DIEUDONNÉ THEORY FOR n-SMOOTH GROUP SCHEMES

CASIMIR KOTHARI AND JOSHUA MUNDINGER

Abstract. For all $n \geq 1$, there is a notion of n-smooth group scheme over any $\mathbb{F}_p$-algebra R, which may be thought of as a "Frobenius analogue" of n-truncated Barsotti-Tate groups over R. We show that the category of n-smooth commutative group schemes over R is equivalent to a certain full subcategory of Dieudonné modules over R. As a consequence, we show that the moduli stack Sm_n of n-smooth commutative group schemes is smooth over $\mathbb{F}_p$ and that the natural truncation morphism $\mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$ is smooth and surjective. These results affirmatively answer conjectures of Drinfeld.

1. Introduction

The goal of this paper is to prove a Dieudonné-theoretic classification for so-called n-smooth group schemes over an arbitrary $\mathbb{F}_p$-algebra. As a consequence, we obtain structural results on the moduli stacks of n-smooth group schemes, resolving conjectures of Drinfeld.

Classification of group schemes in terms of semilinear algebraic data began in 1955, when Dieudonné classified formal Lie groups over a perfect field in terms of modules over what is now known as the Dieudonné ring [Die55]. Since Dieudonné's original papers, many mathematicians have worked to extend this classification to more general groups and base rings. Cartier classified formal Lie groups over a general base ring [Car67]. Grothendieck suggested to attack the problem in general by defining a suitable notion of Dieudonné crystal [Gro74], a task taken up in the following decade [Mes72; MM74; BBM82]. More recently, Gabber and Lau gave a complete classification of commutative finite locally free group schemes of p-power order over a perfect $\mathbb{F}_p$-algebra [Lau13], and Anschütz, Le Bras, and Mondal classified such group schemes over a quasi-syntomic base using prismatic methods [ALB23; Mon24].

In this article, we restrict our attention to a particular kind of finite locally free group scheme called an n-smooth group scheme, and we prove a Dieudonné-theoretic classification of n-smooth commutative group schemes over an arbitrary $\mathbb{F}_p$-algebra. Our formulation and proof are elementary in nature, and make no appeal to crystalline or prismatic Dieudonné theory. Instead, we use homological properties of n-smooth group schemes and Cartier's theory of formal groups.

Fix a prime number p and let R be an $\mathbb{F}_p$-algebra.

Definition 1.0.1. Let $n \in \mathbb{N}$. An R-group scheme G is said to be *n-smooth* if Zariski-locally on $\operatorname{Spec} R$, there exists an isomorphism of R-schemes

$$G \cong \operatorname{Spec} \frac{R[x_1, \ldots, x_r]}{(x_1^{p^n}, \ldots, x_r^{p^n})}$$

for some $r \in \mathbb{N}$ taking the identity section of G to the section $x_1 = \cdots = x_r = 0$.

Such group schemes were studied in [Mes72, Chapter II.2] and [Gro74, Chapter VI §2] in connection with truncated Barsotti-Tate groups. As we will see, one may think of n-smooth group schemes as "Frobenius-analogues" of n-truncated Barsotti-Tate groups, with formal Lie groups playing the role of p-divisible groups. In particular, if $G/\operatorname{Spec} R$ is n-smooth, then G is finite locally free over R and $F^n : G \to G^{(p^n)}$ is zero.

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Let $W(R)$ denote the ring of Witt vectors of R, and $\sigma, v : W(R) \to W(R)$ denote the Frobenius and Verschiebung, respectively. The *Cartier–Dieudonné ring* D_R is by definition the associative ring generated by $W(R)$ and two elements F and V, subject to the relations $FV = VF = p$ and the relations $Fa = \sigma(a)F, V\sigma(a) = aV$, and $VaF = v(a)$ for all $a \in W(R)$.

Definition 1.0.2. Let $n \in \mathbb{N}$. A left D_R-module M is said to be *n-cosmooth* if the following conditions are satisfied:

(1) $V^n = 0$ on M;
(2) M/VM is a finitely generated projective R-module;
(3) For all $i \in \{1, \dots, n-1\}$, the sequence of abelian groups

$$M \xrightarrow{V^i} M \xrightarrow{V^{n-i}} M$$

is exact.

For a finite locally free commutative group scheme $G/\operatorname{Spec} R$, we let $G^\vee$ denote its relative Cartier dual.

Theorem A. *The functor*

$$G \mapsto \operatorname{Hom}(G^\vee, (W_n)_R) \tag{1}$$

defines an equivalence of categories between commutative n-smooth group schemes over R and n-cosmooth D_R-modules.

Remark 1.0.3. A 1-smooth group scheme is a finite locally free group scheme killed by F. Thus Theorem A in the case $n = 1$ is the well-known classification of height 1 group schemes in terms of restricted Lie algebras (see [Dri24, Theorem 2.4.2], which references [SGA3, Exposé VII.A]). At the other extreme, formal Lie groups over R are naturally viewed as ∞-smooth groups over R (see Lemma 2.4.1). Cartier theory classifies commutative formal Lie groups G over R in terms of the Cartier module $\operatorname{Hom}(G^\vee, W_R)$. Thus one may view Theorem A as an interpolation between the classifications for $n = 1$ and $n = \infty$.

Remark 1.0.4. It follows from Theorem A and Lemma 3.1.4 below that the quasi-inverse to the Dieudonné module functor (1) is the functor which associates to an n-cosmooth Dieudonné module M the n-smooth group scheme whose Cartier dual has functor of points $T \mapsto \operatorname{Hom}_{D_R}(M, W_n(T))$. Thus our equivalence is just the naive extension of the classical Dieudonné equivalence for unipotent group schemes over a perfect field [Gro74, Chapter II.4]. The only difference in our formulation is that since we are over an arbitrary base, we must work with the truncated Witt vectors W_n as opposed to the full Witt group $\varinjlim_n W_n$ (see Example 5.2.1 for further explanation of this point).

For all $n \geq 1$, let Sm_n be the moduli stack of commutative n-smooth group schemes. It follows from the definition that if G is $(n+1)$-smooth, then the Frobenius kernel $G[F^n]$ is n-smooth; this defines a truncation morphism $\tau_n : \mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$.

Theorem B. *For all $n \geq 1$, Sm_n is a smooth algebraic stack over $\mathbb{F}_p$, and the truncation morphism $\tau_n : \mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$ is smooth and surjective.*

The results of Theorems A and B affirmatively answer conjectures of Drinfeld [Dri23].

For $n \in \mathbb{N}$, let BT_n denote the $\mathbb{Z}$-stack of n-truncated Barsotti-Tate groups. Theorem B is the n-smooth analogue Grothendieck's smoothness result, which states that for all $n \geq 1$, BT_n is a smooth algebraic stack over $\mathbb{Z}$ and the trunation morphisms $\mathrm{BT}_{n+1} \to \mathrm{BT}_n$ induced by $G \mapsto G[p^n]$ are smooth and surjective [Ill85].

Remark 1.0.5. It is natural to ask how the equivalence of Theorem A compares to the classification results of [ALB23] and [Mon24] in the case when the base R is quasi-syntomic. We hope to address this question in future work.

Let us give some indications about the proofs of Theorems A and B. First, every n-smooth group scheme G is an iterated extension of 1-smooth group schemes, suggesting an inductive approach to the classification result. We begin by establishing that $\mathrm{Hom}(G^\vee, W_n)$ is an iterated extension of the restricted Lie algebra of G (Corollary 3.2.3). The main step is Lemma 3.2.2, which uses Grothendieck's formula for $\tau_{\leq 1} R\,\underline{\mathrm{Hom}}(-, \mathbb{G}_a)$ to deduce that certain Ext^1 classes are zero. To complete the proof of full faithfulness, we carefully compare the extension structure of G and $\mathrm{Hom}(G^\vee, W_n)$, using the snake lemma to conclude that the unit of a certain adjunction is an isomorphism.

To show essential surjectivity of the Dieudonné module functor, we appeal to Cartier's classification of formal Lie groups by ∞-cosmooth modules, which we review in §4.1. First, we show that every n-cosmooth module $\overline{M}$ can be Zariski-locally lifted to an ∞-cosmooth module M (Proposition 4.3.3). We then obtain via Cartier theory a formal Lie group H with Cartier module M, and $G := H[F^n]$ is then an n-smooth group scheme with $\mathrm{Hom}(G^\vee, W_n) \cong \overline{M}$.

Theorem B then follows from Theorem A and Proposition 4.3.1, which describes a local presentation for n-cosmooth modules.

The structure of this article is as follows. In Section 2, we state our conventions and recall some preliminaries on n-smooth group schemes, formal Lie groups, and the co-Lie complex. In Section 3 we set up a general adjunction (Lemma 3.1.4), establish canonical short exact sequences of Dieudonné modules (Corollary 3.2.3), and use these to prove full faithfulness of the Dieudonné module functor on n-smooth group schemes (Theorem 3.3.1). In Section 4 we prove some foundational results on n-cosmooth modules and establish the essential surjectivity of the Dieudonné module functor (Theorem 4.4.3), completing the proof of Theorem A. In Section 5 we establish compatibility of the Dieudonné functor (1) with base change, and make some explicit computations of Dieudonné modules over an arbitrary base. In Section 6, we prove Theorem B.

