ARBITRARILY LARGE JUMPS IN THE DE RHAM COHOMOLOGY OF FAMILIES IN CHARACTERISTIC p

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ABSTRACT. For any $i \geq 1$, we construct smooth projective families of algebraic varieties in characteristic p of relative dimension $i+1$ such that the dimension of the i-th de Rham cohomology groups of the fibers can be made to jump by an arbitrarily large amount. To do this, we first construct an example using the classifying stack of an appropriately constructed finite flat group scheme which deforms $\mathbb{Z}/p^2\mathbb{Z}$ to $\alpha_p \oplus \alpha_p$.

1. INTRODUCTION

If $f : X \to Y$ is a smooth projective family of complex algebraic varieties with Y connected, it is a classical fact that the function on $y \in Y$,

$$y \mapsto \dim_{\mathbb{C}} H^i_{\mathrm{dR}}(X_y; \mathbb{C}),$$

is constant for any $i \geq 0$. However, it is well known that this fails for smooth projective families in mixed and positive characteristic. In [Suh], Suh constructs families of surfaces over a mixed characteristic DVR with an arbitrarily large jump in geometric genus from the generic to the special fiber, while in [CZ], Cotner and Zavyalov construct a family of surfaces in equal characteristic $p > 0$ such that $\dim H^1_{\mathrm{dR}}(X_s/k(s)) = 2$ and $\dim H^1_{\mathrm{dR}}(X_\eta/k(\eta)) = 1$. In both cases, the authors use the method of Godeaux-Serre, in which one obtains such families as quotients by the action of certain group schemes on complete intersections in projective space.

In this article, we use the Godeaux-Serre method to construct families of algebraic varieties in equal characteristic $p > 0$ with an arbitrarily large jump in algebraic de Rham cohomology (of any degree). Let $S = \operatorname{Spec} R$ be the spectrum of a characteristic p DVR with closed point s and generic point η.

Theorem 1.1. Let $e, i \geq 1$ be positive integers. Then there exists a smooth projective family $X \to S$ of relative dimension $i+1$ such that

$$\dim_{k(s)} H^i_{\mathrm{dR}}(X_s/k(s)) \geq \dim_{k(\eta)} H^i_{\mathrm{dR}}(X_\eta/k(\eta)) + e.$$

Remark 1.2. Base changing along the map $\mathbb{F}_p[t]_{(t)} \to R$ which sends t to the uniformizer of R and noting that algebraic de Rham cohomology is compatible with field extensions, we find that it is enough to prove Theorem 1.1 over $\mathbb{F}_p[t]_{(t)}$. Thus for the remainder of this article we work over the base $S = \operatorname{Spec} \mathbb{F}_p[t]_{(t)}$.

Remark 1.3. One can already obtain arbitrarily large de Rham jumps by taking self-products of the surface constructed in [CZ], however this requires the dimension of the family to increase with the prescribed gap e (though see Remark 4.1 for an approach to arbitrary jumps using the [CZ] surface). Thus the content of the theorem is that we

Date: April 16, 2023.

can obtain arbitrarily large jumps in a given cohomological degree in families of fixed dimension.

The strategy of proof is not new, and can be summarized in the following steps. First, construct a finite flat group scheme H/S such that $\dim H^i_{\mathrm{dR}}(BH_s) \geq \dim H^i_{\mathrm{dR}}(BH_\eta) + e$. Second, construct actions of H on projective space such that the complement of the free locus can be made to have arbitrarily large codimension. Finally, choose a complete intersection $Y \subset \mathbb{P}^N_S$ of relative dimension $i+1$ on which H acts freely, and use the Lefschetz hyperplane theorem to deduce that Y/H and $[\mathbb{P}^N/H]$ have the same de Rham cohomology in a range of degrees.

