# Revisiting Failure of Hodge Symmetry in Characteristic $p$

Casimir Kothari
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## 1 Introduction

Let $X/\mathbb{C}$ be a smooth projective complex algebraic variety. For any $n \geq 0$ Hodge theory tells us that there is a decomposition

$$H^n_{\mathrm{dR}}(X/\mathbb{C}) \cong \bigoplus_{i+j=n} H^{i,j}(X;\mathbb{C})$$

where $H^{i,j}(X;\mathbb{C})$ are the Dolbeault cohomology groups

$$H^{i,j}(X;\mathbb{C}) := \frac{\ker(\overline{\partial} : \mathcal{A}^{i,j}(X) \to \mathcal{A}^{i,j+1}(X))}{\mathrm{im}(\overline{\partial} : \mathcal{A}^{i,j-1}(X) \to \mathcal{A}^{i,j}(X))}.$$

Hodge theory moreover tells us that complex conjugation swaps that $(i,j)$ and $(j,i)$ pieces of the decomposition, so that in particular we have *Hodge Symmetry*:

$$\dim_{\mathbb{C}} H^{i,j}(X;\mathbb{C}) = \dim_{\mathbb{C}} H^{j,i}(X;\mathbb{C}).$$

**Question 1.1.** Does Hodge symmetry hold for smooth projective algebraic varieties over fields other than $\mathbb{C}$?

First, we need a way to generalize the objects involved in Hodge theory to a purely algebraic setting. The first key observation is that due to the $\overline{\partial}$-Poincaré lemma (which asserts that any $\overline{\partial}$-closed form is locally an exact form) and an algebraic comparison theorem, we have isomorphisms

$$H^{i,j}(X;\mathbb{C}) \cong H^j(X, \Omega^i_{X,\mathrm{hol}}) \cong H^j(X, \Omega^i_{X,\mathrm{alg}}), \tag{1}$$

where:

1. The latter two groups are sheaf cohomology;
2. $\Omega^i_{X,\mathrm{hol}}$ is the sheaf of holomorphic $i$-forms on $X$ that associates to any open subset $U \subset X$ the holomorphic $i$-forms on $U$;
3. $\Omega^i_{X,\mathrm{alg}}$ is the sheaf of Kähler (or algebraic) differentials of $X$, which roughly speaking record those differential forms which locally look like $f(x)dx_1 \wedge \cdots \wedge dx_i$ with $f$ a polynomial. Note that $\Omega^0_X = \mathcal{O}_X$, the sheaf of regular functions on $X$. It is important to note here that $\Omega^i_{X,\mathrm{alg}}$ is a sheaf on $X$ in the *Zariski* topology, whereas the others were sheaves in the usual complex topology.

The benefit of making these comparisons is that now we have a purely algebraic way to speak about Hodge theoretic considerations, since the sheaf $\Omega^i_{X,\mathrm{alg}}$ can be defined for any algebraic variety. We will denote this sheaf by $\Omega^i_X$ going forward and will forget the other objects involved above. With this, we can formulate our main question of interest purely algebraically:

**Question 1.2.** If $X$ is a smooth projective variety over a field $k$, is it always true that

$$\dim_k H^i(X, \Omega^j_X) = \dim_k H^j(X, \Omega^i_X)$$

for all $i,j$?

As we've seen above, Hodge theory and the comparison isomorphisms (1) imply that the answer is yes for $k = \mathbb{C}$, and in fact for any $k$ of characteristic 0. So from now on we will let $k$ be an algebraically closed field of characteristic $p > 0$, and for simplicity assume $p \geq 5$. The main result is that Hodge symmetry need not hold:

**Theorem 1.3** (Serre, 1958)**.** There exists a smooth projective algebraic surface $X$ over $k$ such that $H^0(X, \Omega^1_X) = 0$ and $H^1(X, \mathcal{O}_X) \neq 0$. In particular, Hodge symmetry need not hold in characteristic $p > 0$.

In this talk, we will begin by briefly explaining Serre's original argument. Then we will explain how to interpret it using a more modern perspective using the classifying space of a finite group. [1]

# 2 Serre's Example

The starting point for Serre's example is the following:

**Proposition 2.1.** There exists a nontrivial action of the group $G = \mathbb{Z}/p\mathbb{Z}$ on $\mathbb{P}^3_k$ and a smooth $G$-stable hypersurface $Y \subset \mathbb{P}^3$ on which the action is free.

