Differential Forms in Positive Characteristic and the Cartier Isomorphism

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1 Introduction: Differential Forms and The Story Over $\mathbb{C}$

Let $X/\mathbb{C}$ be a complex manifold. One can consider the *holomorphic de Rham complex*, which is the complex of sheaves $(\Omega^*_{X,\text{hol}}, d)$. The sections of $\Omega^i_{X,\text{hol}}$ over an open subset $U \subset X$ are the holomorphic i-forms on U, namely those which in a holomorphic chart look like sums of $f(z)dz_{j_1} \wedge \cdots \wedge dz_{j_i}$ with $f(z)$ a holomorphic function. We are interested in the question of computing cohomology, i.e. understanding the closed mod exact forms on X. To this end, we can form the *cohomology sheaves* associated to this complex

$$\mathcal{H}^i(\Omega^*_{X,\text{hol}}) := \frac{\ker(\partial : \Omega^i_{\text{hol}} \to \Omega^{i+1}_{\text{hol}})}{\text{im}(\partial : \Omega^{i-1}_{\text{hol}} \to \Omega^i_{\text{hol}})},$$

which locally measure the failure of closed i-forms to be exact. But recall the holomorphic Poincare lemma tells us that every holomorphic differential form on a complex manifold is locally exact, so we conclude

$$\mathcal{H}^i(\Omega^*_{X,\text{hol}}) = \begin{cases} \underline{\mathbb{C}} & i = 0 \\ 0 & i > 0. \end{cases}$$

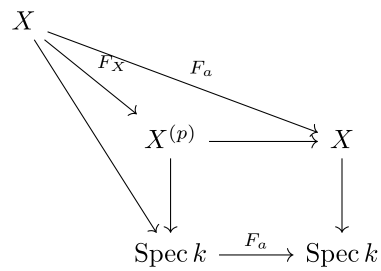
If $X/\mathbb{C}$ is a smooth variety with its Zariski topology, we can instead consider the *algebraic de Rham complex* of $X/\mathbb{C}$, which is the complex of sheaves $(\Omega^*_{X/\mathbb{C}}, d)$. Here $\Omega^i_{X/\mathbb{C}}$ is the sheaf of i-th Kahler differentials of X (or i-th regular differentials of X), which locally look like $f(x)dx_{j_1} \wedge \cdots \wedge dx_{j_n}$ with f a polynomial in the coordinates x_i. We can then form the cohomology sheaves $\mathcal{H}^i(\Omega^*_X)$, whose sections over an open affine $U \subset X$ are $\{$closed i-forms on $U\}/\{$exact i-forms on $U\}$, but note that there is no analogue of the Poincare lemma in the algebraic world: the open sets in the Zariski topology are too large! In this case, we still have $\mathcal{H}^0(\Omega^*_X) = \underline{\mathbb{C}}$, but the sheaves $\mathcal{H}^i(\Omega^*_X)$ for $i > 0$ don't have a simple description.

Note that the definition of the algebraic de Rham complex works nicely for a smooth variety over any field, including those of positive characteristic. The Cartier isomorphism gives us an explicit description of the sheaves $\mathcal{H}^i(\Omega^*_X)$ when X is a smooth scheme over a field of characteristic $p > 0$. The key points are that in characteristic p, $dx^p = px^{p-1}dx = 0$, and that we have a Frobenius morphism which we can leverage to say something nontrivial.

2 Frobenii

From now on, let $k = \overline{\mathbb{F}}_p$. If X/k is a smooth variety, then there are two kinds of Frobenius morphisms that will be relevant to us. First, we have the *absolute Frobenius* $F_a : X \to X$, which is the identity on underlying topological spaces and which is the p-th power morphism on structure sheaves. On $U = \operatorname{Spec} A \subset X$, we see that F_a comes from the ring map $a \mapsto a^p$ on A. Note that as a consequence, F_a is $\mathbb{F}_p$-linear but not k-linear, since if $a \in k \setminus \mathbb{F}_p$ then $a^p \neq a$.