The structure of this paper is as follows. In section 2 we construct a desirable finite flat G/S and compute the de Rham cohomology of the classifying stacks of its fibers. In fact, the group H above will be a power of G depending on the gap e. In section 3 we construct actions of powers of G on projective spaces for which the complement of the free locus can be made to have arbitrarily large codimension, and in section 4 we combine these to prove Theorem 1.1.

ACKNOWLEDGEMENTS

I would like to thank my advisor Matthew Emerton for his guidance and encouragement. Thanks also to Sean Cotner, Luc Illusie, Ray Li, Akhil Mathew, Callum Sutton, and Bogdan Zavyalov for helpful conversations and comments relating to this work. I would especially like to thank Sean Cotner for allowing me to include his construction (Remark 2.2) in this work. This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. 2140001.

2. A GROUP SCHEME IN CHARACTERISTIC p

In this section, we construct a group scheme G/S which will have desirable cohomological properties by modifying a construction of Reid [Reid, Section 2.1] of a group scheme which deforms $\mathbb{G}_m$ to $\mathbb{G}_a$. Recall that $S = \operatorname{Spec} \mathbb{F}_p[t]_{(t)}$.

Lemma 2.1. There exists a finite flat commutative group scheme G/S of order p^2 with $G_\eta \cong \mathbb{Z}/p^2\mathbb{Z}$ and $G_s \cong \alpha_p \oplus \alpha_p$.

Proof. We begin by constructing the Cartier dual $G^\vee$ of G. First, let $R = \mathbb{F}_p[t]_{(t)}$ and consider the R-algebra

$$A := \frac{R[x,y]}{(x^p, y^p - tx)}.$$

A is freely generated as an R-module by monomials $x^a y^b$ with $0 \leq a, b \leq p-1$. We endow A with a Hopf algebra structure $\Delta : A \to A \otimes A, e : A \to R, i : A \to A$ given by

$$\Delta(y) = 1 \otimes y + y \otimes 1 + ty \otimes y$$
$$e(y) = 0$$
$$i(y) = \frac{-y}{1+ty},$$

which by the relation $tx = y^p$ forces $\Delta(x) = 1 \otimes x + x \otimes 1 + t^{p+1} x \otimes x, e(x) = 0$, and $i(x) = \frac{-x}{1+t^{p+1}x}$. One checks by hand that axioms for a commutative Hopf algebra hold (note that one only has to check these relations on the generators).

When $t = 0$, we get $k[x, y]/(x^p, y^p)$, with Hopf algebra structure

$$\begin{aligned} \Delta(x) &= 1 \otimes x + x \otimes 1, \Delta(y) = 1 \otimes y + y \otimes 1 \\ e(x) &= e(y) = 0 \\ i(x) &= -x, i(y) = -y, \end{aligned}$$

which we recognize as $\alpha_p \oplus \alpha_p$, while when t is invertible we get $k(t)[y]/((1+ty)^{p^2} - 1)$ with Hopf algebra structure

$$\begin{aligned} \Delta(1 + ty) &= (1 + ty) \otimes (1 + ty) \\ e(1 + ty) &= 1 \\ i(1 + ty) &= \frac{1}{1 + ty}, \end{aligned}$$

which we recognize as μ_{p^2}. Taking the Cartier dual gives the desired G/S. □

In the above proof, note that $A/(x) \cong R[y]/(y^p)$ with its given Hopf algebra structure recovers the deformation of μ_p to α_p constructed in [Reid, Proposition 3.1]. It is also interesting to note that the G/S constructed above is a group scheme which is a truncated Barsotti-Tate group generically but which is far from being truncated Barsotti-Tate on the special fiber, making it an equal characteristic p example of Raynaud's "affaissement" or "drooping" of truncated Barsotti-Tate groups [Ray2, Section 3].

Remark 2.2. An alternate geometric construction of a deformation of $\mathbb{Z}/p^2\mathbb{Z}$ to $\alpha_p \oplus \alpha_p$ due to Sean Cotner is as follows: begin with a degeneration $\mathcal{E}$ of an ordinary elliptic curve into a supersingular one such that $\mathcal{E}$ has rational p^2-torsion generically. Then the pullback of the short exact sequence

$$0 \to \ker V \to \ker V^2 \to \ker V \to 0$$

by the Frobenius map $F : \ker V \to \ker V$ yields such a deformation.