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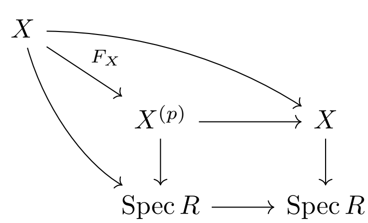
2. Preliminaries

2.1. **Notation and conventions.** All group schemes in this article are assumed commutative.

Throughout, p is a fixed prime number and R is a fixed $\mathbb{F}_p$-algebra. If $X/\operatorname{Spec} R$ is a scheme, then the *Frobenius twist* $X^{(p)}$ is defined to be the pullback of X along the absolute Frobenius morphism $\operatorname{Spec} R \to \operatorname{Spec} R$. The relative Frobenius morphism $F_X : X \to X^{(p)}$ is the R-linear morphism defined by the diagram



where $X \to X$ is the absolute Frobenius morphism. We will often write $F : X \to X^{(p)}$ for the relative Frobenius morphism F_X. For $n \geq 1$, $F^n : X \to X^{(p^n)}$ denotes the composition of relative Frobenius morphisms $X \to X^{(p)} \to \cdots \to X^{(p^n)}$. If G is a group scheme over $\operatorname{Spec} R$, then $F : G \to G^{(p)}$ is a morphism of groups, which is functorial in G ([SGA3, Expose VII_A, 4.2]).

Let $(W_n)_R$ be the ring scheme over $\operatorname{Spec} R$ of length n Witt vectors, and $W_R = \varprojlim_n (W_n)_R$ be the ring scheme of Witt vectors. Let $\sigma : W_R \to W_R$ be the canonical Frobenius lift. Since W_R is obtained via base change from $W_{\mathbb{F}_p}$, we can identify W_R and $W_R^{(p)}$, and under this identification, σ

is identified with the relative Frobenius F_{W_R}. As R is a fixed $\mathbb{F}_p$-algebra, we will omit subscripts in the sequel and write W_n and W in place of $(W_n)_R$ and W_R, respectively.

The Frobenius lift $\sigma : W(R) \to W(R)$ extends to a morphism $\sigma : D_R \to D_R$ which sends $V \mapsto V$ and $F \mapsto F$. If M is a left D_R-module, then $M|_{\sigma^i}$ denotes the restriction of scalars of M along $\sigma^i : D_R \to D_R$.

2.2. **Cartier duality and Verschiebung.** If $G/\operatorname{Spec} R$ is a finite locally free commutative group scheme, its relative Cartier dual, denoted $G^\vee$, is the group scheme $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$. Concretely, the Hopf algebra of functions of $G^\vee$ is the R-linear dual of $\mathcal{O}(G)$. Cartier duality induces an involutive anti-equivalence from the category of finite locally free R-group schemes to itself[1].

For all locally free group schemes $G/\operatorname{Spec} R$, there exists a Verschiebung homomorphism $V_G : G^{(p)} \to G$ ([SGA3, Expose VII$_A$, 4.3]); if the group G is understood, we write V for V_G. The homomorphism V_G is functorial in G, and we have $V_G \circ F_G = [p] : G \to G$ and $F_G \circ V_G = [p] : G^{(p)} \to G^{(p)}$. Moreover, when G is finite, V_G agrees with the Cartier dual of the Frobenius morphism $F_{G^\vee} : G^\vee \to (G^\vee)^{(p)}$. Furthermore, we can concretely identify the Verschiebung operator on Witt vectors with the usual shift operator:

Lemma 2.2.1 ([Dem72] III.3 Proposition, p. 57)**.** *Under the identification $(W_n)^{(p)} \cong W_n$, V_{W_n} is identified with the shift homomorphism $(x_1, \ldots, x_n) \mapsto (0, x_1, \ldots, x_{n-1})$.*

2.3. ***n*-smooth group schemes.** Recall the definition of n-smooth group schemes from Definition 1.0.1: an affine R-group scheme G is n-smooth if its ring of functions is locally isomorphic to $R[x_1, \ldots, x_r]/(x_1^{p^n}, \ldots, x_r^{p^n})$ where the identity section corresponds to $x_1 = \cdots = x_r = 0$. The following proposition makes precise the idea that n-smooth group schemes are "Frobenius-analogues" of n-truncated Barsotti-Tate groups (c.f. characterizations of BT_n groups in [Ill85, Remark 1.3]).

Lemma 2.3.1 ([Gro74], Chap. VI, Proposition 2.1)**.** *Let n be a positive integer and $G/\operatorname{Spec} R$ be a group scheme. Then the following are equivalent:*

(a) G is n-smooth;
(b) G is finite locally free, $F^n G = 0$, and the sequence

$$G \xrightarrow{F^i} G^{(p^i)} \xrightarrow{F^{n-i}} G^{(p^n)} \tag{2}$$

is exact[2] for all $i \in \{1, \ldots, n-1\}$;
(c) G is finite locally free, $F^n G = 0$, and the sequence (2) is exact for some $i \in \{1, \ldots, n-1\}$.

Example 2.3.2 (n-smooth group schemes of order p and p^2)**.** Over an algebraically closed field of characteristic p, the 1-smooth group schemes of order p are μ_p and α_p, and the 2-smooth group schemes of order p^2 are μ_{p^2}, α_{p^2}, and the p-torsion subgroup scheme of a supersingular elliptic curve.

If $G/\operatorname{Spec} R$ is an n-smooth group scheme, then it follows from the definition that the co-Lie algebra $\omega_G := e^*\Omega^1_{G/R}$ is a vector bundle over $\operatorname{Spec} R$, where $e : \operatorname{Spec} R \to G$ is the identity section. By duality, $\mathrm{Lie}(G) = \mathrm{Hom}(G^\vee, \mathbb{G}_a)$ is a vector bundle over R of the same rank.

Definition 2.3.3. The *rank* of an n-smooth group scheme G over $\operatorname{Spec} R$ is the rank of $\mathrm{Lie}(G)$, a locally constant function on $\operatorname{Spec} R$.

Definition 2.3.4. A group scheme $G/\operatorname{Spec} R$ is *n-cosmooth* if it is finite locally free and its Cartier dual $G^\vee$ is n-smooth.

[1]More generally, it is known that Cartier duality induces an involutive anti-equivalence between certain categories of group schemes and formal group schemes. For example, the duality between locally free R-modules and topological R-modules locally of the form $\prod_I R$ induces an anti-equivalence between locally free R-group schemes and locally free formal R-group schemes. For a discussion over a field, see [Dem72, §II.4].

[2]A sequence of affine R-group schemes is said to be exact if the corresponding sequence of fppf sheaves over $\operatorname{Spec} R$ is exact.

By definition, the *rank* of an n-cosmooth group scheme G is the rank of its Cartier dual $G^\vee$ in the sense of Definition 2.3.3. Cartier duality induces an anti-equivalence of categories between n-smooth and n-cosmooth group schemes over $\operatorname{Spec} R$. Since the equivalence in Theorem A factors through Cartier duality, we will work primarily with n-cosmooth group schemes.

For $r \geq 1$, let $\mathrm{Sm}_n^r/\mathbb{F}_p$ denote the moduli prestack of commutative n-smooth group schemes of rank r.

Lemma 2.3.5. *Sm_n^r is an fpqc algebraic stack of finite type over $\mathbb{F}_p$.*

Proof. Let $\mathcal{X}_n^r$ denote the stack of finite locally free group schemes of order p^{nr}. Then $\mathrm{Sm}_n^r \subset \mathcal{X}_n^r$ is a locally closed substack of $\mathcal{X}_n^r$: by [Dri24, Lemma 2.2.3] and upper semicontinuity of the rank of the Lie algebra, Sm_n^r is an open substack of the closed substack of $\mathcal{X}_n^r$ where $F^n = 0$. Since $\mathcal{X}_n^r$ is an fpqc algebraic stack of finite type over $\mathbb{F}_p$ (as can be seen via the presentation obtained from fixing a basis of the Hopf algebra of a group scheme of order p^{nr}), we obtain the lemma. □

We let Sm_n denote the $\mathbb{F}_p$-stack of n-smooth group schemes. It is the disjoint union of Sm_n^r over all $r \geq 1$.

2.4. **The relation to formal Lie groups.** From the definition, it follows that if G is N-smooth and $n \leq N$, then $G[F^n]$ is n-smooth. Thus, $G \mapsto G[F^n]$ defines the *truncation morphism* $\mathrm{Sm}_N \to \mathrm{Sm}_n$, making the stacks $\{\mathrm{Sm}_n\}_{n\geq 1}$ into a projective system. Let Sm_∞ denote the stack of commutative formal Lie groups over $\mathbb{F}_p$.

Lemma 2.4.1 ([Dri24], 2.2.10 and [Mes72], Chapter II)**.** *If $G/\operatorname{Spec} R$ is a commutative formal Lie group, then $G[F^n]$ is n-smooth over R. The assignment $G \mapsto \{G[F^n]\}_{n\geq 1}$ defines an equivalence*

$$\mathrm{Sm}_\infty \simeq \varprojlim_n \mathrm{Sm}_n .$$

Although it will not be used in this article, the following theorem makes precise the idea that in characteristic p, formal groups are "F-divisible".