To form a k-linear Frobenius morphism (the *relative Frobenius morphism*), we consider the variety $X^{(p)}/k$ and morphism $F_X : X \to X^{(p)}$ of varieties over k defined by the fiber product diagram



Concretely, if X locally looks like $k[x_i]/(f_j)$, then $X^{(p)}$ looks like $k[x_i]/(f_j^{(p)})$, where $f_j^{(p)}$ denotes the polynomial f_j with coefficients raised to the p-th power, and the map $F_X : X \to X^{(p)}$ comes from the ring map taking $x_i \mapsto x_i^p$. Some observations:

1. Since X/k is smooth, so is $X^{(p)}$, being the base change along the absolute Frobenius map of k.
2. The schemes X and $X^{(p)}$ are isomorphic (over $\mathbb{F}_p$) via the natural isomorphism $k[x_i]/(f_j) \to k[x_i]/(f_j^{(p)}), g \mapsto g^{(p)}$.
3. The map F_X is a homeomorphism of underlying topological spaces, since $X^{(p)} \to X$ and $F_a : X \to X$ are homeomorphisms. Notation: if $U \subset X$, let $U^{(p)} := F_X(U) \subset X^{(p)}$. If U has coordinate functions $x_1, \ldots, x_n$, then let $x_1^{(p)}, \ldots, x_n^{(p)}$ denote the corresponding coordinate functions on $U^{(p)}$.

As a consequence of the final point above, we can view $F_X^{-1}\mathcal{O}_{X^{(p)}}$ as a subsheaf of $\mathcal{O}_X$, since over an open set we can compose a section $s : U^{(p)} \to \mathbb{A}_k^1$ with the homeomorphism $F_X : U \to U^{(p)}$. Can we identify what subsheaf of $\mathcal{O}_X$ the sheaf $F_X^{-1}\mathcal{O}_{X^{(p)}}$ is? If we take sections over an open affine $U = \operatorname{Spec} k[x_i]/(f_j)$ of X, then we find

$$\Gamma(U, F_X^{-1}\mathcal{O}_{X^{(p)}}) = \frac{k[x_i]}{(f_j^{(p)})} \xrightarrow{x_i \mapsto x_i^p} \frac{k[x_i]}{(f_j)} = \Gamma(U, \mathcal{O}_X),$$

i.e. the image is the p-th power subsheaf of $\mathcal{O}_X$. So we can conclude $F_X^{-1}\mathcal{O}_{X^{(p)}} = \mathcal{O}_X^p$. Recall that in characteristic p, $dx^p = px^{p-1}dx = 0$, i.e. p-th power things are closed. This leads us to the following guess:

Conjecture 2.1. We have an equality of subsheaves of $\mathcal{O}_X$, $F_X^{-1}\mathcal{O}_{X^{(p)}} = \ker(d : \mathcal{O}_X \to \Omega_X^1)$ (which we called $\mathcal{H}^0(\Omega_X^*)$).

We give the proof for $X = \mathbb{A}^n$, and note that the general case follows from covering X by open affines admitting étale maps to $\mathbb{A}^n$.

Proof. Let $X = \mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. We want to show that the kernel of $d : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n] \langle dx_1, \ldots, dx_n \rangle$ is precisely the p-th powers of $k[x_1, \ldots, x_n]$. Given $f \in k[x_1, \ldots, x_n]$, we can write

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i,$$

so $df = 0$ if and only if $\partial f / \partial x_i = 0$ for each i. Viewing f as a polynomial in x_i, we see that this happens precisely if all the exponents of x_i occurring in f are divisible by p. Since every element of k is a p-th power, we conclude the result in this case.

□

One consequence of this result is that while the differentials in the complex Ω_X^* are not $\mathcal{O}_X$-linear (Leibniz), they are $F_X^{-1}\mathcal{O}_{X^{(p)}}$-linear, so that (Ω_X^*, d) is really a complex of $F_X^{-1}\mathcal{O}_{X^{(p)}}$-modules. Consequently, the sheaves $\mathcal{H}^i(\Omega_X^*)$ are also $F_X^{-1}\mathcal{O}_{X^{(p)}}$-modules.

