the fact that F, F_1 are Frobenius-linear, while the factor of p in front comes from the relation $pF_1 = F$ on L. The map H is an isomorphism if the associated matrix is invertible.

Let $H^{(d,h-d)}(R)$ denote the subgroup of $\mathrm{GL}_h(W(R))$ consisting of matrices of the form $\begin{pmatrix} X & v(Y) \\ Z & T \end{pmatrix}$, and let $\Phi(H)$ denote the matrix $\begin{pmatrix} \sigma(X) & Y \\ p\sigma(Z) & \sigma(T) \end{pmatrix}$. Given $A \in \mathrm{GL}_h(W(R))$, we define the Φ-conjugation of A by H to be $H^{-1} A \Phi(H)$. The above analysis then gives

Proposition 3.1. The moduli stack $\mathrm{Disp}_\infty^{h,d}$ of displays of height h and dimension d over $\mathrm{Spf}\, \mathbb{Z}_p$ is isomorphic to the quotient stack $[\mathrm{GL}_h(W)/H^{(d,h-d)}]$ (where the action is Φ-conjugation).

3.2 Truncated displays

The notion of truncated displays is defined by Lau [L1], in the spirit of Zink's original definition of displays. However, this definition is rather complicated: it consists of the data of two $W_n(R)$-modules P and Q together with additional maps satisfying a bunch of properties. The advantage of the matrix approach to displays defined above is that we may define truncated displays in a much easier fashion. This notion ends up being equivalent to Lau's original definition, as shown by Lau and Zink [LZ].

For truncated displays, we will work over $\mathbb{F}_p$. This is because Frobenius will induce an honest endomorphism of $W_n(R)$ (as opposed to a map $W_{n+1}(R) \to W_n(R)$), so it still makes sense to talk about Frobenius-linear maps.

Intuitively, we will define n-truncated displays of height h and dimension d to be as in the definition of displays, except with $W_n(R)$ in place of $W(R)$ everywhere. However, there is one important caveat in this definition. Namely, in the case of the full Witt vectors, $v : W(R) \to W(R)$ is injective, and thus the Y entry of H is uniquely recoverable from $v(Y)$. However since $v : W_n(R) \to W_n(R)$ need not be injective anymore, we cannot uniquely recover Y from $v(Y)$. We want to remember the data of Y, so the way to fix this is by working with a subgroup $BP_n^{h,d} \subset \mathrm{GL}_h(W_n) \times \mathrm{GL}_h(W_n)$ (notation for Bueltel-Pappas, due to Drinfeld) which we define to be those pairs of block matrices $\left(\begin{pmatrix} \sigma(X) & Y \\ p\sigma(Z) & \sigma(T) \end{pmatrix}, \begin{pmatrix} X & v(Y) \\ Z & T \end{pmatrix}\right)$; Φ-conjugation above just becomes $(g,h) \cdot X = h^{-1}Xg$, and we get

Definition 3.2 (Definition/Proposition)**.** The moduli stack $\mathrm{Disp}_n^{h,d}$ of n-truncated displays of height h and dimension d is the quotient $[\mathrm{GL}_h(W_n)/BP_n^{h,d}]$.

As a consequence of the above descriptions, we can deduce a few things about the stacks of displays and truncated displays:

Corollary 3.3. Disp_n is a smooth algebraic stack over $\mathbb{F}_p$ of dimension 0. The natural truncation morphisms $\tau_n : \mathrm{Disp}_{n+1} \to \mathrm{Disp}_n$ are smooth and surjective.

As remarked above, n-truncated displays admit a vector bundle-type description as with usual displays, however we don't go into that here. Instead, we analyze the case of 1-truncated displays in more detail.

3.3 Example: 1-truncated displays and BT_1 groups

We discuss explicit descriptions of the functor of points of $\mathrm{Disp}_1^{h,d}$ and $\mathrm{BT}_1^{h,d}$ on $\mathbb{F}_p$-schemes which reveal more about their relationship, following [D]. In particular we will find that points of $\mathrm{BT}_1^{h,d}$ "look like" the de Rham cohomology of smooth proper schemes in positive characteristic, while points of $\mathrm{Disp}_1^{h,d}$ "forget the Gauss-Manin connection" associated to de Rham cohomology.

