THE HODGE NUMBERS OF PROJECTIVE HYPERSURFACES

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In this note, we describe how to use properties of coherent sheaf cohomology to obtain the Hodge numbers of nonsingular hypersurfaces in projective space over a field k. Given a variety X over k of dimension n, recall that the Hodge numbers $h^{p,q}(X), 0 \leq p, q \leq n$ are defined by

$$h^{p,q}(X) := \dim_k H^q(X, \Omega_X^p),$$

where Ω_X is the sheaf of Kähler differentials of X over k and $\Omega_X^p := \wedge^p \Omega_X$.

0.1. **Hodge Numbers of Projective Space.** Let $X = \mathbb{P}_k^n$ be projective n-space over k. To begin, we will prove that $h^{p,q} = \delta_{p=q}$ for any $0 \leq p, q, \leq n$.

We proceed by induction on p; our strategy is to reduce this computation to twisting sheaves on X. Our starting point is the following exact sequence relating the 1-forms on X to a direct sum of twisted sheaves:

Lemma 0.1 (Euler Exact Sequence)**.** *We have an exact sequence of sheaves on X*

$$0 \to \Omega_X^1 \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0.$$

We explain the intuition for this sequence over $\mathbb{C}$ as follows. Dualizing, it is equivalent to the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus n+1} \to \mathcal{T}_X \to 0.$$

Now on $\mathbb{C}^{n+1} - 0$, one has the standard coordinate vector fields $\partial_i = d/dx_i$, and a tangent vector field $\sum_i f_i \partial_i$ descends to a vector field on $\mathbb{P}^n$ when each f_i is a linear form in the x_i; this yields a surjection $\mathcal{O}_X(1)^{\oplus n+1} \to \mathcal{T}_X$, sending $[f_0 : \cdots : f_n] \to \sum_i f_i \partial_i$. The kernel of this map is those forms $\sum_i a_i \partial_i$ which point normal to the unit sphere in $\mathbb{C}^{n+1}$, and hence is a free $\mathcal{O}_X$-module of rank 1, generated by $\sum_i x_i \partial_i$.

To get from Ω_X^1 to Ω_X^p, we also need to understand wedge powers of exact sequences of sheaves, which is handled by the following lemma:

Lemma 0.2. *Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. For any $p \geq 0$, there is a filtration*

$$\wedge^p \mathcal{F} = \mathcal{G}^0 \supset \mathcal{G}^1 \supset \cdots \supset \mathcal{G}^{p+1} = 0$$

such that

$$\mathcal{G}^i / \mathcal{G}^{i+1} \cong (\wedge^i \mathcal{F}') \otimes (\wedge^{p-i} \mathcal{F}'')$$

for each i.

Proof. First, let us assume that $\mathcal{F}', \mathcal{F}$, and $\mathcal{F}''$ are all free on X. This allows us to choose a splitting $\mathcal{F} \cong \mathcal{F}' \oplus \mathcal{F}''$, yielding an isomorphism

$$\wedge^p \mathcal{F} \cong \bigoplus_{i=0}^{p} \left(\wedge^i \mathcal{F}' \otimes \wedge^{p-i} \mathcal{F}'' \right).$$

For each $0 \leq j \leq p+1$, define $\mathcal{G}^j$ to be the image of $\bigoplus_{i \geq j} (\wedge^i \mathcal{F}' \otimes \wedge^{p-i} \mathcal{F}'')$ in $\wedge^p \mathcal{F}$. Then clearly the $\mathcal{G}^j$ yield the desired filtration in the free case.

Now observe that the construction of $\mathcal{G}^j$ above is in fact independent of our choice of splitting. Indeed, let ψ denote the map $\mathcal{F}' \to \mathcal{F}$, and suppose we had two sections $\phi, \phi' : \mathcal{F}'' \to \mathcal{F}$. Fix a basis $x_1, \ldots, x_n$ of $\mathcal{F}'$ and $y_1, \ldots, y_m$ of $\mathcal{F}''$. Via ϕ, the image of $\wedge^j \mathcal{F}' \otimes \wedge^{p-j} \mathcal{F}''$ in $\wedge^p \mathcal{F}$ is spanned by elements of the form

$$\psi(x_{i_1}) \wedge \cdots \wedge \psi(x_{i_j}) \wedge \phi(y_{k_1}) \wedge \cdots \wedge \phi(y_{k_{p-j}}).$$

Date: Winter 2022.

But for each $1 \leq k \leq m$, we can write $\phi(y_j) = \sum_i a_i\psi(x_i) + \sum_j b_j\phi'(y_j)$, and substituting into the above expression shows that the image of $\bigoplus_{i\geq j}(\wedge^i\mathcal{F}' \otimes \wedge^{p-i}\mathcal{F}'')$ in $\wedge^p\mathcal{F}$ is independent of the choice of section $\mathcal{F}'' \to \mathcal{F}$, as desired.