3 The Cartier Isomorphism

Let X/k be as above. We've shown that $F_X^{-1}\mathcal{O}_{X^{(p)}} \cong \mathcal{H}^0(\Omega_X^*)$. But as observed above, both of these sheaves fit into complexes of $F_X^{-1}\mathcal{O}_{X^{(p)}}$-modules, which are moreover graded algebras because we can take the wedge product of differential forms or cohomology classes. The Cartier isomorphism generalizes what we have shown to this setting in a canonical way:

Theorem 3.1 (Cartier Isomorphism)**.** There is a unique isomorphism of graded $F_X^{-1}\mathcal{O}_{X^{(p)}}$-algebras

$$\mathscr{C}^{-1} : \bigoplus_i F_X^{-1}\Omega^i_{X^{(p)}} \to \bigoplus_i \mathcal{H}^i(\Omega_X^*)$$

such that if $x^{(p)}$ is a local coordinate on $X^{(p)}$ corresponding to local coordinate x on X, then $\mathscr{C}^{-1}(dx^{(p)}) = [x^{p-1}dx]$.

Let us unpack what this statement and the local description mean. First, to give an isomorphism $F_X^{-1}\Omega^i_{X^{(p)}} \cong \mathcal{H}^i(\Omega_X^*)$ is to specify, for each affine open $U \subset X$, an isomorphism

$$\{i\text{-forms on } U^{(p)}\} \cong \frac{\{\text{closed } i\text{-forms on } U\}}{\{\text{exact } i\text{-forms on } U\}}$$

in a compatible way. To say that this isomorphism is $F_X^{-1}\mathcal{O}_{X^{(p)}}$-linear is to say that, for $f \in \Gamma(U^{(p)}, \mathcal{O}_{X^{(p)}})$ and $\omega \in \Omega^1_{X^{(p)}}(U^{(p)})$, we have

$$\mathscr{C}^{-1}(f\omega) = (F_X^* f)\mathscr{C}^{-1}(\omega),$$

since F_X^* is how we view sections of $F_X^{-1}\mathcal{O}_{X^{(p)}}$ inside $\mathcal{O}_X$. Finally, one thinks of the relation $\mathscr{C}^{-1}(dx^{(p)}) = [x^{p-1}dx]$ as sending $dx^{(p)}$ to "$\frac{dx^p}{p}$", and in this sense we recover the archetypal closed but not exact form $x^{p-1}dx$ on our variety X.

The proof proceeds by reducing to $\mathbb{A}^n$ via étale maps, then reducing to $\mathbb{A}^1$ via Kunneth, and is similar in spirit to our earlier proof of the degree 0 case. Rather than give it, we verify the Cartier isomorphism for $X = \mathbb{A}^1_k$.

Example 3.2. Let $X = \mathbb{A}^1_k$ be the affine line. Then $X^{(p)} \cong \mathbb{A}^1_k$, and $F_X : X \to X$ is determined by $k[t^{(p)}] \to k[t], t^{(p)} \mapsto t^p$. Since everything is affine here, we can just work with modules instead of coherent sheaves. We first have that $F_X^{-1}\mathcal{O}_X = k[t^p] \subset k[t]$, and the de Rham complex of X is the complex of $k[t^p]$-modules

$$0 \to k[t] \to k[t]\langle dt \rangle \to 0.$$

Note that the image of d is generated (as a $k[t^p]$-module) by $\{dt, tdt, \ldots, t^{p-2}dt\}$ and thus $\mathscr{H}^1(\Omega_X^*)$ is a free $k[t^p]$-module generated by $t^{p-1}dt$. Now $F_X^{-1}\Omega^1_{X^{(p)}}$ is the free $k[t^p]$-module generated by $dt^{(p)}$, and according to the Cartier isomorphism we have $\mathscr{C}^{-1}(dt^{(p)}) = t^{p-1}dt$, so indeed $\mathscr{C}^{-1}$ maps the generator to the generator, confirming the Cartier isomorphism in this case.

4 Curves

For the remainder, let us concentrate on the case when X is a smooth projective curve over k. Then the de Rham complex has length 2, and

$$\mathcal{H}^1(\Omega_X^*) = \text{coker}(d : \mathcal{O}_X \to \Omega^1_X).$$

The global sections of $\mathcal{H}^1(\Omega_X^*)$ are somewhat mysterious as sheafification is required. However, being a quotient of Ω^1_X it does admit a map $\Omega^1_X \to \mathcal{H}^1(\Omega_X^*)$, and taking global sections of the composition of this map and the Cartier isomorphism yields a map

$$\mathscr{C} : H^0(X, \Omega^1_X) \to H^0(X^{(p)}, \Omega^1_{X^{(p)}})$$

which is called the Cartier operator (note: this map need not be an isomorphism).

