Small quotients of surface braid groups

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Nearly Carbon Neutral Geometric Topology Conference
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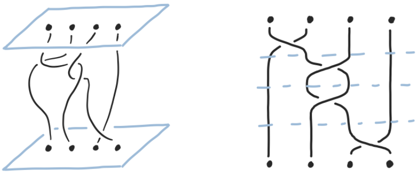
Summary

What are the smallest nonabelian quotients of braid groups and surface braid groups?

- For the braid group B_n, the answer is (almost always) S_n
- Surface braid groups admit a class of Heisenberg-like quotients which do not have analogues in the B_n story

Braid groups

- Braided strands



Braid groups

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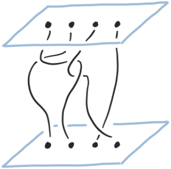
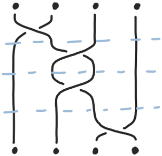
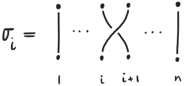


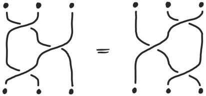


- Generators: $\sigma_1, \ldots, \sigma_{n-1}$



Relations:

1. $[\sigma_i, \sigma_j] = 1$ for $|i - j| \geq 2$
2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



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Braid groups

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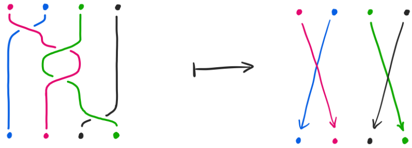
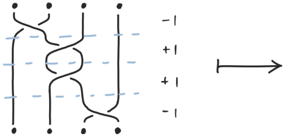


2. $B_n' \hookrightarrow B_n \overset{\text{ab}}{\twoheadrightarrow} \mathbb{Z}$ (signed crossing number)



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Braid groups

- Loops in configuration space of $\mathbb{C}$

$$\begin{aligned} \mathrm{Conf}_n(\mathbb{C}) &= \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \neq x_j \text{ if } i \neq j\} \\ &= \mathbb{C}^n - \mathrm{Diag}\, \mathbb{C}^n \\ \mathrm{UConf}_n(\mathbb{C}) &= \mathrm{Conf}_n(\mathbb{C})/S_n \end{aligned}$$

$$B_n := \pi_1(\mathrm{UConf}_n(\mathbb{C}))$$

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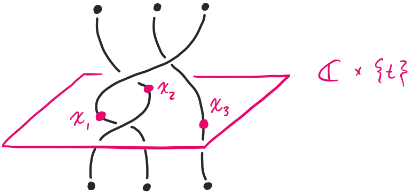
$\gamma : S^1 \to UConf_n(\mathbb{C})$

$\gamma(t) = \{x_1, x_2, x_3\}$



Surface braid groups

- Loops in configuration space of a surface Σ

$$\mathrm{Conf}_n(\Sigma) = \{(x_1, \ldots, x_n) \in \Sigma^n : x_i \neq x_j \text{ if } i \neq j\}$$
$$= \Sigma^n - \mathrm{Diag}\,\Sigma^n$$
$$\mathrm{UConf}_n(\Sigma) = \mathrm{Conf}_n(\Sigma)/S_n$$

$$B_n(\Sigma) := \pi_1(\mathrm{UConf}_n(\Sigma))$$

$\gamma : S^1 \to \mathrm{UConf}_n(\Sigma)$

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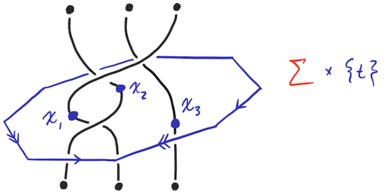
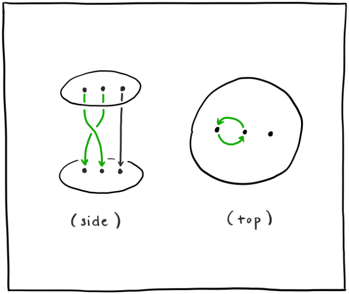