In the general case, we fix an open cover $\{U_i\}$ of X on which each of $\mathcal{F}, \mathcal{F}'$, and $\mathcal{F}''$ are free. The splitting-independence implies that the local constructions $\mathcal{G}^i|_{U_i}$ glue to a well-defined sheaf $\mathcal{G}^i$ on X and the local isomorphisms $\mathcal{G}^i/\mathcal{G}^{i+1}|_{U_j} \cong (\wedge^i\mathcal{F}'|_{U_j}) \otimes (\wedge^{p-i}\mathcal{F}''|_{U_j})$ glue to a global isomorphism, yielding the desired filtration. □

In our applications, we will always have one of $\mathcal{F}'$ or $\mathcal{F}''$ locally free of rank 1, and hence it is useful to introduce the following corollary.

Corollary 0.3. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of locally free sheaves and $\mathcal{F}'$ has rank 1, then for any $p \geq 0$ we have an exact sequence*

$$0 \to \mathcal{F}' \otimes \wedge^{p-1}\mathcal{F}'' \to \wedge^p\mathcal{F} \to \wedge^p\mathcal{F}'' \to 0.$$

If instead $\mathcal{F}''$ has rank 1, then for any $p \geq 0$ we have an exact sequence

$$0 \to \wedge^p\mathcal{F}' \to \wedge^p\mathcal{F} \to \wedge^{p-1}\mathcal{F}' \otimes \mathcal{F}'' \to 0.$$

Proof. Let $\wedge^p\mathcal{F} = \mathcal{G}^0 \supset \cdots \supset \mathcal{G}^{p+1} = 0$ be the filtration guaranteed by the previous lemma. Then since $\mathcal{F}'$ is invertible, $\mathcal{G}^i/\mathcal{G}^{i+1} = 0$ for all $i \geq 2$, and hence $\mathcal{G}^2 = \mathcal{G}^3 = \cdots = \mathcal{G}^{p+1} = 0$. Thus we have a filtration $\wedge^p\mathcal{F} \supset \mathcal{G}^1 \supset 0$ with $\wedge^p\mathcal{F}/\mathcal{G}^1 = \wedge^p\mathcal{F}''$ and $\mathcal{G}^1 = \mathcal{F}' \otimes \wedge^{p-1}\mathcal{F}''$, giving the desired exact sequence. The second sequence follows by an analogous argument. □

Returning to the Euler exact sequence, we may apply Corollary 0.3 to obtain

Proposition 0.4 (Key Exact Sequence)**.** *Let $X = \mathbb{P}^n_k$. Then for $p \geq 0$ we have an exact sequence*

$$0 \to \Omega^p_X \to \wedge^p\left(\mathcal{O}_X(-1)^{\oplus n+1}\right) \to \Omega^{p-1}_X \to 0. \tag{0.1}$$

We can now compute the Hodge numbers of projective space by induction on $p \leq n$. For $p = 1$, we take the long exact sequence in cohomology associated to the Euler sequence, obtaining segments

$$H^{i-1}(X, \mathcal{O}_X) \to H^i(X, \Omega^1_X) \to H^i(X, \mathcal{O}_X(-1))^{n+1}.$$

If $2 \leq i \leq n-1$, we know that the left and right terms vanish, and hence we conclude $H^i(X, \Omega^1_X) = 0$ for such i. We also know $H^n(X, \mathcal{O}_X(-1)) = 0$, so that the end of this long exact sequence shows $H^n(X, \Omega^1_X) = 0$ if $n > 1$. Finally, the beginning of the long exact sequence reads

$$0 \to H^0(X, \Omega^1_X) \to H^0(X, \mathcal{O}(-1))^{n+1} \to H^0(X, \mathcal{O}_X) \to H^1(X, \Omega^1_X) \to 0,$$

from which we deduce $H^0(X, \Omega^1_X) = 0$ and $H^1(X, \Omega^1_X) = k$. This proves the base case.

Let us now assume we know that $H^i(X, \Omega^{p-1}_X)$ is k in dimension $p-1$ and 0 otherwise. Taking the long exact sequence in cohomology associated to the Key Exact Sequence (0.1), we obtain pieces

$$H^{i-1}(X, \wedge^p\mathcal{O}_X(-1)^{n+1}) \to H^{i-1}(X, \Omega^{p-1}_X) \to H^i(X, \Omega^p_X) \to H^i(X, \wedge^p\mathcal{O}_X(-1)^{n+1}).$$

Now a direct calculation by induction and the isomorphism

$$\wedge^p\mathcal{O}_X(-1)^{n+1} \cong \wedge^p\mathcal{O}(-1)^n \oplus \left(\mathcal{O}(-1) \otimes \wedge^{p-1}\mathcal{O}(-1)^n\right)$$

shows that $\wedge^p\mathcal{O}_X(-1)^{n+1}$ is a direct sum of $\mathcal{O}_X(-p)$'s, and hence has vanishing cohomology in all dimensions since $p \leq n$. We therefore obtain

$$H^i(X, \Omega^p_X) = H^{i-1}(X, \Omega^{p-1}_X)$$

for all i, completing our computation.