Remark 4.1. Since X and $X^{(p)}$ are isomorphic as schemes over $\mathbb{F}_p$, one has an $\mathbb{F}_p$-linear isomorphism $H^0(X^{(p)}, \Omega^1_{X^{(p)}}) \to H^0(X, \Omega^1_X)$. Composing with the above map, we get an $\mathbb{F}_p$-linear (but not $\overline{\mathbb{F}}_p$-linear) map also known as the Cartier operator $\mathscr{C} : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$. In what follows, we sometimes use $\mathscr{C}$ to refer to this map instead.

Remark 4.2. The definition of $\mathscr{C}$ might seem rather ad hoc, however it arises naturally in a couple of ways. For one, it turns out that $\mathscr{C} : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ is adjoint to Frobenius $F^* : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ under Serre duality. Additionally, we will see below that when X is a modular curve with good reduction at p, then the Cartier operator acts as the usual Hecke operator T_p on modular forms.

4.1 The Case of Elliptic Curves

Let E/k be an elliptic curve given by a Weierstrass equation $Y^2Z = F(X, Z)$ with F a cubic. We will write $y^2 = f(x)$ for E in the chart $\{Z \neq 0\}$. Note then that $E^{(p)}$ is also an elliptic curve, as $f^{(p)}$ has discriminant equal to $\Delta(f)^p$.

To avoid confusion, let $x^{(p)}$ and $y^{(p)}$ denote the coordinates on $E^{(p)}$. Then we know that both $H^0(E, \Omega^1_E)$ and $H^0(E^{(p)}, \Omega^1_{E^{(p)}})$ are both 1-dimensional, spanned by the invariant differentials ω, ω' which look like dx/y and $dx^{(p)}/y^{(p)}$ respectively on the chart $U = \{Z \neq 0\} \cap \{Y \neq 0\}$. Note that to study whether or not $\mathscr{C}(\omega) = 0$, it suffices to work on the open subset U, since a regular 1-form which is 0 over a dense open subset is 0. In this chart, we furthermore have the Ω^1_U is coherent, with global sections equal to

$$H^0(U, \Omega^1_U) = \frac{k[x, y, y^{-1}]}{(y^2 = f(x))} \left\langle dx, dy | 2ydy = f'(x)dx \right\rangle .$$

Note that $x^n dx$ is an exact form if $n \not\equiv -1 \pmod p$, and thus in the group $\mathcal{H}^1(\Omega^*_X)(U)$ we have that $x^n dx = 0$ if $n \not\equiv -1 \pmod p$. Now we can compute

$$\mathscr{C}\left(\frac{dx}{y}\right) = \mathscr{C}\left(\frac{y^{p-1}}{y^p} dx\right) = \frac{1}{y^{(p)}} \mathscr{C}\left(f(x)^{\frac{p-1}{2}} dx\right)$$

Write $f(x)^{\frac{p-1}{2}} = \sum_{i \geq 0} c_i x^i$, and note that since $\deg(f) = 3$ the only term with exponent i congruent to -1 mod p is c_{p-1}. Thus we find

$$\frac{1}{y^{(p)}} \mathscr{C}\left(f(x)^{\frac{p-1}{2}} dx\right) = \frac{1}{y^{(p)}} \sum_i \mathscr{C}(c_i x^i dx) = \frac{1}{y^{(p)}} c_{p-1}^{1/p} \mathscr{C}(x^{p-1} dx) = c_{p-1}^{1/p} \frac{dx^{(p)}}{y^{(p)}}.$$

Thus we find

$$\omega \text{ is exact} \iff \mathscr{C}\left(\frac{dx}{y}\right) = 0 \iff c_{p-1} = 0 \iff E/k \text{ is supersingular},$$

where the final implication follows from counting $\#E(\mathbb{F}_q)$ via character sums. Explicitly, if $E : y^2 = f(x)$ is supersingular, then we can integrate $f(x)^{\frac{p-1}{2}} dx$ to a polynomial $F(x)$ (since the only obstruction to integration is a nonzero coefficient of x^{p-1}), and then we have

$$d\left(\frac{F(x)}{y^p}\right) = \frac{dx}{y}.$$

Remark 4.3. Using the duality between $\mathscr{C}$ and Frobenius, we have that $\mathscr{C} = 0$ if and only if $F^* = 0$ on $H^1(X, \mathcal{O}_X)$. This can also be related to supersingularity using the fact that E is supersingular if and only if $E[p]$ is a connected subgroup scheme, in addition to some Dieudonné theory.