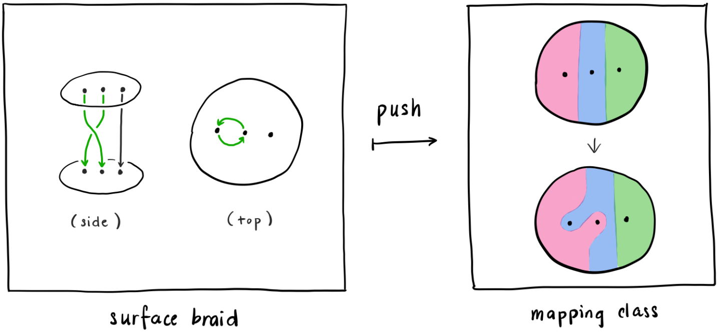
Σ oriented connected finite type

Braids as mapping classes

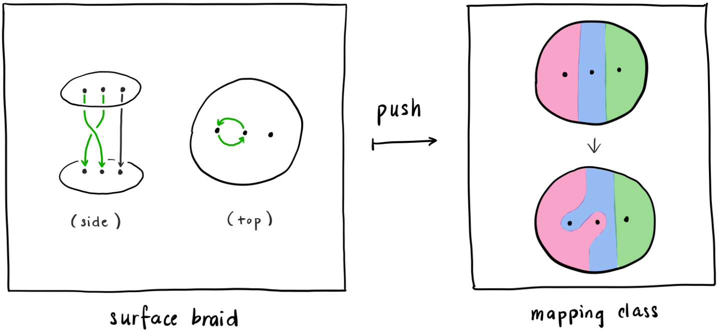


surface braid

Braids as mapping classes



Braids as mapping classes



This is an isomorphism $B_n(D) \xrightarrow{\cong} \mathrm{Mod}(D, n)$.

$\mathrm{Mod}(\Sigma) = \{$oriented homeomorphisms of Σ fixing $\partial\Sigma\}$ / isotopy
$(\Sigma, n) = \Sigma - \{n$ points$\}$

Surface braids as mapping classes

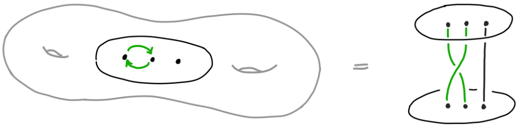
For $\Sigma \neq S^2, T^2$, there is a short exact sequence

$$1 \to B_n(\Sigma) \xrightarrow{\text{push}} \mathrm{Mod}(\Sigma, n) \xrightarrow{\text{forget}} \mathrm{Mod}(\Sigma) \to 1.$$

- $B_n \cong \mathrm{Mod}(D, n)$

Structure of the surface braid group

1. Regular braids



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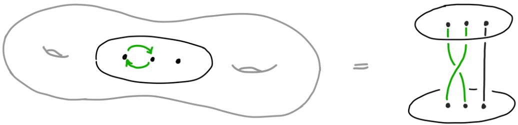
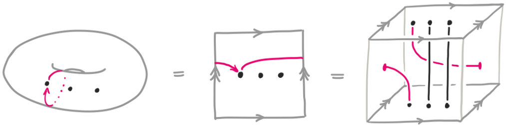
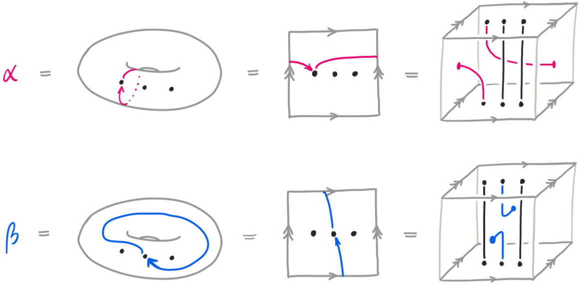


2. Braids from loops



Structure of the surface braid group

- $B_n(\Sigma_g)$ is generated by B_n and a (choice of) homology basis:

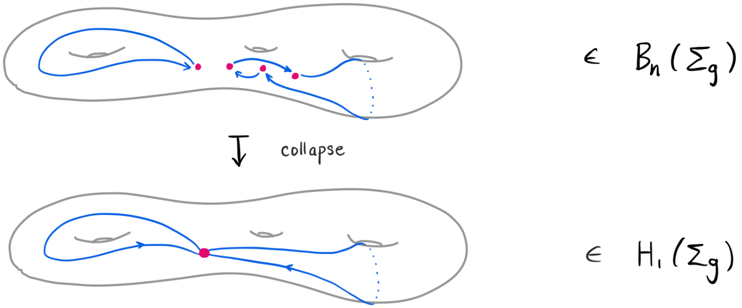


Homology basis generators of $B_n(T^2)$

Structure of the surface braid group

- There is a short exact sequence

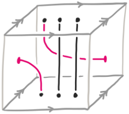
$$1 \longrightarrow \langle\langle B_n \rangle\rangle \longrightarrow B_n(\Sigma_g) \xrightarrow{\text{collapse}} H_1(\Sigma_g, \mathbb{Z}) \longrightarrow 1$$