Theorem 2.4.2 ([Mes72], Theorem 2.1.7)**.** *Let G be an fppf sheaf of groups over* $\operatorname{Spec} R$*. Then G is a formal Lie group if and only if the following three conditions hold:*

(1) $G = \varinjlim G[F^n]$;
(2) $F : G \to G^{(p)}$ is an epimorphism;
(3) For all $n \geq 1$, $G[F^n]$ is a finite locally free group scheme.

Thus we may think of n-smooth group schemes as truncations of formal Lie groups. As such truncations are formed by taking Frobenius kernels, we only have this perspective in characteristic p.

2.5. **The co-Lie complex.** We recall the co-Lie complex of a group scheme, following [Ill85, §2.1].

Definition 2.5.1. Let S be a scheme and G be a finite locally free S-group scheme. The *co-Lie complex* of G is the complex

$$\ell_G := Le^* L_{G/S} \in D(S)$$

where $L_{G/S}$ is the cotangent complex of G and $e : S \to G$ is the identity section.

The co-Lie complex ℓ_G is a perfect complex of amplitude in $[-1, 0]$. The following result is well-known [Mes72, Remark 2.1.4].

Lemma 2.5.2. *Let R be an $\mathbb{F}_p$-algebra. If $G/\operatorname{Spec} R$ is n-smooth, then $H^{-1}(\ell_G)$ and $H^0(\ell_G)$ are finite locally free R-modules of rank equal to the rank of G.*

Proof. This follows immediately from the definitions, as every n-smooth group scheme of rank r is locally isomorphic to $\operatorname{Spec} R[x_1, \ldots, x_r]/(x_1^{p^n}, \ldots, x_r^{p^n})$ where the identity section corresponds to $x_1 = \cdots = x_r = 0$. □

The following result of Grothendieck will be crucial for cohomological calculations with n-smooth group schemes.

Lemma 2.5.3 (Grothendieck, [MM74] §14)**.** *If S is a scheme, G is a finite locally free commutative S-group scheme, and M is a quasi-coherent $\mathcal{O}_S$-module, then*

$$R\,\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\ell_G, M) \simeq \tau_{\leq 1} R\,\underline{\mathrm{Hom}}(G^\vee, M)$$

where the latter $R\,\underline{\mathrm{Hom}}$ is in the category of fppf sheaves of abelian groups over S.

3. Full Faithfulness of the Dieudonné Functor

In this section, we establish that the Dieudonné functor $G \mapsto \mathrm{Hom}(G, W_n)$ is fully faithful on n-cosmooth group schemes over R. Our strategy is to construct an adjoint to the Dieudonné functor and to show that the unit of this adjunction is an isomorphism, using key homological properties of n-cosmooth group schemes.

3.1. **The Dieudonné adjunction.** Let Ab_R be the category of fppf sheaves of abelian groups over $\mathrm{Spec}\, R$.

Definition 3.1.1. The *level n Dieudonné functor* $\underline{M}_n : \mathrm{Ab}_R^{op} \to (D_R\text{-mod})$ is defined by

$$\underline{M}_n(A) = \mathrm{Hom}_{\mathrm{Ab}_R}(A, W_n),$$

where D_R acts by postcomposing by the action on $W \twoheadrightarrow W_n$.

Define the functor

$$\underline{G}_n : (D_R\text{-mod})^{op} \to \mathrm{Ab}_R$$

by $\underline{G}_n(M)(S) = \mathrm{Hom}_{D_R}(M, W_n(S))$ for all R-algebras S. We will sometimes denote this as $\underline{G}_n(M) = \mathrm{Hom}_{D_R}(M, W_n)$.

Lemma 3.1.2. *$S \mapsto \mathrm{Hom}_{D_R}(M, W_n(S))$ is an fppf sheaf over* $\mathrm{Spec}\, R$.

Proof. $S \mapsto W_n(S)$ is a sheaf of D_R-modules. Since $\mathrm{Hom}_{D_R}(M, -)$ preserves limits, it preserves the sheaf condition. □

Remark 3.1.3. By writing down an explicit representing algebra, one can see that $\underline{G}_n(M)$ is an affine group scheme over R, which is of finite type if M is finitely generated. We will not use this observation in what follows.

Lemma 3.1.4. *$\underline{M}_n$ and $\underline{G}_n$ are adjoint functors.*

Proof. We wish to construct an isomorphism

$$\mathrm{Hom}_{\mathrm{Ab}_R}(A, \underline{G}_n(M)) \cong \mathrm{Hom}_{D_R}(M, \underline{M}_n(A))$$

natural in $A \in \mathrm{Ab}_R$ and $M \in D_R$-mod. If B is any sheaf of associative rings, X is a sheaf of abelian groups, and Y and Z are sheaves of left B-modules, then the adjunction of the tensor product and the internal Hom functor $\underline{\mathrm{Hom}}$ gives an isomorphism

$$\mathrm{Hom}(X, \underline{\mathrm{Hom}}_B(Y, Z)) \cong \mathrm{Hom}_B(Y \otimes X, Z) \cong \mathrm{Hom}_B(Y, \underline{\mathrm{Hom}}(X, Z)),$$

natural in X, Y, and Z. In our case, we take B to be the constant sheaf on D_R, Y to be the constant sheaf on M, and Z to be W_n. □

3.2. **Homological properties of n-cosmooth group schemes.** Let G be an n-cosmooth group scheme over R. The main goal of this subsection is Corollary 3.2.3, which establishes a canonical exact sequence exhibiting $\mathrm{Hom}(G, W_n)$ as an extension of $\mathrm{Hom}(G, W_{n-j})$ by $\mathrm{Hom}(G, W_j)$ for all $j \in \{1, \ldots, n-1\}$. Using this, we conclude that $\mathrm{Hom}(G, W_n)$ is an n-cosmooth D_R-module in the sense of Definition 1.0.2.

Fix $i \in \{1, \ldots, n-1\}$. Let $Q_i = (G^\vee[F^i])^\vee$, a quotient of G, and let K_{n-i} be the kernel of $G \to Q_i$. We have a short exact sequence

$$0 \longrightarrow K_{n-i} \longrightarrow G \longrightarrow Q_i \longrightarrow 0\ , \tag{3}$$

and by Lemma 2.3.1, Q_i is i-cosmooth and K_{n-i} is $(n-i)$-cosmooth.

Lemma 3.2.1. *If $i \geq j$, then the quotient map $G \to Q_i$ induces an isomorphism*

$$\mathrm{Hom}(Q_i, W_j) \to \mathrm{Hom}(G, W_j)$$

Proof. Since $G \to Q_i$ is surjective, the map is injective. Now suppose $f : G \to W_j$ is given. Since G is n-cosmooth, we have $\ker(V^i : G^{(p^i)} \to G) = \mathrm{im}(V^{n-i} : G^{(p^n)} \to G^{(p^i)}) = K_{n-i}^{(p^i)}$. By naturality of the Verschiebung, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{n-i}^{(p^i)} & \longrightarrow & G^{(p^i)} & \xrightarrow{V^i} & G & \longrightarrow & Q_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & \ker(V^i) & \longrightarrow & W_j^{(p^i)} & \xrightarrow{V^i} & W_j & \longrightarrow & \mathrm{coker}(V^i) & \longrightarrow & 0 \end{array}$$

with exact rows. Since $i \geq j$, Lemma 2.2.1 gives that V^i is zero on W_j, and the map from W_j to the cokernel of $V^i : W_j^{(p^i)} \to W_j$ is an isomorphism. We thus find that f factors through $G \to Q_i$, as desired. ☐

Lemma 3.2.1 implies that G and Q_i have the same rank. Since G is n-cosmooth, we have an identification $K_{n-i} \cong Q_{n-i}^{(p^i)}$, so K_{n-i} and G also have the same rank.

Lemma 3.2.2. *If $G/\operatorname{Spec} R$ is an n-cosmooth group scheme, then the coboundary map*

$$\mathrm{Hom}(G, W_{n-1}) \to \mathrm{Ext}^1(G, \mathbb{G}_a)$$

induced by the short exact sequence $0 \to \mathbb{G}_a \to W_n \to W_{n-1} \to 0$ is zero.

Proof. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(G, W_{n-1}) & \longrightarrow & \mathrm{Ext}^1(G, \mathbb{G}_a) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(K_1, W_{n-1}) & \longrightarrow & \mathrm{Ext}^1(K_1, \mathbb{G}_a) \end{array} \tag{4}$$

induced by the short exact sequence $0 \to \mathbb{G}_a \to W_n \to W_{n-1} \to 0$. By Lemma 3.2.1, the map $G \to Q_{n-1}$ induces an isomorphism $\mathrm{Hom}(Q_{n-1}, W_{n-1}) \cong \mathrm{Hom}(G, W_{n-1})$, so restriction along $K_1 \to G$ induces the zero map $\mathrm{Hom}(G, W_{n-1}) \to \mathrm{Hom}(K_1, W_{n-1})$.