4.2 The Case of Modular Curves and the Relation to Hecke Operators

Let $X = X_1(N)$ with $p \nmid N$, so that X is a fine moduli space over $\mathbb{Z}[1/N]$, coming equipped with a universal elliptic curve $\mathscr{E}$. Recall that if R is a $\mathbb{Z}[1/N]$-algebra, the Kodaira-Spencer isomorphism lets us think of modular forms of weight 2 and level N over R as sections of $\Omega^1_X(\log C)_R$, i.e. those meromorphic differentials with at worst logarithmic singularities at the cusps C of X. Moreover, we obtain the q-expansion of such a form by evaluating on the Tate curve over R.

Let us now work over $k = \overline{\mathbb{F}}_p$. Let $A = k((q))$, and let $g : U := \operatorname{Spec} A \to X/k$ be the punctured infinitesimal neighborhood of the cusp at infinity. Then the Tate curve T is obtained via the pullback square

$$\begin{array}{ccc} T & \longrightarrow & \mathscr{E}/k \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & X/k, \end{array}$$

and a modular form $\omega \in H^0(X/k, \Omega^1_X(\log C))$ with q-expansion $f(q)$ pulls back under g to $f(q)\frac{dq}{q}$. Now the definition of the Cartier operator extends to U, where we have

$$\mathscr{C}\left(q^n \frac{dq}{q}\right) = \begin{cases} q^{n/p}\frac{dq}{q} & n \equiv 0 \pmod p \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, functoriality of $\mathscr{C}$ gives us a square

$$\begin{array}{ccc} H^0(X/k, \Omega^1_X(\log C)) & \xrightarrow{\mathscr{C}} & H^0(X/k, \Omega^1_X(\log C)) \\ \downarrow{g^*} & & \downarrow{g^*} \\ H^0(U, \Omega^1_X(\log C)|_U) & \xrightarrow{\mathscr{C}} & H^0(U, \Omega^1_X(\log C)|_U). \end{array}$$

Therefore, if $\omega \in H^0(X/k, \Omega^1_X(\log C))$ is associated to the modular form f with q-expansion $f(q) = \sum a_n q^n \in k[[q]]$, we can compute

$$\begin{aligned} \mathscr{C}(f^*\omega) &= \mathscr{C}(f(q)dq/q) \\ &= \sum_{n=0}^{\infty} \mathscr{C}(a_n q^n \frac{dq}{q}) \\ &= \sum_{n=0}^{\infty} a_{np}^{1/p} q^n \frac{dq}{q}, \end{aligned}$$

and thus the q-expansion of $\mathscr{C}(\omega)$ is $\sum_n a_{np}^{1/p} q^n$, which we recognize as $T_p(f)$.

A particularly interesting case of the above computation arises when the differential ω is associated to a modular form $\overline{f}$ which is the reduction modulo p of a (normalized) cuspidal eigenform f with rational coefficients. Recall then that there is an elliptic curve $E_f/\mathbb{Q}$ with good reduction at p such that $L(E_f, s) = L(f, s)$, and in particular such that $a_p(f) = p + 1 - \#E(\mathbb{F}_p)$. Then we find

$$\mathscr{C}(\omega) = 0 \iff T_p(\overline{f}) = 0 \iff \overline{a_p f} = 0 \iff a_p \equiv 0 \pmod p \iff E_f \text{ is supersingular at } p!$$

The apparent similarity with our earlier conclusion that an elliptic curve is supersingular if and only if its invariant differential is killed by Cartier is not coincidental, and can be explained by modular parametrizations. Recall that we have a non-constant morphism $X_1(N) \to E_f$, under which the invariant differential ω_{E_f} of E_f pulls back to a nonzero scalar multiple of $f(q)\frac{dq}{q}$. Functoriality of the Cartier map then shows that $\mathscr{C}\left(f(q)\frac{dq}{q}\right) = 0 \iff \mathscr{C}(\omega_{E_f}) = 0$, relating this example with the previous one.

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