When $n = 1$, we have that $W_1(R) = R$ and $V = 0$. View elements of GL_h as block matrices corresponding to $(d, h-d)$ as above, and consider the following block subgroups:

$$M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, P^+ = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, P^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

The fact that $V = p = 0$ immediately imply

$$BP_1^{h,d} = \{(g,h) \in P^+ \times P^- : g_M = \sigma(h_M)\},$$

where g_M, h_M are the images of g and h in M (i.e. forget all but the diagonal blocks). Thus we have a short exact sequence

$$1 \to BP_1^{h,d} \to P^+ \times P^- \overset{g_M * \sigma(h_M)^{-1}}{\to} M \to 1.$$

To analyze the quotient stack $\mathrm{Disp}_1^{h,d} = [\mathrm{GL}_h / BP_1^{h,d}]$ further, we use the following two general facts about classifying stacks:

1. If $G \to H$ is a morphism of group schemes over a scheme S, then $[H/G]$ is an H-torsor on BG which induces a morphism $BG \to BH$ and thus a cartesian square

$$\begin{array}{ccc} [H/G] & \longrightarrow & S \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BH \end{array}$$

2. If $1 \to K \to G \to Q \to 1$ is an exact sequence of group schemes over a scheme S, then we have

$$BK = BG \times_{BQ} S.$$

Using these two facts, we find

$$[\mathrm{GL}_h / BP_1^{h,d}] \cong B(BP_1^{h,d}) \times_{B(\mathrm{GL}_h)} \mathbb{F}_p$$

and

$$B(BP_1^{h,d}) \cong B(P^+ \times P^-) \times_{BM} \mathbb{F}_p.$$

Combining, we get

Proposition 3.4. A 1-truncated display of height h and dimension d over an $\mathbb{F}_p$-scheme S consists of

1. a $P^\pm$-torsor $X^\pm$ on S;
2. an isomorphism $X_M^+ \cong F_S^*(X_M^-)$;
3. an isomorphism between the GL_h-torsors corresponding to X^+ and X^-.

Let us understand these conditions a bit more concretely. Recall that GL_h-torsors over a scheme S can be identified with vector bundles on S, via $X \mapsto (X \times \mathbb{A}_S^h)/\mathrm{GL}_h$. Since the subgroup P^+ of GL_h is the stabilizer of a d-dimensional subspace, we find that P^+-torsors over a scheme S can be identified with pairs $(\mathcal{E}, \mathcal{E}')$, where $\mathcal{E}$ is a vector bundle of rank h an $\mathcal{E}'$ is a subbundle of rank d. Similarly P^--torsors are pairs $(\mathcal{E}, \mathcal{E}')$ with $\mathcal{E}'$ a subbundle of rank $h - d$.

It follows that a $P^\pm$-torsor $X^\pm$ together with an isomorphism of the corresponding GL_h-bundles can be thought of as a vector bundle on S, together with two short filtrations coming from P^+ and P^- which are related via Frobenius. Thinking of the analogy with de Rham cohomology, a typical example of such a situation occurs by looking at the de Rham cohomology of a smooth proper S-scheme, whose fibers are such that the Hodge to de Rham spectral sequence degenerates. In this setting, the two filtrations will be the Hodge and conjugate filtrations.

The de Rham cohomology of algebraic varieties also comes equipped with a flat connection ∇, the Gauss-Manin connection.

Theorem 3.5 (Conjectured by Drinfeld [D], Proven by Lau (unpublished))**.** An object of $\mathrm{BT}_1^{h,d}$ over an $\mathbb{F}_p$-scheme S consists of the 3 pieces of data described above, and additionally a flat connection ∇ on X^+ satisfying certain additional conditions (e.g. the Katz p-curvature condition).

Under these descriptions, the morphism $\mathrm{BT}_1^{h,d} \to \mathrm{Disp}_1^{h,d}$ is simply the map which forgets the connection.

4 From Barsotti-Tate Groups to Displays

4.1 The Display Functor $\phi_\infty : \mathrm{BT}_\infty^{h,d} \to \mathrm{Disp}_\infty^{h,d}$ over $\mathrm{Spf}\,\mathbb{Z}_p$

We sketch how to associate a display to a p-divisible group over a p-nilpotent ring R, following Lau [L1], [L2, Construction 3.16]. The construction proceeds in two steps:

1. First, one uses crystalline Dieudonné theory to associate to a p-divisible group G/R an object called a filtered F-V module over R. We will explain this step in a special case below.

2. Then one shows that actually this filtered F-V module can be upgraded to a display over R. As with the setting of Dieudonné modules, this is not so hard to do if $W(R)$ has no p-torsion, but Lau [L1] shows that this can always be done. His proof uses the Grothendieck theorem of the smoothness of the truncation morphisms $\mathrm{BT}_{n+1} \to \mathrm{BT}_n$.