We will show $\mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(K_1, \mathbb{G}_a)$ is an isomorphism. Consider the fiber sequence of co-Lie complexes $\ell_{K_1^\vee} \to \ell_{G^\vee} \to \ell_{Q_{n-1}^\vee} \to^{+1}$. The associated long exact sequence of cohomology groups is of the form

$$\begin{array}{l} 0 \longrightarrow H^{-1}(\ell_{K_1^\vee}) \xrightarrow{*} H^{-1}(\ell_{G^\vee}) \longrightarrow H^{-1}(\ell_{Q_{n-1}^\vee}) \\ \xrightarrow{*} H^0(\ell_{K_1^\vee}) \longrightarrow H^0(\ell_{G^\vee}) \xrightarrow{*} H^0(\ell_{Q_{n-1}^\vee}) \longrightarrow 0 \end{array},$$

where all the groups involved are locally free R-modules of the same rank by Lemma 2.5.2. Climbing the exact sequence shows that the maps marked by $*$ are isomorphisms; in particular, $H^{-1}(\ell_{K_1^\vee}) \to H^{-1}(\ell_{G^\vee})$ is an isomorphism. Now apply the Grothendieck formula $\tau^{\leq 1} R\,\underline{\mathrm{Hom}}(A, \mathbb{G}_a) \cong R\,\underline{\mathrm{Hom}}(\ell_{A^\vee}, \mathcal{O})$ of Lemma 2.5.3. If A is a j-cosmooth group scheme, then $\ell_{A^\vee}$ has locally free cohomology groups by Lemma 2.5.2, so $\mathrm{Ext}^i(A, \mathbb{G}_a) \cong \mathrm{Hom}_R(H^{-i}(\ell_{A^\vee}), R)$ for $i \in \{0, 1\}$. Thus $\mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(K_1, \mathbb{G}_a)$ is an isomorphism.

The composition $\mathrm{Hom}(G, W_{n-1}) \to \mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(K_1, \mathbb{G}_a)$ is zero and $\mathrm{Ext}^1(G, \mathbb{G}_a) \to \mathrm{Ext}^1(K_1, \mathbb{G}_a)$ is an isomorphism. Now the commutative square (4) shows the morphism in the statement is zero. □

Corollary 3.2.3. *If $G/\operatorname{Spec} R$ is an n-cosmooth group scheme, then for all $j \in \{1, \ldots, n-1\}$, the short exact sequence $0 \to W_j \to W_n \to W_{n-j} \to 0$ induces a short exact sequence*

$$0 \to \mathrm{Hom}(G, W_j) \to \mathrm{Hom}(G, W_n) \to \mathrm{Hom}(G, W_{n-j}) \to 0.$$

Proof. Since Hom is left exact, it suffices to show that $\mathrm{Hom}(G, W_n) \to \mathrm{Hom}(G, W_{n-j})$ is surjective. By induction, it suffices to show $\mathrm{Hom}(G, W_i) \to \mathrm{Hom}(G, W_{i-1})$ is surjective for $i \leq n$. By Lemma 3.2.1, it suffices to check when $i = n$, in which case surjectivity follows from Lemma 3.2.2. □

Recall from Definition 1.0.2 that a left D_R-module M is n-cosmooth if $V^n M = 0$, $\ker(V^i) = \mathrm{im}(V^{n-i})$ for all $i \in \{1, \ldots, n-1\}$, and M/VM is a finitely generated projective R-module.

Definition 3.2.4. The *rank* of an n-cosmooth module M is the rank of M/VM as an R-module.

The rank of an n-cosmooth group scheme G was defined in §2.3 to be the rank of the Lie algebra of $G^\vee$. Using Corollary 3.2.3, we can then establish that the Dieudonné modules of n-cosmooth group schemes are n-cosmooth D_R-modules of the appropriate rank.

Corollary 3.2.5. *If $G/\operatorname{Spec} R$ is an n-cosmooth group scheme, then $\underline{M}_n(G)$ is an n-cosmooth D_R-module. The rank of G agrees with the rank of $\underline{M}_n(G)$.*

Proof. This is a direct application of Corollary 3.2.3. Let $M = \underline{M}_n(G)$. Clearly $V^n = 0$ on M. Moreover, by Corollary 3.2.3, $M/VM = \mathrm{Hom}(G, \mathbb{G}_a) = \mathrm{Lie}(G^\vee)$. By Lemma 2.5.2, M/VM is a finitely generated projective R-module and the rank of G is the rank of M/VM. Finally, the exactness of Corollary 3.2.3 is the statement that $\mathrm{im}(V^j) = \ker(V^{n-j})$ on M for all $j \in \{1, \ldots, n-1\}$, as desired. □

3.3. **The Dieudonné functor is fully faithful.** Corollary 3.2.3 implies that if G is n-cosmooth, then $\underline{M}_n(G)$ is an iterated extension of the restricted Lie algebra of $G^\vee$. We now prove that $\underline{M}_n$ is fully faithful on n-cosmooth group schemes inductively using this structure of an iterated extension.

Theorem 3.3.1. *For all $n \geq 1$, the functor $\underline{M}_n$ is fully faithful on n-cosmooth group schemes over* $\operatorname{Spec} R$.

Proof. By the adjunction of $\underline{G}_n$ and $\underline{M}_n$ of Lemma 3.1.4, it suffices to show that if $G/\operatorname{Spec} R$ is n-cosmooth, then the unit map $G \to \underline{G}_n \underline{M}_n(G)$ is an isomorphism.

The proof is by induction on n. The case $n = 1$ is the classical correspondence between restricted Lie algebras and 1-smooth group schemes [SGA3, Exposé VII.A]. Now suppose G is $(n+1)$-cosmooth. We have a short exact sequence of D_R-module schemes

$$0 \to (\mathbb{G}_a)|_{\sigma^n} \to W_{n+1} \to W_n \to 0,$$

where $(-)|_\sigma$ means restriction of scalars along Frobenius and the map $(\mathbb{G}_a)|_{\sigma^n} \to W_{n+1}$ is the inclusion of the image of V^n. By Corollary 3.2.3, we obtain a short exact sequence of D_R-modules

$$0 \to \underline{M}_1(G)|_{\sigma^n} \to \underline{M}_{n+1}(G) \to \underline{M}_n(G) \to 0.$$

Now apply $\underline{G}_{n+1}$ to obtain an exact sequence

$$0 \to \underline{G}_{n+1}\underline{M}_n(G) \to \underline{G}_{n+1}\underline{M}_{n+1}(G) \to \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n}). \tag{5}$$

Let $G \to Q_n$ be the cokernel of V^n and let $G \to Q_1$ be the cokernel of V. Then $Q_n^{(p)} = G^{(p)}/\ker(V) \cong \mathrm{im}(V : G^{(p)} \to G)$. We will compare (5) to the exact sequence $0 \to Q_n^{(p)} \to G \to Q_1 \to 0$.

We first construct a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & Q_n^{(p)} & \longrightarrow & G \\ & & {\scriptstyle\cong}\downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{G}_{n+1}\underline{M}_n(G) & \longrightarrow & \underline{G}_{n+1}\underline{M}_{n+1}(G) \end{array} \tag{6}$$

where $G \to \underline{G}_{n+1}\underline{M}_{n+1}(G)$ is the unit map. Observe $V^n\underline{M}_n(G) = 0$, so

$$\begin{aligned} \underline{G}_{n+1}\underline{M}_n(G) &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, W_n), W_{n+1}) \\ &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, W_n), W_{n+1}[V^n]) \\ &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, W_n), W_n|_\sigma). \end{aligned}$$

Consider the map $\tilde{\varphi} : G^{(p)} \to \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, W_n), W_n|_\sigma)$ defined by $\tilde{\varphi}(g) : \bar{f} \mapsto \bar{f}^{(p)}(g)$. Postcomposing with $\mathrm{im}(V) = W_n|_\sigma \to W_{n+1}$ gives $G^{(p)} \to \underline{G}_{n+1}\underline{M}_n(G)$ which assigns $g \in G^{(p)}$ and $\bar{f} : G \to W_n$ to $V\bar{f}^{(p)}(g) = (0, \bar{f}^{(p)}(g))$. If $f : G \to W_{n+1}$ is a homomorphism, then by naturality of the Verschiebung the following square commutes:

$$\begin{array}{ccc} G^{(p)} & \xrightarrow{V} & G \\ {\scriptstyle f^{(p)}}\downarrow & & \downarrow{\scriptstyle f} \\ W_{n+1}^{(p)} & \xrightarrow[V]{} & W_{n+1} \end{array}.$$

Thus $Vf^{(p)} = fV$, so the following square commutes:

$$\begin{array}{ccccc} & & G^{(p)} & \xrightarrow{V} & G \\ & & {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow \\ 0 & \longrightarrow & \underline{G}_{n+1}\underline{M}_n(G) & \longrightarrow & \underline{G}_{n+1}\underline{M}_{n+1}(G) \end{array}$$

Further, $\tilde{\varphi}$ factors through $G^{(p)} \to Q_n^{(p)}$, yielding a map $\varphi : Q_n^{(p)} \to \underline{G}_{n+1}\underline{M}_n(G)$. By Lemma 3.2.1, $\underline{M}_n(Q_n) \to \underline{M}_n(G)$ is an isomorphism, and $\underline{G}_{n+1}\underline{M}_n(Q_n) \cong \underline{G}_n\underline{M}_n(Q_n^{(p)})$. Under this identification, φ is the unit map of the adjunction for $(\underline{G}_n, \underline{M}_n)$ applied to $Q_n^{(p)}$. By induction, $\varphi : Q_n^{(p)} \to \underline{G}_{n+1}\underline{M}_n(G)$ is an isomorphism.