*Proof.* To construct an action of $G$ on $\mathbb{P}^3_k$, we can specify an action of $G$ on $k^4$ and then projectivize, or equivalently give a map $G \to \mathrm{GL}_4(k) \to \mathrm{PGL}_4(k)$. Thus consider the 4-dimensional representation of $G = \mathbb{Z}/p\mathbb{Z}$ which sends the generator to

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 1 + N$$

Since $N^4 = 0$, we have that since $k$ has characteristic $p$ that $A^p = (1+N)^p = 1$, and so this is indeed a representation of $G$. Since the only eigenvector of $A$ is $(0,0,0,1)$, we find that the only fixed point in the action of $G$ on $\mathbb{P}^3_k$ is $[0:0:0:1]$. Now the quotient $Q := \mathbb{P}^3_k/G$ turns out to be a projective variety, and by Bertini's theorem we can find a smooth hyperplane section $X$ of $Q$ which does not contain the image of the point $[0:0:0:1]$; it follows that the preimage $Y$ of $Z$ in $\mathbb{P}^3$ is a smooth, $G$-stable hypersurface which doesn't contain the unique fixed point $[0:0:0:1]$, as desired. □

Since the action of $G$ on $Y$ is free, the quotient variety $X := Y/G$ is a smooth, projective algebraic surface over $k$.

**Theorem 2.2.** Hodge symmetry fails for $X$.

*Proof.*
1. $H^0(X, \Omega^1_X) = 0$, since global holomorphic 1-forms on $X$ are the same as $G$-invariant global holomorphic 1-forms on its finite cover $Y$. But $Y$ is a hypersurface in $\mathbb{P}^3$ and thus has no global holomorphic 1-forms.
2. One way to see that $H^1(X, \mathcal{O}_X) \neq 0$ is to use the Hochschild-Serre spectral sequence for the étale $G$-cover $Y \to X$.
$$E_2^{pq} = H^p(G, H^q(Y, \mathcal{O}_Y)) \implies H^{p+q}(X, \mathcal{O}_X)$$
which yields an isomorphism
$$H^1(X, \mathcal{O}_X) \cong H^1(G, k) \cong k.$$

□

In the remainder of the talk, I want to explain an alternate approach to computing the cohomology of our quotient $X = Y/G$, which to me provides a more satisfying explanation as to why Hodge symmetry fails for $X$. So as not to keep you in suspense, let me sketch the idea behind this.

[1] Serre's original argument can be found in [S]. The relation of Serre's method to the cohomology of classifying spaces is discussed in [ABM].

1. There is a classifying space (stack) $BG$ over $k$, which comes equipped with sheaves $\mathcal{O}$ and $\Omega^1$ of regular functions and algebraic differentials, respectively. It turns out that

$$H^0(BG, \Omega^1) = 0 \text{ and } H^1(BG, \mathcal{O}) = k,$$

so Hodge symmetry fails for $BG$. We would like to produce from this example an example of a *smooth projective algebraic variety* for which Hodge symmetry fails.

2. Since $X = Y/G$ is a hyperplane section of the projective variety $\mathbb{P}^3/G$, the Lefschetz hyperplane theorem implies that

$$H^0(X, \Omega^1_X) \cong H^0(\mathbb{P}^3/G, \Omega^1) \text{ and } H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3/G, \mathcal{O}).$$

3. The action of $G$ on $\mathbb{P}^3$ makes the quotient $\mathbb{P}^3/G$ a $\mathbb{P}^3$-bundle over the classifying space $BG$. Since the cohomology of a projective space bundle depends only on the base space and the fiber, we can now compute the cohomology of $X$ completely in terms of the cohomology of $BG$. It will turn out that

$$H^0(X, \Omega^1_X) = H^0(BG, \Omega^1) = 0 \text{ and } H^1(X, \mathcal{O}_X) = H^1(BG, \mathcal{O}) \cong k.$$

The point: allow ourselves to work with "complicated" objects like stacks which are easier to compute with, and then approximate by classical objects (varieties). Now I would like to explain how the above works in more detail. We will begin by reviewing some material about classifying spaces and cohomology.

**Remark 2.3.** The correct language to work with on the algebraic side is that of stacks. In order to convey the essential ideas of the argument, we choose not to use the language of stacks, and as a result there are some creative liberties taken with the discussion of classifying stacks and their cohomology in subsection 3.2.