Definition 4.1. Let R be a p-nilpotent ring. A filtered F-V module over R is a quadruple $(P, Q, F^\#, V^\#)$ where P is a finitely generated projective $W(R)$-module with a filtration $I_R P \subset Q \subset P$ such that P/Q is projective over R, and $F^\# : P^{(1)} \to P, V^\# : P \to P^{(1)}$ are $W(R)$-linear maps composing to p.

Thus a filtered F-V module is just a Dieudonné module with a short filtration (which doesn't interact with F or V at all). We can use crystalline Dieudonné theory to construct a functor $\Theta_R : \{p\text{-divisible groups}/R\} \to \{\text{filtered } F\text{-}V\text{-modules}/R\}$. To avoid using general constructions involving the crystalline Dieudonné functor, we will suppose that $G = \mathcal{A}[p^\infty]$, where $\mathcal{A}/R$ is an abelian scheme.

A key observation is that $W(R)$ is a PD thickening of R and of R/p, and moreover if $\mathcal{A}_0$ denotes the fiber of $\mathcal{A}$ over R/p, we have $H^1_{\mathrm{cris}}(\mathcal{A}/W(R)) = H^1_{\mathrm{cris}}(\mathcal{A}_0/W(R))$. Let $P = H^1_{\mathrm{cris}}(\mathcal{A}/W(R))$, a finite free $W(R)$-module. Let $Q_0 \subset H^1_{\mathrm{dR}}(\mathcal{A}/R)$ be the first step of the Hodge filtration, i.e. $Q_0 = H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/R})$. Then we may let $Q \subset P$ be the unique lift of Q_0 to a $W(R)$-submodule of P which contains $I_R P$. Now we may let $F^\#$ and $V^\#$ be induced from the natural maps $F : \mathcal{A}_0 \to \mathcal{A}_0^{(1)}, V : \mathcal{A}_0^{(1)} \to \mathcal{A}_0$, and in this way we obtain our filtered F-V module.

Note that the tuple $(P, F^\#, V^\#)$ really only depends on the restriction of G to R/p, while the submodule Q really depends on G over all of R. This is analogous to the phenomenon where lifts of abelian varieties to characteristic 0 determine filtrations on the crystalline cohomology (via the identification of crystalline cohomology with de Rham cohomology of a lift).

In the general case, we do something similar, where we use properties of the crystalline Dieudonné functor as a substitute for properties of de Rham cohomology of abelian schemes.

There is a natural functor $\Upsilon_R : (\mathrm{Disp}/R) \to \{\text{filtered } F\text{-}V \text{ modules over } R\}$, given by $\Upsilon_R(P, Q, F, F_1) = (P, Q, F^\#, V^\#)$, where $F^\#$ is the linearization of F and $V^\#$ is determined by $V^\#(F_1(x)) = 1 \otimes x$ for $x \in Q$. As in the case of Dieudonné modules, we can go backwards when $W(R)$ has no p-torsion. Using Grothendieck's smoothness theorem, Lau reduces to the p-torsion free case and shows that for each p-divisible group G/R with associated filtered F-V module $(P, Q, F^\#, V^\#)$, there is a unique map $F_1 : Q \to P$ which is functorial in G and R such that (P, Q, F, F_1) is a display which induces $V^\#$. The point is that it suffices to construct a display associated to a filtered F-V module in the universal case, and Grothendieck's theorem allows us to make sure that the universal case is such that there is no p-torsion.

4.2 The Truncated Display Functor $\phi_n : \overline{\mathrm{BT}}_n^{h,d} \to \mathrm{Disp}_n^{h,d}$ over $\mathbb{F}_p$

For each $\mathbb{F}_p$-algebra R, there is a functor from $\mathrm{BT}_n(R) \to \mathrm{Disp}_n(R)$ which is compatible with base change in R and with the natural truncation functors on both sides, and thus a morphism $\mathrm{BT}_n^{h,d} \to \mathrm{Disp}_n^{h,d}$ of algebraic stacks over $\mathbb{F}_p$.

The rough idea for the construction of the above morphism is as follows. Assume that our n-truncated Barsotti-Tate group G_0/R can be written as the kernel of a morphism $G \to H$ of p-divisible groups over R. Then the idea is to pass to the associated displays of G and H, truncate, and take the cokernel. One reduces to this case using Grothendieck's smoothness theorem; Grothendieck tells us that for $G_0 \in \mathrm{BT}_n(R)$, there is a sequence of faithfully flat ring maps $R = R_0 \to R_1 \to \ldots$ such that $G \otimes_R \varinjlim R_i$ is the p^n torsion of a p-divisible group over R. For more details, we refer to [L1].

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