Now we calculate the de Rham cohomology of the classifying stacks BG_s and BG_η. For definitions and basic properties of de Rham cohomology of stacks, we refer the reader to [ABM, Section 2] and [CZ, Section 2.4], though we remark that for our calculations one only needs to know Totaro's theorem ([Tot, Theorem 3.1]) and the fact that de Rham cohomology still satisfies versions of the Lefschetz hyperplane theorem and projective bundle formula in this setting, as in [ABM, Section 5].

Proposition 2.3. For the group scheme G/S of Lemma 2.1, we have

$$\begin{aligned} \dim H^i_{\mathrm{dR}}(B(G^n_\eta))) &= \binom{n+i-1}{i}, \\ \dim H^i_{\mathrm{dR}}(B(G^n_s)) &= \binom{2n+i-1}{i}. \end{aligned}$$

Proof. First, since $G_\eta \cong \mathbb{Z}/p^2\mathbb{Z}$ is a discrete group, one has by [Tot, Lemma 10.2] that

$$H^i_{\mathrm{dR}}(BG_\eta) \cong H^i(\mathbb{Z}/p^2\mathbb{Z}, k(\eta)) \cong k(\eta)$$

for all $i \geq 0$. Next, by [ABM, Proposition 4.12], we have

$$\dim H^i_{\mathrm{dR}}(B\alpha_p) = 1$$

for all $i \geq 0$. The result then follows by the Künneth formula. □

3. Group Actions

3.1. **Generalities.** In this subsection, we give an exposition of the definitions and facts we will need concerning actions of finite flat group schemes on varieties. Let S be a locally noetherian scheme, G a finite flat S-group scheme, and X an S-scheme. An action of G on X is an S-map $G \times_S X \to X$ inducing a group action on T-valued points for all S-schemes T.

Definition 3.1. If T is an S-scheme and $x \in X(T)$, the stabilizer of x in G is the functor which sends a T-scheme U to $\{g \in G(U) : gx_U = x_U\}$. The free locus is the functor sending an S-scheme T to those points in $X(T)$ with trivial stabilizer.

Lemma 3.2. If X/S is separated, the free locus is represented by an open subscheme of X, and is compatible with arbitrary base change. In particular, the free locus is determined by its values on affine S-schemes.

Proof. See [CZ2, Lemma 2.1]. □

If X/S is separated, we will refer to the complement of the free locus as the *fixed locus*, which by Lemma 3.2 is closed in X. In this situation, we say that the action of G on X is free if the free locus is all of X. The relevance of free actions for us is contained in the following theorem.

Theorem 3.3 (Existence and Properties of Quotients)**.** Let X be a quasiprojective S-scheme with an action of a finite flat S-group scheme G. Then

(a) The ringed space quotient X/G is a quasiprojective S-scheme and the natural quotient map $\pi : X \to X/G$ is finite and surjective;
(b) If S is quasi-compact and quasi-separated and X is projective over S, then X/G is projective over S;
(c) If the action of G on X is free and X is smooth over S, then so is X/G;
(d) If G acts freely on X and Y/S is any separated S-scheme, then the diagonal action of G on $X \times_S Y$ is free.

Proof. For (a) and (c), see [CZ2, Theorem 2.2] and the discussion which follows. Part (d) follows readily from the definitions.

For (b), we argue as follows. First, X/G is proper over S by part (a) and [Sta, Tag 03GN]. Then since X/G is quasiprojective and proper over S, it is projective by [Sta, Tag 0BCL]. □

For the purposes of finding projective varieties with free actions, we will be interested in group actions on projective spaces. First, we study the situation over a field k. The following arguments are classical and may be found in e.g. [Suh, Lemma 2.1.1], but we include them for completeness.