Now consider the map $\underline{G}_{n+1}\underline{M}_{n+1}(G) \to \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n})$. Similarly, $VM_1(G) = 0$, so

$$\begin{aligned} \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n}) &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, (\mathbb{G}_a)|_{\sigma^n}), W_{n+1}) \\ &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, (\mathbb{G}_a)|_{\sigma^n}), W_{n+1}[V]) \\ &= \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, (\mathbb{G}_a)|_{\sigma^n}), (\mathbb{G}_a)|_{\sigma^n}). \end{aligned}$$

Now Lemma 3.2.1 implies $\underline{M}_1(Q_1) \to \underline{M}_1(G)$ is an isomorphism. Further, since every D_R-linear map is also linear after restriction of scalars along σ^n, we have an injection

$$\underline{G}_1\underline{M}_1(G) = \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, \mathbb{G}_a), \mathbb{G}_a) \to \mathrm{Hom}_{D_R}(\mathrm{Hom}(G, (\mathbb{G}_a)|_{\sigma_n}), (\mathbb{G}_a)|_{\sigma^n}) = \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n})$$

Thus we have a commutative square

$$\begin{array}{ccccc} G & \longrightarrow & Q_1 & \longrightarrow & 0 \\ \Big\downarrow & & \downarrow & & \\ & & \underline{G}_1\underline{M}_1(G) & & \\ \Big\downarrow & & \hookdownarrow & & \\ \underline{G}_{n+1}\underline{M}_{n+1}(G) & \longrightarrow & \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n}) & & \end{array}.$$

We claim that there exists a map

$$T : \underline{G}_{n+1}\underline{M}_{n+1}(G) \to \underline{G}_1\underline{M}_1(G)$$

which makes the following diagram commute:

$$\begin{array}{ccccc} G & \longrightarrow & Q_1 & \longrightarrow & 0 \\ \Big\downarrow & & \Big\downarrow & & \\ & & \underline{G}_1\underline{M}_1(G) & & \\ & \overset{T}{\dashrightarrow} & \Big\downarrow & & \\ \underline{G}_{n+1}\underline{M}_{n+1}(G) & \longrightarrow & \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n}) & & \end{array} \tag{7}$$

If $\psi \in \underline{G}_{n+1}\underline{M}_{n+1}(G)$ and $f \in \underline{M}_1(G)$, then define $T\psi(f) \in \mathbb{G}_a$ as follows: by Corollary 3.2.3, there exists $\tilde{f} \in \underline{M}_{n+1}(G)$ lifting $f \in \underline{M}_1(G)$ along $\underline{M}_{n+1}(G) \to \underline{M}_1(G)$. Now $T\psi(f)$ is defined to be the image of $\psi(\tilde{f})$ under $W_{n+1} \to \mathbb{G}_a$. This is well-defined: lifts $\tilde{f}$ of f are defined up to V, and ψ is V-linear.

Now we show (7) commutes. The quadrilateral commutes by definition of the unit morphisms. To address the triangle, observe that the map $\underline{G}_1\underline{M}_1(G) \to \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n})$ takes $\psi \in \underline{G}_1\underline{M}_1(G)$ and $f \in \operatorname{Hom}(G, \mathbb{G}_a)|_{\sigma^n}$ to $(0, \ldots, 0, \psi(f)) \in V^n W_{n+1}$. But $(0, \ldots, 0, f) = V^n \tilde{f}$ for any lift $\tilde{f} : G \to W_{n+1}$ of $f : G \to \mathbb{G}_a$. Thus, if $\tilde{\psi} \in \underline{G}_{n+1}\underline{M}_{n+1}(G)$ and $f \in \underline{M}_1(G)|_{\sigma^n}$, then

$$(T\tilde{\psi})((0, \ldots, 0, f)) = \tilde{\psi}(V^n\tilde{f}) = V^n\tilde{\psi}(\tilde{f}).$$

as $\tilde{\psi}$ is V-linear. Thus the triangle commutes.

Joining (6) and (7) gives a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_n^{(p)} & \longrightarrow & G & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \underline{G}_{n+1}\underline{M}_n(G) & \longrightarrow & \underline{G}_{n+1}\underline{M}_{n+1}(G) & \xrightarrow[T]{} & \underline{G}_1\underline{M}_1(G) & & \end{array}$$

The first row is exact. Moreover, since $\underline{G}_1\underline{M}_1(G) \to \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n})$ is injective, the kernel of T is the kernel of $\underline{G}_{n+1}\underline{M}_{n+1}(G) \to \underline{G}_{n+1}(\underline{M}_1(G)|_{\sigma^n})$. Since (5) is exact, the second row is exact. The first and third vertical maps are isomorphisms by induction. The snake lemma shows $G \to \underline{G}_{n+1}\underline{M}_{n+1}(G)$ is an isomorphism, as desired. □

4. Essential Image

As outlined in the introduction, our strategy for calculating the essential image of the Dieudonné functor $\underline{M}_n$ is to compare the Dieudonné modules of n-smooth group schemes with the Cartier modules of formal Lie groups. The key technical result is Proposition 4.3.1, which describes a Zariski-local presentation for n-cosmooth modules analogous to the presentation of a Cartier module by structure equations. This is applied in Proposition 4.3.3 to lift n-cosmooth modules to Cartier modules of formal Lie groups. We then combine this result with Cartier theory and descent to describe the essential image of $\underline{M}_n$ in §4.4.

4.1. **Review of Cartier theory.** The *p-typical Cartier ring* $\mathbb{E} = \mathbb{E}_R$ is the V-adic completion of the ring D_R. By definition, an $\mathbb{E}$-module M is *V-reduced* if $V : M \to M$ is injective and $M = \varprojlim M/V^nM$ (i.e. M is V-adically complete and V-torsion free), and M is *∞-cosmooth* if M is V-reduced and M/VM is a finite locally free R-module. As usual, we define the *rank* of an ∞-cosmooth module M to be the rank of M/VM as an R-module.

The following is a reformulation of the second main theorem of local Cartier theory, following the exposition of [Zin84, Chapter IV]. Because the formulation is nonstandard, we give a proof with references to the literature.

Theorem 4.1.1 (Cartier theory)**.** *The functor $G \mapsto \mathrm{Hom}(G^\vee, W)$ defines an equivalence of categories between formal Lie groups over R and ∞-cosmooth $\mathbb{E}_R$-modules. The rank of G as a formal Lie group agrees with the rank of $\mathrm{Hom}(G^\vee, W)$ as an ∞-cosmooth module.*

Proof. The standard formulation of Cartier theory gives us an equivalence by sending a formal Lie group G to the $\mathbb{E}$-module of p-typical curves in G ([Zin84, Theorem 4.23]). By [Zin84, Theorem 4.15], we may identify the p-typical curves in G with the $\mathbb{E}$-module $\mathrm{Hom}(\hat{W}, G)$ where $\hat{W}$ is the Witt vector formal group and the $\mathbb{E}$-module structure arises via the identification $\mathbb{E} = \mathrm{End}(\hat{W})$ ([Zin84, Corollary 4.16]). Since $\hat{W}$ is the Cartier dual of W [DG70, Chapter V, §4.4.6], we find a natural isomorphism $\mathrm{Hom}(\hat{W}, G) \cong \mathrm{Hom}(G^\vee, W)$, which finishes the proof. □

4.2. **Preliminaries on modules over the Cartier ring.** For $n \geq 1$, let $\mathbb{E}_n := \mathbb{E}/V^n\mathbb{E} = D_R/V^nD_R$ be the truncated Cartier–Dieudonné ring over R.

Lemma 4.2.1. *Left multiplication by $V : \mathbb{E} \to \mathbb{E}$ is injective.*

Proof. By [Zin84, Definition/Theorem 4.17], any element of $\mathbb{E}$ has a unique representation of the form

$$\sum_{r,s\geq 0} V^r[x_{r,s}]F^s$$

with $x_{r,s} \in R$ and $x_{r,s} = 0$ for fixed r and s sufficiently large, and the lemma follows immediately. □

The following lemma tells us that the rings $\mathbb{E}_n$ share some formal properties with n-cosmooth D_R-modules.

Lemma 4.2.2. *The rings $\mathbb{E}_n$ satisfy $V^i\mathbb{E}_n = \mathbb{E}_n[V^{n-i}]$ for all $i \in \{1, \ldots, n-1\}$. Multiplication by V^{n-i} is an isomorphism $(\mathbb{E}_n/V^i\mathbb{E}_n)|_{\sigma^{n-i}} \cong V^{n-i}\mathbb{E}_n$ of left $\mathbb{E}_n$-modules.*

Proof. Clearly $V^i\mathbb{E}_n \subset \mathbb{E}_n[V^{n-i}]$. Conversely, if $x \in \mathbb{E}_n[V^{n-i}]$, lift x to $\tilde{x} \in \mathbb{E}$; then $V^{n-i}\tilde{x} = V^n y$ for some $y \in \mathbb{E}$. By Lemma 4.2.1, left multiplication by V is injective on $\mathbb{E}$, so $\tilde{x} = V^i y$ and thus $x \in V^i\mathbb{E}_n$. Thus $V^i\mathbb{E}_n = \mathbb{E}_n[V^{n-i}]$. The second statement then follows, as

$$(\mathbb{E}_n/V^i\mathbb{E}_n)|_{\sigma^{n-i}} = \mathbb{E}_n|_{\sigma^{n-i}}/V^i = \mathbb{E}_n|_{\sigma^{n-i}}/\ker(V^{n-i}) = V^{n-i}\mathbb{E}_n.$$ □

The following Lemmas establish that certain properties of $\mathbb{E}_n$-modules can be checked modulo V.