0.2. **Hodge Numbers of a Nonsingular Projective Hypersurface.** Let $i : Y \hookrightarrow X = \mathbb{P}^n_k$ be a nonsingular hypersurface of degree d; we wish to compute $H^q(Y, \Omega^p_Y)$.

Our starting point is the dual to the conormal exact sequence of Y in X, which reads

$$0 \to i^{-1}\left(I_Y/I_Y^2\right) \to i^*\Omega^1_X \to \Omega^1_Y \to 0.$$

Since $I_Y = \mathcal{O}_X(-d)$, we compute

$$i^{-1}(I_Y/I_Y^2) = i^{-1}(I_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X/I_Y) = i^*(\mathcal{I}_Y) = \mathcal{O}_Y(-d).$$

Thus we have an exact sequence

$$0 \to \mathcal{O}_Y(-d) \to i^*\Omega^1_X \to \Omega^1_Y \to 0.$$

Since $\mathcal{O}_Y(-d)$ is locally free of rank 1 on Y, we can apply Corollary 0.3 to obtain

Proposition 0.5 (Key Exact Sequence 2)**.** *Let $i : Y \to X = \mathbb{P}^n_k$ be a nonsingular hypersurface of degree d. Then for $p \geq 0$ we have an exact sequence*

$$0 \to \Omega^{p-1}_Y(-d) \to i^*\Omega^p_X \to \Omega^p_Y \to 0.$$

Taking the associated long exact sequence in cohomology yields segments

$$H^q(Y, \Omega^{p-1}_Y(-d)) \to H^q(Y, i^*\Omega^p_X) \to H^q(Y, \Omega^p_Y) \to H^{q+1}(Y, \Omega^{p-1}_Y(-d)).$$

Arguing by induction on p, we obtain for $p+q < n-1$ an isomorphism

$$H^q(Y, \Omega^p_Y) \cong H^q(Y, i^*\Omega^p_X) \cong H^q(X, \Omega^p_X \otimes i_*\mathcal{O}_Y) = \delta_{p=q}.$$

By Serre duality, we also obtain the same result for $p+q > n-1$, so it remains to consider $p+q = n-1$. In this case, the definition of Euler characteristic gives

$$h^{p,q} = (-1)^n + (-1)^q \chi(\Omega^p_Y) + \delta_{p=q}. \tag{0.2}$$

Taking Euler Characteristics in Key Exact Sequence 2 gives (after an arbitrary twist)

$$\chi(\Omega^p_Y(i)) = \chi(i^*\Omega^p_X(i)) - \chi(\Omega^{p-1}_Y(i-d)).$$

Therefore we obtain

$$\chi(\Omega^p_Y(i)) = \chi(\Omega^p_X(i)) - \chi(\Omega^p_X(i-d)) - \chi(\Omega^{p-1}_Y(i-d)), \tag{0.3}$$

which allows us to inductively compute this Euler characteristic and hence the middle Hodge numbers.

Example 0.6. To illustrate the general method, we compute the Hodge number $h^{2,0}$ of a degree 4 hypersurface Y in $X = \mathbb{P}^3$. First, by (0.2) we have

$$h^{2,0}(Y) = \chi(\Omega^2_Y) - 1.$$

Using (0.3), we can write

$$\chi(\Omega^2_Y) = \chi(\Omega^2_X) - \chi(\Omega^2_X(-4)) - \chi(\Omega^1_Y(-4))$$
$$\chi(\Omega^1_Y(-4)) = \chi(\Omega^1_X(-4)) - \chi(\Omega^1_X(-8)) - \chi(\mathcal{O}_Y(-8)).$$

We then use the Key Exact Sequence (0.1) for $\mathbb{P}^3$ to compute

$$\chi(\Omega^1_X(-4)) = 4\chi(\mathcal{O}_X(-5)) - \chi(\mathcal{O}_X(-4)) = -15$$
$$\chi(\Omega^1_X(-8)) = 4\chi(\mathcal{O}_X(-9)) - \chi(\mathcal{O}_X(-8)) = -189$$
$$\chi(\Omega^2_X(-4)) = 6\chi(\mathcal{O}_X(-6)) - \chi(\Omega^1_X(-4)) = -45.$$

Finally, the exact sequence

$$0 \to \mathcal{O}_X(-12) \to \mathcal{O}_X(-8) \to \mathcal{O}_Y(-8) \to 0$$

shows that $\chi(\mathcal{O}_Y(-8)) = 130$. Putting these computations together yields

$$\chi(\Omega^2_Y) = 1 + 45 - 44 = 2,$$

so $h^{2,0}(Y) = 2 - 1 = 1$.

References

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[Ha77] Robin Hartshorne. *Algebraic Geometry.* Springer Volume 52, 1977.