Lemma 3.4. Let G/k be a finite group scheme over a field which acts on $\mathbb{P}^N_k$, and let r be the codimension of the fixed locus. Then for any $1 \leq r' \leq r-1$, there is an r'-dimensional complete intersection Y in $\mathbb{P}^N$ on which G acts freely, such that the quotient $X = Y/G$ is a smooth projective variety.

Proof. Let Z be the fixed locus of the action of G. Since $\mathbb{P}^N$ is projective, the quotient $\mathbb{P}^N/G$ exists as a projective variety P by Theorem 3.3 (b). Let $\pi : \mathbb{P}^N \to P$ be the quotient map and $i : P \to \mathbb{P}^M$ be a closed embedding. By applying Gabber's Bertini theorem [Gab, Corollary 1.7] in combination with Theorem 3.3 (c), we find that there exists a complete intersection $L \subset \mathbb{P}^M$, cut out by $N - r'$ equations, such that L does not intersect $i(\pi(Z))$ and $X := L \cap i(P)$ is smooth projective of dimension r'. Pulling back equations defining L to $\mathbb{P}^N$, we obtain a complete intersection Y of dimension r' on which G acts freely with quotient isomorphic to X. $\square$

Now, we relativize by lifting the construction on the special fiber. Let S be a DVR with residue field k.

Proposition 3.5. Let G/S be a finite flat group scheme which acts on $\mathbb{P}^N_S$, and let r be the relative codimension of the fixed locus. Then for any $1 \leq r' \leq r-1$, there is a complete intersection $Y \subset \mathbb{P}^N_S$ of relative dimension r' on which G acts freely, such that the quotient $X = Y/G$ is a smooth projective variety over S.

Proof. Let $Z \subset \mathbb{P}^N_S$ be the fixed locus, and let $\pi : \mathbb{P}^N_S \to P = \mathbb{P}^N/G$ and $i :\ P \to \mathbb{P}^M$ be the quotient map and a closed immersion, respectively. As in the proof of the previous lemma applied to the special fiber, we have a complete intersection $\overline{L} = V(\overline{f}_1, \ldots, \overline{f}_{N-r'}) \subset \mathbb{P}^M_k$. Choose lifts f_i of the equations defining $\overline{L}$ to S, and let $L \subset \mathbb{P}^M_S$ be the corresponding complete intersection. Then $X := L \cap i(P)$ doesn't intersect $i(\pi(Z))$ since there is no intersection on the special fiber and this intersection is proper. Moreover X is smooth of relative dimension r' over S because the smooth locus is open on S by [EGAIV, Theorem 12.2.4], and the special fiber is smooth of the correct dimension. Pulling back the sections f_i to $\mathbb{P}^N$ then gives the desired Y with a free action of G and quotient X. $\square$

Remark 3.6. As the above proof shows, it suffices to work on the special fiber, so we need only require Z_s to have codimension r in $\mathbb{P}^N_s$.

In the context of finding arbitrary de Rham jumps, the proposition shows that, for our group of interest G, we need to find actions of G on projective space such that the codimension of the complement of the free locus can be made arbitrarily large.

3.2. **An explicit representation.** We now return to the notation of Section 2, in which $S = \operatorname{Spec} \mathbb{F}_p[t]_{(t)}$ and G/S is the group scheme constructed in Lemma 2.1. In this section we construct projective spaces with an action of G^n with large free locus by taking powers of the regular representation of G^n, i.e. the action of G^n on its ring of regular functions $\mathcal{O}(G^n)$. This study has been undertaken in full generality by Raynaud in [Ray], however the group G is simple enough that we can explicitly carry out the analysis in our case. By Remark 3.6, it suffices to study the regular representation of the special fiber so we focus on the action of α_p^n on $\mathcal{O}(\alpha_p^n) \cong \mathbb{A}^{p^n}$. If A is a k-algebra and B is an A-algebra, the action of $b = (b_1, \ldots, b_n) \in \alpha_p^n(B)$ on $f(x_1, \ldots, x_n) \in$