Lemma 4.2.3. *Let $u : T \to L$ be a map of free $\mathbb{E}_n$-modules. If the induced map $\overline{u} : T/VT \to L/VL$ is injective, then so is u.*

Proof. We proceed by induction. Suppose the Lemma holds for $\mathbb{E}_n$-modules, and let $u : T \to L$ be a map of free $\mathbb{E}_{n+1}$-modules. Then $u' : T/V^nT \to L/V^nL$ is injective by the inductive hypothesis. Furthermore there is a commutative square

$$\begin{array}{ccc} (T/VT)|_{\sigma^n} & \xrightarrow{u} & (L/VL)|_{\sigma^n} \\ {\scriptstyle V^n}\downarrow & & {\scriptstyle V^n}\downarrow \\ V^nT/V^{n+1}T & \xrightarrow{u} & V^nL/V^{n+1}L \end{array}$$

where the vertical arrows are the isomorphisms of Lemma 4.2.2. It follows that $V^nT \to V^nL$ is injective, and consequently u is injective. □

Let $W(R)\{V\}$ denote the associative ring generated by $W(R)$ and V subject to $aV = V\sigma(a)$ for all $a \in W(R)$. By construction there is a natural map $W(R)\{V\} \to D_R$.

Lemma 4.2.4. *Let M be an $\mathbb{E}_n$-module, and suppose that $e_1, \ldots, e_r \in M$ are such that their images modulo V generate M/VM as an R-module. Then M is generated over $W(R)\{V\}$ by $e_1, \ldots, e_r$.*

Proof. As $e_1, \ldots, e_r$ generate M/VM over R, $V^i e_1, \ldots, V^i e_r$ generate $V^i M/V^{i+1}M$ over R. Therefore $e_1, \ldots, e_r$ generate $\mathrm{gr}_V(M)$ over $W(R)\{V\}$. As V is nilpotent on M, the V-adic filtration on M is finite and exhaustive, so $e_1, \ldots, e_r$ generate M over $W(R)\{V\}$. □

Finally, we show that for n-cosmooth modules, checking that a morphism is an isomorphism can be done modulo V.

Lemma 4.2.5. *Let $f : M' \to M$ be a map of n-cosmooth D_R-modules such that the induced morphism $\overline{f} : M'/VM' \to M/VM$ is an isomorphism. Then f is an isomorphism.*

Proof. Since M and M' are both n-cosmooth, for $i \leq n-1$ we have a commutative square

$$\begin{array}{ccc} (M'/VM')|_{\sigma^i} & \xrightarrow{f} & (M/VM)|_{\sigma^i} \\ \downarrow{\scriptstyle V^i} & & \downarrow{\scriptstyle V^i} \\ V^i M'/V^{i+1}M' & \xrightarrow{f} & V^i M/V^{i+1}M \end{array}$$

where the vertical maps are isomorphisms. Since f induces an isomorphism $M'/VM' \to M/VM$, f also induces an isomorphism on V-adic associated graded, so f is an isomorphism. □

4.3. **Lifting of Dieudonné modules.** In this subsection, we give a local presentation of n-cosmooth modules and apply this to prove that Zariski-locally, we may lift any n-cosmooth module to an ∞-cosmooth module.

Recall from Definition 3.2.4 that the rank of an n-cosmooth D_R-module M is defined to be the rank of M/VM as an R-module. The following proposition then gives a Zariski-local description of n-cosmooth modules which may be thought of as a truncated version of the notion of structure equations for a Cartier module, c.f. [Zin84, Chapter IV.8].

Proposition 4.3.1. *Let M be a D_R-module. Then M is n-cosmooth of rank r if and only if Zariski-locally on* $\operatorname{Spec} R$, *there is a short exact sequence*

$$0 \to T \xrightarrow{u} L \to M \to 0$$

where T and L are free $\mathbb{E}_n$-modules of rank r with bases $h_1, \ldots, h_r$ and $g_1, \ldots, g_r$, respectively, and

$$u(h_i) := Fg_i - \sum_{j=1}^{r} a_{ij}(V) g_j$$

where $a_{ij}(V) \in W(R)\{V\}$.

Before proving Proposition 4.3.1, we establish the following Lemma on injectivity of the map u used to present an n-cosmooth module.

Lemma 4.3.2. *Let T and L be free $\mathbb{E}_n$-modules of rank r with bases h_i and g_i respectively, and consider a map $u : T \to L$ defined by $u(h_i) = Fg_i - \sum_j a_{ij}(V) g_j$, where $a_{ij}(V) \in W(R)\{V\}$. Then u is injective.*

Proof. By Lemma 4.2.3, it suffices to consider $n = 1$. Then if $t = \sum_i r_i h_i$ is a general element of T with $r_i \in R\{F\}$, we can write $u(t) = \sum_i c_i g_i$ with

$$c_i = r_i F - \sum_j r_j a_{ji}.$$

By considering the degree of F, we find that $c_i = 0$ for all i implies $r_i = 0$ for all i, and therefore $u : T \to L$ is injective. □

Proof of Proposition 4.3.1. First, suppose that $M = L/u(T)$ where u is as in the statement. Then $V^n = 0$ on M since M is an $\mathbb{E}_n$-module. Moreover

$$M/VM = \mathbb{E}_1^r/(F - (\overline{a_{ij}(V)})) \cong R^r$$

is a free R-module of rank r. For all $i \in \{1, \ldots, n-1\}$, Lemma 4.2.2 implies $u : T/V^iT \to L/V^iL$ is injective, so applying the Snake Lemma to multiplication by V^i on the short exact sequence $0 \to T \to L \to M \to 0$ implies

$$M[V^i] = \frac{L[V^i]}{T[V^i]}.$$

But by Lemma 4.2.2, $L[V^i] = V^{n-i}L$, and the image of $V^{n-i}L$ in M is $V^{n-i}M$, so $M[V^i] \subseteq V^{n-i}M$ and hence M is n-cosmooth.

Conversely suppose that M is n-cosmooth over R of rank r. Working locally, assume that M/VM is a free R-module. Let $e_1, \ldots, e_r \in M$ be such that their images in M/VM form a basis over R. By Lemma 4.2.4, M is generated by $e_1, \ldots, e_r$ over $W(R)\{V\}$, so we can write

$$Fe_i = \sum_j a_{ij}(V)e_j,$$

where $a_{ij}(V) \in W(R)\{V\}$. Let T and L be free $\mathbb{E}_n$-modules of rank r with bases h_i and g_i, respectively, and define $u : T \to L$ by $u(h_i) = Fg_i - \sum_j a_{ij}(V)g_j$. By the reverse direction of the Proposition, the map u is injective and $M' := \operatorname{coker}(u)$ is n-cosmooth. By definition $g_i \mapsto e_i$ defines a map $f : M' \to M$. Note that M'/VM' and M/VM are free R-modules with basis $g_1, \ldots, g_r$ and $e_1, \ldots, e_r$, so that the induced map $\overline{f} : M'/VM' \to M/VM$ is an isomorphism. Thus f is an isomorphism by Lemma 4.2.5, and we are done. □

Proposition 4.3.3. *Let $\overline{M}$ be an n-cosmooth D_R-module of rank r. Then Zariski-locally on* $\operatorname{Spec} R$ *there exists an ∞-cosmooth D_R-module M of rank r such that $M/V^nM \cong \overline{M}$.*

Proof. Let

$$0 \to T \xrightarrow{u} L \to \overline{M} \to 0$$

be a local presentation of $\overline{M}$ as in Proposition 4.3.1. Let T', L' be free $\mathbb{E}$-modules with bases h_i' and g_i', respectively. Then we let M be the $\mathbb{E}$-module defined by $M := \operatorname{coker}(u' : T' \to L')$, where $u'(h_i') = Fg_i' - \sum a_{ij}(V)g_j'$. By [Zin84, Lemma 4.37], $T' \to L'$ is injective with V-reduced cokernel, so we conclude that the sequence

$$0 \to T' \xrightarrow{u'} L' \to M \to 0$$

is exact and that M is V-reduced. By definition $M/V^nM = \overline{M}$, so M has rank r. □

4.4. **Essential image of Dieudonné module functor.** By Corollary 3.2.5, if $G/\operatorname{Spec} R$ is an n-cosmooth group scheme, then $\underline{M}_n(G)$ is an n-cosmooth D_R-module. In this subsection, we show that every n-cosmooth D_R-module arises in this way. Our strategy is to combine Proposition 4.3.3 with Cartier theory, and for that we make the following definition.

Definition 4.4.1. A group scheme H over R is said to be ∞-*cosmooth* if it is Cartier dual to a formal Lie group over R, i.e. if we have an isomorphism $H \cong \underline{\operatorname{Hom}}(G, \hat{\mathbb{G}}_m)$ for G a formal Lie group over R.