# 3 Classifying Spaces and Cohomology

## 3.1 The topological story

Let $G$ be a finite group with the discrete topology (much of what we say here goes through for any topological group but only finite groups matter for our situation). We have the *classifying space* $BG$, which is the quotient of a weakly contractible space $EG$ (i.e. $\pi_i(EG) = 0$ for all $i \geq 0$) by a free $G$-action. It has the important classification property that any normal (i.e. transitive action on a fiber) covering map $f : M \to N$ of "nice" spaces with deck group $G$ (or more generally any *principal $G$-bundle*) is obtained via pullback

$$\begin{array}{ccc} M & \longrightarrow & EG \\ \downarrow & & \downarrow \\ N & \longrightarrow & BG. \end{array}$$

**Proposition 3.1.** Suppose that $G$ is a discrete group (e.g. $G$ is a finite group with the discrete topology). Then $BG$ is a $K(G, 1)$ space.

*Proof.* We have a fibration

$$G \hookrightarrow EG \to BG.$$

Taking the long exact sequence in homotopy, we get segments

$$\pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G).$$

Since $G$ is discrete and $EG$ is weakly contractible, we have $\pi_i(BG) = 0$ for $i = 0$ and $i \geq 2$. When $i = 1$, we find

$$\pi_1(BG) \cong \pi_0(G) \cong G.$$

□

**Example 3.2.** $B\mathbb{Z} = S^1, E\mathbb{Z} = \mathbb{R}$. $B\mathbb{Z}/2\mathbb{Z} = RP^\infty, E\mathbb{Z}/2\mathbb{Z} = S^\infty$, the unit sphere in $\mathbb{R}^\infty$.

It follows that we can describe $BG$ via the usual cellular decomposition for a $K(G,1)$ space, which we now recall.

We want to build a connected topological space $X$ with $\pi_1(X) = G$ and $\pi_i(X) = 0$ for all $i \geq 2$. So we start with a point, then add a loop labeled by every element of $G$. But now we've added extraneous elements of $\pi_1$, since traversing the loop $g$ then $h$ is distinct from traversing $gh$. To remedy this, we need to attach 2-cells with sides $g, gh, h$ for every pair of elements $(g,h) \in G \times G$. But now we've added nontrivial $\pi_2$, in the form of tetrahedra with sides $g_2, g_3, g_2g_3, g_1, g_1g_2g_3, g_1g_2$. So we need to kill this homotopy by gluing in some 3-cells with these sides. But now we've introduced $\pi_3$, so we need to add some 4-cells, etc.....

We can now illustrate the power of a simplicial decomposition in computing cohomology:

**Proposition 3.3.** For a group $G$, we have an isomorphism

$$H^*_{\text{sing}}(BG; \mathbb{Z}) \cong H^*(G; \mathbb{Z}),$$

where the left hand side is singular cohomology and the right hand side is group cohomology.

*Proof.* By the universal coefficients theorem, it suffices to show that the homology of $BG$ is the same as the group homology of $G$. Recall that group homology of a group $G$ is defined as the left derived functor of the coinvariants functor, and we can compute it by resolving $\mathbb{Z}$ by projective $\mathbb{Z}[G]$ modules, tensoring with $\mathbb{Z}$ over $\mathbb{Z}[G]$, and taking homology.

On the other hand, we can compute singular homology of $BG$ as cellular homology using the above decomposition. We find that the homology of $BG$ is computed as the homology of the complex

$$\cdots \to \mathbb{Z}[G \times G \times G] \to \mathbb{Z}[G \times G] \to \mathbb{Z}[G] \xrightarrow{0} \mathbb{Z}.$$

where the differential is the alternating sum of the face maps in the simplicial decomposition of $BG$. But we can recognize this complex as a standard complex ([B, Section II.3]) which computes group homology (namely it is what happens when one applies the coinvariants functor to the "standard" resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module), and this gives the desired result. ☐

We will again make use of this simplicial decomposition when we go to compute sheaf cohomology of $BG$ algebraically.

## 3.2 The algebraic story

Now let $G$ be a finite abelian group with the discrete topology, and let $k$ be a field. Note that we can view $G$ as a group variety over $k$, consisting of a finite set of points which correspond to the elements of the group.