$\mathcal{O}(\alpha_p^n)(A) = A[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$ is given by $b \cdot f(x) = f(x+b)$. From this description, a simple calculation shows that the free locus contains a hyperplane complement, namely the locus of polynomials with nonzero leading coefficient (i.e. nonzero coefficient of $x_1^{p-1} \dots x_n^{p-1}$ when we express elements of $\mathcal{O}(\alpha_p^n)(A)$ in the standard basis). It follows that the fixed locus has codimension at least 1, so taking direct sums of the regular representation, applying Theorem 3.3 (d), and projectivizing, we find

Lemma 3.7. For any $m > 0$, there exists an action of G^n on a projective space $\mathbb{P}^N_S$ such that the fixed locus has codimension at least m.

Combining this with Proposition 3.5, we obtain

Corollary 3.8. For any $n, m > 0$, there is a an $N > 0$ and a complete intersection $Y \subset \mathbb{P}^N_S$ of relative dimension m on which G^n acts freely, such that the quotient Y/G^n is a smooth projective variety over S.

4. Proof of Theorem 1.1

Let $e, i \geq 1$ be positive integers, and let G/S be the group scheme constructed in Lemma 2.1, so $G_s \cong \alpha_p \oplus \alpha_p$ and $G_\eta \cong \mathbb{Z}/p^2\mathbb{Z}$. By proposition 2.3, we may choose an $n > 0$ such that

$$\dim H^i_{\mathrm{dR}}(B(G_s^n)) \geq \dim H^i_{\mathrm{dR}}(B(G_\eta^n)) + e. \tag{4.1}$$

By Corollary 3.8, there is a projective space $\mathbb{P}^N_S$ with an action of G^n and a complete intersection $Y \subset \mathbb{P}^N_S$ of relative dimension $i+1$ on which the action is free, such that $X := Y/G^n$ is a smooth projective variety over S. By the Lefschetz hyperplane theorem [ABM, Proposition 5.3], we have for each $t \in S$ that

$$H^i_{\mathrm{dR}}(Y_t/k(t)) \cong H^i_{\mathrm{dR}}(\mathbb{P}^N_{k(t)}),$$

and by [ABM, Proposition 5.10] we obtain

$$H^i_{\mathrm{dR}}(X_t/k(t)) \cong H^i_{\mathrm{dR}}([\mathbb{P}^N_{k(t)}/G_t^n]), \tag{4.2}$$

where $[\mathbb{P}^N_{k(t)}/G_t^n]$ denotes the quotient stack. But by construction, $[\mathbb{P}^N_{k(t)}/G_t^n]$ is the projectivization of a vector bundle over BG_t^n, and therefore by the projective bundle formula[1] we can compute

$$H^i_{\mathrm{dR}}([\mathbb{P}^N_{k(t)}/G_t^n]) \cong H^i_{\mathrm{dR}}(\mathbb{P}^N_{k(t)} \times BG_t^n) \cong H^i_{\mathrm{dR}}(BG_t^n) \oplus H^{i-2}_{\mathrm{dR}}(BG_t^n) \oplus \dots.$$

Combining this with (4.1) and (4.2), we find

$$\dim H^i_{\mathrm{dR}}(X_s) \geq \dim H^i_{\mathrm{dR}}(X_\eta) + e,$$

as desired.

Remark 4.1. As pointed out to the author by Sean Cotner, one can also obtain arbitrarily large jumps in de Rham cohomology by taking self products of the surface constructed in [CZ] and slicing the result by hyperplanes. However, our approach was motivated by the search for families of group schemes in equal characteristic p that have visibly different cohomological behavior in all degrees on the generic and special fibers (e.g. non-split generically and split on the special fiber).

[1]The projective bundle formula for the de Rham cohomology of smooth stacks follows from the classical version for schemes [Sta, Tag 0FMS] as in [ABM, Proposition 5.11].

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