Lemma 4.4.2. *Let H be an ∞-cosmooth group scheme. Then the short exact sequence* $0 \to W|_{\sigma^n} \to W \to W_n \to 0$ *induces a short exact sequence of* $\mathbb{E}$*-modules*

$$0 \longrightarrow \operatorname{Hom}(H, W|_{\sigma^n}) \longrightarrow \operatorname{Hom}(H, W) \longrightarrow \operatorname{Hom}(H, W_n) \longrightarrow 0$$

Proof. To show the Lemma, it suffices to show that $\operatorname{Hom}(H, W) \to \operatorname{Hom}(H, W_n)$ is surjective. Since $W = \varprojlim_m W_m$ as a group scheme, by induction it suffices to show $\operatorname{Hom}(H, W_{m+1}) \to \operatorname{Hom}(H, W_m)$ is surjective. Since $H^\vee$ is a formal Lie group, $H^\vee = \varinjlim_r H^\vee[F^r]$, so $H = \varprojlim_r H_r$ where H_r is the cokernel of V^r. Following Lemma 3.2.1, if $r \geq m$, then pullback along $H \to H_r$ induces an isomorphism

$$\mathrm{Hom}(H_r, W_m) \cong \mathrm{Hom}(H, W_m)$$

By Corollary 3.2.3, the map $\mathrm{Hom}(H_{m+1}, W_{m+1}) \to \mathrm{Hom}(H_{m+1}, W_m)$ is surjective, and thus the map $\mathrm{Hom}(H, W_{m+1}) \to \mathrm{Hom}(H, W_m)$ is surjective. □

Theorem 4.4.3. *The functor $\underline{M}_n$ on n-cosmooth group schemes has essential image the n-cosmooth D_R-modules.*

Proof. Let $\overline{M}$ be an n-cosmooth D_R-module. By Proposition 4.3.3, there is a Zariski open cover $U \to \operatorname{Spec} R$ and an ∞-cosmooth $\mathbb{E}_U$-module M such that $M/V^n M \cong \overline{M}_U$. By Cartier theory (Theorem 4.1.1), $M = \mathrm{Hom}(H, W)$ for some ∞-cosmooth group scheme H/U. Define G_U to be the cokernel of V^n on H. Then G_U/U is n-cosmooth, and by Lemma 4.4.2,

$$\mathrm{Hom}(G, W_n) \cong \mathrm{Hom}(H, W_n) \cong \mathrm{Hom}(H, W)/V^n \mathrm{Hom}(H, W) = M/V^n M \cong \overline{M}_U.$$

Thus there exists an n-cosmooth group scheme G_U/U such that $\underline{M}_n(G_U) \cong \overline{M}_U$. By Theorem 3.3.1, $\underline{M}_n$ is fully faithful, so $G_U \cong \underline{G}_n \underline{M}_n(G_U) \cong \underline{G}_n(M_U)$. Since $\overline{M}_U$ descends along $U \to \operatorname{Spec} R$, apply $\underline{G}_n$ to the descent data to obtain descent data for G_U. By Lemma 2.3.5, Sm_n is a stack, so G_U descends to an n-cosmooth group scheme $G/\operatorname{Spec} R$ with $\underline{M}_n(G) \cong M$. □

5. Base Change and Dieudonné Module Computations

With Theorem A established, in this section we turn our attention to the compatibility of our Dieudonné theory with base change. We also make some explicit computations of Dieudonné modules over an arbitrary base.

5.1. **Base Change.** Let $R \to R'$ be a map of $\mathbb{F}_p$-algebras. We write $\underline{M}_{n,R}$ (resp. $\underline{M}_{n,R'}$) for the level n Dieudonné functor over R (resp. R') of Definition 3.1.1. If $G/\operatorname{Spec} R$ is an n-smooth group scheme, base change induces a natural map

$$D_{R'} \otimes_{D_R} \underline{M}_{n,R}(G^\vee) \to \underline{M}_{n,R'}(G^\vee_{R'}), \tag{8}$$

of $D_{R'}$-modules, where $G^\vee_{R'} = G^\vee \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is the base change of $G^\vee$ to R'. The domain of the morphism above is n-cosmooth, for example because it has a presentation as in Proposition 4.3.1. As both modules in (8) are n-cosmooth $D_{R'}$-modules, we will show that this map is an isomorphism by establishing that it is an isomorphism modulo V. For that, we begin with the following lemma.

Lemma 5.1.1. *Suppose that $G/\operatorname{Spec} R$ is n-smooth. Then the formation of $\mathrm{Lie}(G)$ commutes with arbitrary base change. More precisely, for any map $R \to R'$ of $\mathbb{F}_p$-algebras, the natural map*

$$R' \otimes_R \mathrm{Lie}(G) \to \mathrm{Lie}(G_{R'})$$

is an isomorphism.

Proof. Since G is locally free over $\operatorname{Spec} R$, formation of the co-Lie complex ℓ_G commutes with arbitrary base change. Since G is n-smooth, the cohomology groups hence formation of the co-Lie algebra $\omega_G = H^0(\ell_G)$ does as well. Moreover, ω_G is locally free of finite rank over $\operatorname{Spec} R$, so we conclude by duality. □

Proposition 5.1.2. *Formation of the Dieudonné module for n-smooth group schemes commutes with arbitrary base change. More precisely, for any n-smooth $G/\operatorname{Spec} R$ and morphism $R \to R'$ of $\mathbb{F}_p$-algebras, the comparison morphism (8) is an isomorphism.*

Proof. By Lemma 4.2.5, it suffices to show that the morphism (8) is an isomorphism modulo V. By Corollary 3.2.3, we have an isomorphism

$$\underline{M}_{n,R}(G^\vee)/V\underline{M}_{n,R}(G^\vee) \cong \mathrm{Hom}(G^\vee, \mathbb{G}_a) = \mathrm{Lie}(G),$$

and similarly for $G_{R'}$. Reducing modulo V, the comparison morphism is then just the isomorphism of Lemma 5.1.1, and we are done. □

5.2. **Dieudonné module computations.** To illustrate the explicit nature of our Dieudonné theory, we make some direct computations of Dieudonné modules over an arbitrary base.

Example 5.2.1. Consider $(\mathbb{Z}/p^n\mathbb{Z})_R$, an n-cosmooth group scheme over R with i-cosmooth quotient equal to $\mathbb{Z}/p^i\mathbb{Z}$ for all $1 \leq i \leq n$. Writing $p^n = F^n \circ V^n$ on the Witt vectors, we have

$$\underline{\mathrm{Hom}}(\mathbb{Z}/p^n\mathbb{Z}, W_m) = W_m[p^n].$$

Thus the Dieudonné module of $\mathbb{Z}/p^n\mathbb{Z}$ over R is $W_n(R)$ with the standard action of F and V. For $m \leq n$, we get the expected Dieudonné module of the m-cosmooth quotient $\mathbb{Z}/p^m\mathbb{Z}$, while for $m > n$ we pick up extra contributions from nilpotents in the ring R. This explains why in our formulation of Dieudonné theory, we must work with the truncated Witt vectors as opposed to the full Witt group $\varinjlim_n W_n$.

Example 5.2.2. Let $R = \mathbb{F}_p[\lambda]$ and let $\hat{\mathbb{G}}_\lambda$ be the formal Lie group over R with group law $X, Y \mapsto X + Y + \lambda XY$. If $\lambda = 0$, the formal group $\hat{\mathbb{G}}_\lambda$ is isomorphic to $\hat{\mathbb{G}}_a$, while if λ is invertible, $\hat{\mathbb{G}}_\lambda$ is isomorphic to $\hat{\mathbb{G}}_m$. Let $G_n = (\hat{\mathbb{G}}_\lambda[F^n])^\vee$, identified with the cokernel of V^n for $\hat{\mathbb{G}}_\lambda^\vee$. Thus G_n is an n-cosmooth group scheme over $\operatorname{Spec} R$. Its fiber at $\lambda = 0$ is isomorphic to $\alpha_{p^n}^\vee$, while its fiber over $\lambda = 1$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$.

We first compute the restricted Lie algebra of $G_1^\vee = \hat{\mathbb{G}}_\lambda[F]$ by hand. Note that $\hat{\mathbb{G}}_\lambda$ has invariant differential $dT/(1+\lambda T)$ and thus its Lie algebra is spanned by $(1+\lambda T)\frac{d}{dT}$. According to Hochschild's identity [Hoc55, Lemma 1], if ∂ is a derivation of a commutative ring A and $f \in A$, then $(f\partial)^p = f^p\partial^p + (f\partial)^{p-1}(f)\partial$. Thus

$$\left((1+\lambda T)\frac{d}{dT}\right)^p = \left(\left((1+\lambda T)\frac{d}{dT}\right)^{p-1}(1+\lambda T)\right)\frac{d}{dT} = \lambda^{p-1}(1+\lambda T)\frac{d}{dT}.$$

We conclude that the Dieudonné module of G_1 is $R[F]/R[F](F - \lambda^{p-1})$.