It turns out that in this setting, one can make sense of the classifying space $BG$ over the field $k$ purely algebraically, and $BG$ in this setting still has many of the same desirable properties:

1. $BG$ can still be described via the same simplicial decomposition, which we can use to understand its cohomology.[2]

2. Finite $G$-coverings of varieties over $k$ are still classified by maps to $BG$, and in particular a variety $Y$ with a $G$-action leads to a quotient space $Y/G$ which maps to $BG$. Key point: because of stackiness, this still holds true even when the action is not free, and in particular $\mathbb{P}^n/G$ always carries the structure of a $\mathbb{P}^n$ bundle over $BG$ when considered as an algebraic stack. This last fact is essentially due to the diagram

$$\begin{tikzcd} \mathbb{P}^n \arrow[r] \arrow[d] & EG \arrow[d] \\ \mathbb{P}^n/G \arrow[r] & BG. \end{tikzcd}$$

   which exhibits $\mathbb{P}^n$ as the fiber of the map $\mathbb{P}^n/G \to BG$.

[2] More precisely, we can compute sheaf cohomology of $BG$ via the Cech nerve of the smooth cover $\operatorname{Spec} k \to BG$, as in [Sta, Tag 06XJ].

Being algebraic, $BG$ comes equipped with regular functions and Kähler differentials, and so it makes sense to talk about the cohomology groups $H^i(BG, \Omega^j)$. We can compute these cohomology groups using the simplicial decomposition in much the same way as before.

**Proposition 3.4.** Let $G$ be a finite abelian group (with the discrete topology). Then

$$H^i(BG, \Omega^j) = 0 \text{ for all } j > 0$$

and

$$H^i(BG, \mathcal{O}) = H^i(G; k),$$

where the latter is group cohomology with coefficients in $k$.

*Proof.* Since $G$ is discrete and therefore 0-dimensional, we have

$$\Omega^j_G = 0$$

for all $j > 0$, which implies that $H^i(BG, \Omega^j) = H^i(BG, 0) = 0$ for all $j > 0$.

Next, we would like to use the simplicial decomposition of $BG$ to compute $H^i(BG, \mathcal{O})$. Morally, we consider the simplicial decomposition, apply $\mathcal{O}$ to everything, and totalize. Note that $\mathcal{O}(G^p)$ is by definition just regular functions on $G^p$ with values in $k$, and so is isomorphic to $k[G^p]$ (since $G$ is finite). Thus we find that $H^*(BG, \mathcal{O})$ is the cohomology of the complex

$$k \to k[G] \to k[G^2] \to \dots$$

which is just a standard complex for computing $H^*(G; k)$. $\square$

**Corollary 3.5.** For the group $G = \mathbb{Z}/p\mathbb{Z}$ and $k$ a field of characteristic $p$, we have

$$H^0(BG, \Omega^1) = 0 \text{ and } H^1(BG, \mathcal{O}) \cong k.$$

In particular, Hodge symmetry fails for the classifying space $B\mathbb{Z}/p\mathbb{Z}$ over $k$.

*Proof.* The fact that $H^0(BG, \Omega^1) = 0$ follows directly from the above proposition, while the second part follows from the above proposition and an explicit calculation of the group cohomology of a group of order $p$ with coefficients in a characteristic $p$ field. $\square$

# 4 Serre's Example Revisited

We are now ready to revisit Serre's example with our new perspective. Recall that $G = \mathbb{Z}/p\mathbb{Z}$ acts on $\mathbb{P}^3_k$, and $Y \subset \mathbb{P}^3_k$ is a smooth hypersurface on which the action is free, such that the quotient $X := Y/G$ is a smooth projective variety (obtained as a hyperplane section of $\mathbb{P}^3/G$). By the Lefschetz hyperplane theorem, we have isomorphisms

$$H^0(X, \Omega^1) = H^0(\mathbb{P}^3/G, \Omega^1) \text{ and } H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3/G, \mathcal{O}_X).$$

But now $\mathbb{P}^3/G$ is a projective bundle over $BG$, so we can apply the Projective Bundle Formula, which says that the cohomology of $\mathbb{P}^3/G$ is the same as that of $\mathbb{P}^3 \times BG$. So

$$H^0(\mathbb{P}^3/G, \Omega^1) = H^0(BG, \Omega^1) = 0$$

and

$$H^1(\mathbb{P}^3/G, \mathcal{O}) = H^1(BG, \mathcal{O}) \cong k,$$

and the result follows.

## References


[ABM] Benjamin Antieau, Bhargav Bhatt, and Akhil Mathew. Counterexamples to Hochschild-Kostant-Rosenberg in characteristic $p$. *Forum Math. Sigma*, 9:Paper No. e49, 26, 2021.

[B] Kenneth S. Brown. Cohomology of Groups. Springer GTM Volume 87, 1982.

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