Now we calculate the Dieudonné module of G_n for all $n \geq 1$. By [SS01] the group $\hat{\mathbb{G}}_\lambda^\vee$ is the kernel of the epimorphism $F - [\lambda^{p-1}] : W \to W$. On account of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\hat{\mathbb{G}}_\lambda^\vee)^{(p^n)} & \longrightarrow & W & \xrightarrow{F-[\lambda^{p^n(p-1)}]} & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle V^n} & & \downarrow{\scriptstyle V^n} & & \downarrow{\scriptstyle V^n} & & \\
0 & \longrightarrow & \hat{\mathbb{G}}_\lambda^\vee & \longrightarrow & W & \xrightarrow{F-[\lambda^{p-1}]} & W & \longrightarrow & 0
\end{array}$$

we find $G_n = \ker(F - [\lambda^{p-1}] : W_n \to W_n)$. Thus we have an exact sequence

$$0 \to \mathrm{Hom}(W_n, W_n) \to \mathrm{Hom}(W_n, W_n) \to \mathrm{Hom}(G_n, W_n).$$

The map $\mathbb{E}_n = \mathrm{Hom}(W_n, W_n) \to \mathrm{Hom}(G_n, W_n)$ reduced modulo V is the corresponding map for $n = 1$ by Lemma 3.2.1. Thus, $\mathbb{E}_n \to \mathrm{Hom}(G_n, W_n)$ is surjective modulo V, hence is surjective. The induced map $\mathrm{Hom}(W_n, W_n) \to \mathrm{Hom}(W_n, W_n)$ is right multiplication by $F - [\lambda]^{p-1}$. Thus the Dieudonné module of G_n is

$$\underline{M}_n(G_n) = \mathbb{E}_n/\mathbb{E}_n(F - [\lambda^{p-1}]).$$

By Proposition 5.1.2, we can calculate the Dieudonné modules of $\alpha_{p^n}^\vee$ and $\mathbb{Z}/p^n\mathbb{Z}$ through base change. At $\lambda = 0$, we find $\underline{M}_n(\alpha_{p^n}^\vee) = \mathbb{E}_n/\mathbb{E}_nF$. The underlying $W(R)$-module is $M = \bigoplus_{i=0}^{n-1} R|_{\sigma^i}$, and $V : M|_\sigma \to M$ acts as the shift. Similarly, for $\lambda = 1$, we find $G_n = (\mu_{p^n})^\vee = \mathbb{Z}/p^n\mathbb{Z}$, so $\underline{M}_n(\mathbb{Z}/p^n\mathbb{Z}) = \mathbb{E}_n/\mathbb{E}_n(F-1)$. The map $W_n \to \mathbb{E}_n/\mathbb{E}_n(F-1)$ is an isomorphism of $W_n\{V\}$-modules, and F acts on the image of W_n by

$$Fx = x^\sigma F \equiv x^\sigma \mod \mathbb{E}_n(F-1).$$

We find $\underline{M}_n(\mathbb{Z}/p^n\mathbb{Z}) \cong W_n$ with the usual actions of F and V, agreeing with Example 5.2.1.

6. Smoothness of stacks and truncation morphisms

Let τ_n denote the truncation $\mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$. For $n \geq 1$, Theorem A allows us to identify Sm_n^r with the moduli stack of n-cosmooth Dieudonné modules of rank r. Moreover under these identifications, Lemma 3.2.1 and Corollary 3.2.3 identify the truncation morphisms $\tau_n : \mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$ with the truncations $M \mapsto M/V^n M$ on Dieudonné modules. Theorem B follows from Propositions 6.0.1 and 6.0.3, which are applications of Proposition 4.3.1.

Proposition 6.0.1. *Sm_n is a smooth algebraic stack over $\mathbb{F}_p$.*

Proof. Since $\mathrm{Sm}_n = \sqcup_{r\geq 1} \mathrm{Sm}_n^r$ and Sm_n^r is of finite type over $\mathbb{F}_p$ by Lemma 2.3.5, it suffices to show that Sm_n^r is formally smooth over $\mathbb{F}_p$ ([Stacks, Tag 0DP0]). Let $A \to B$ be a surjection of local rings, and suppose that we are given an n-cosmooth rank r module M over B. Then by Proposition 4.3.1 we can write

$$M \cong \frac{L}{(Fe_i - \sum_{j=1}^r a_{ij}(V)e_j)}$$

where L is a free $\mathbb{E}_{n,B}$-module with basis $e_1, \ldots, e_r$ and the $a_{ij}(V) \in W(B)\{V\}$. Let $a'_{ij}(V)$ be lifts of the $a_{ij}(V)$ to $W(A)\{V\}$. Then if L' denotes a free $\mathbb{E}_{n,A}$-module with basis $e'_1, \ldots, e'_r$, Proposition 4.3.1 gives that the module

$$\frac{L'}{(Fe'_i - \sum_j a'_{ij}(V)e'_j)}$$

is a lift of M to an n-cosmooth rank r module over A. ◻

To prove smoothness of the truncation morphism $\tau_n : \mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$, we need more control on the matrix of F on a cosmooth module.

Lemma 6.0.2. *Let M be an n-cosmooth $\mathbb{E}_R$-module and $e_1, \ldots, e_r \in M$ such that their images form a basis over R for M/VM. If $[a] \in W(R)$ is the multiplicative lift of $a \in R$, then for all $m \in M$ there exist unique $a_i^j \in R$ for $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, r\}$ such that*

$$m = \sum_{i=0}^{n-1} \sum_{j=1}^{r} V^i [a_i^j] e_j.$$

Proof. The proof is by induction on n. The statement holds for $n = 1$. If M is n-cosmooth, then $M/V^{n-1}M$ is $(n-1)$-cosmooth, so for $m \in M$ there exist unique $a_i^j \in R$ for $i < n-1$ such that

$$m - \sum_{i=0}^{n-2} \sum_{j=1}^{r} V^i [a_i^j] e_j \in V^{n-1}M.$$

Now left multiplication by V^{n-1} defines an isomorphism of abelian groups $M/VM \to V^{n-1}M$, so there exist unique $a_{n-1}^j \in R$ such that

$$m - \sum_{i=0}^{n-2} \sum_{j=1}^{r} V^i [a_i^j] e_j = \sum_{j=1}^{r} V^{n-1} [a_{n-1}^j] e_j.$$ ◻

Proposition 6.0.3. *The truncation morphism $\tau_n : \mathrm{Sm}_{n+1} \to \mathrm{Sm}_n$ induced by $G \mapsto G[F^n]$ is smooth and surjective.*

Proof. The surjectivity of τ_n follows from Proposition 4.3.1: over a field k, every n-cosmooth module M can be written in the form

$$M \cong \frac{\mathbb{E}_{n,k}^r}{(Fe_i - \sum a_{ij}(V)e_j)}$$

with $a_{ij}(V) \in W(k)\{V\}$. Then

$$\frac{\mathbb{E}_{n+1,k}^r}{(Fe_i - \sum a_{ij}(V)e_j)}$$

is a lift of M to an $(n+1)$-cosmooth module over k.

To show that τ_n is smooth, it suffices to show that the restriction $\tau_n : \mathrm{Sm}^r_{n+1} \to \mathrm{Sm}^r_n$ is smooth. Since this is a map between finite type stacks over $\mathbb{F}_p$, it is locally of finite presentation, so it suffices to show that τ_n is formally smooth ([Stacks, Tag 0DP0]). Suppose we have a 2-commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} B & \longrightarrow & \mathrm{Sm}^r_{n+1} \\ \big\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \big\downarrow{\scriptstyle \tau_n} \\ \operatorname{Spec} A & \longrightarrow & \mathrm{Sm}^r_n \end{array} \tag{9}$$

where $A \to B$ is a surjection of Artinian local rings. Thus we are given an $(n+1)$-cosmooth module N over B, an n-cosmooth module $\overline{M}$ over A, and an isomorphism $\alpha : \overline{M}|_B \to N/V^n N$. Pick $\{e_1, \ldots, e_r\} \subseteq M$ and $\{f_1, \ldots, f_r\} \subseteq N$ such that $\{e_1, \ldots, e_r\}$ maps to a basis of $\overline{M}/V\overline{M}$ and $\alpha(e_j) \equiv f_j \mod V^n$ for all j. By Lemma 6.0.2, there exist unique $a^j_{ik} \in A$ and $b^j_{ik} \in B$ such that

$$Fe_k = \sum_{i=0}^{n-1} \sum_{j=1}^{r} V^i[a^j_{ik}]e_j, \qquad Ff_k = \sum_{i=0}^{n} \sum_{j=1}^{r} V^i[b^j_{ik}]f_j.$$

Since $\alpha(e_j) \equiv f_j \mod V^n$, the uniqueness of Lemma 6.0.2 implies that $a^j_{ik} \mapsto b^j_{ik}$ for $i < n$. Pick $a^j_{nk} \in A$ such that $a^j_{nk} \mapsto b^j_{nk}$ under $A \to B$, and define

$$M := \bigoplus_{j=1}^{r} \mathbb{E}_{n+1,A} e_j \Bigg/ \left(Fe_k - \sum_{i=0}^{n} \sum_{j=1}^{r} V^i[a^j_{ik}]e_j \right).$$

Then $M/V^n M \cong \overline{M}$ and $M|_B \cong N$ compatibly with α. Thus M defines a lift $\operatorname{Spec} A \to \mathrm{Sm}^r_{n+1}$ in the diagram (9). □

Remark 6.0.4. Note that the proofs of Propositions 6.0.1 and 6.0.3 yield a stronger lifting property than what is needed for smoothness. In particular, the Artinian hypothesis in the proof of Proposition 6.0.3 is not used.

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University of Chicago, Chicago, IL
Email address: ckothari@uchicago.edu

University of Wisconsin-Madison, Madison, WI
Email address: jmundinger@wisc.edu