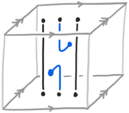
Collapsing map in $B_n(\Sigma_3)$

Structure of the surface braid group

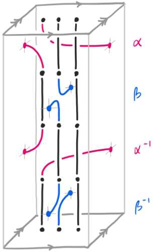
$\alpha =$



$\beta =$



$[\alpha, \beta] =$



A commutator in $B_n(T^2)$

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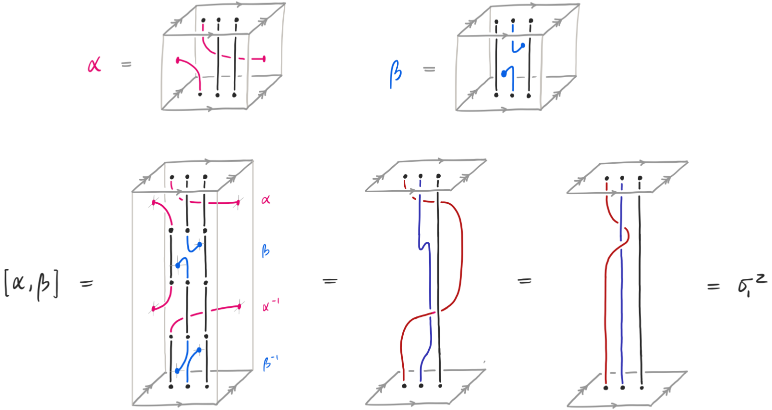
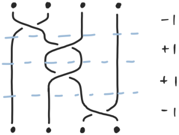


A commutator in $B_n(T^2)$

Finite quotients

Quotients of braid groups

1. $B_n \overset{\text{ab}}{\twoheadrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/(d)$ (signed crossing number mod d)



$\longmapsto$ 0 mod d

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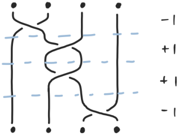

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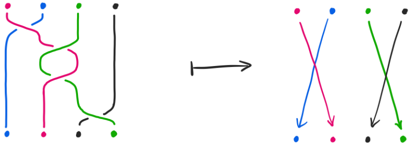
2. $B_n \twoheadrightarrow S_n$ (permutation of a braid)



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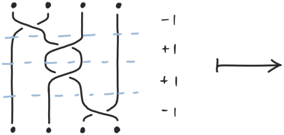

$0 \bmod d$

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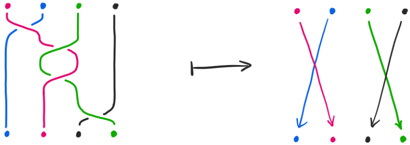

3. $B_4 \twoheadrightarrow S_4 \twoheadrightarrow S_3$

Theorem (Kolay, 2021).

Let $n = 3$ or $n \geq 5$. If G is a noncyclic quotient of B_n then

$$|G| \geq n!$$

with equality if and only if $G \cong S_n$ and the quotient map is the standard projection post-composed with an automorphism of S_n.

Quotients of surface braid groups

- $B_n(\Sigma_g) \overset{\text{ab}}{\twoheadrightarrow} \mathbb{Z}/(2) \oplus \mathbb{Z}^{2g}$
- $B_n(\Sigma_g) \twoheadrightarrow S_n$ (permutation of a braid)
- $B_4(\Sigma_g) \twoheadrightarrow S_3$

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Is S_n the smallest ~~noncyclic~~ **nonabelian** quotient of $B_n(\Sigma_g)$?

Claim.

Given an odd divisor d of n, there is a surjection

$$B_n(T^2) \twoheadrightarrow \mathcal{H}_3(d) = \left\{ \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} : * \in \mathbb{Z}/(d) \right\}.$$

- Some of these quotients are smaller than S_n, for example

$$B_7(T^2) \twoheadrightarrow \mathcal{H}_3(7)$$

and $|\mathcal{H}_3(7)| = 7^3 < 7! = |S_7|$

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- In general, $B_n(\Sigma_g)$ surjects onto similar nonabelian nilpotent groups, some of which are smaller than S_n

Construction of $B_n(T^2) \twoheadrightarrow \mathcal{H}_3(d)$

Given the presentation

$$\mathcal{H}_3(d) = \langle X, Y, Z \in \mathbb{Z}/(d) : [Z, X] = [Z, Y] = 1,\ [X, Y] = Z \rangle$$

the map on generators is

$$\begin{aligned} B_n(T^2) &\twoheadrightarrow \mathcal{H}_3(d) \\ \alpha &\mapsto X \\ \beta &\mapsto Y \\ \sigma_i &\mapsto Z^{\frac{d+1}{2}} \end{aligned}$$

Is S_n the smallest ~~nonabelian~~ **non-nilpotent** quotient of $B_n(\Sigma_g)$?

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Theorem 1 (T, 2023).

Let $n = 3$ or $n \geq 5$ and $g \geq 0$. The smallest non-nilpotent quotient of $B_n(\Sigma_g)$ is S_n and the quotient map is unique up post-composition with an automorphism of S_n.

Theorem 2 (T, 2023).

Let G be a nonabelian nilpotent quotient. Let p be the smallest prime dividing $g + n - 1$. Then

1. if $p = 2$ then $|G| \geq 2^{2g+2}$, and
2. if p is odd then $|G| \geq p^{2g+1}$.

In each case equality is attained by exactly two nonisomorphic groups.

Corollary.

Let $g \geq 1$ and $n \geq 5$. Then the order of any nonabelian quotient of $B_n(\Sigma_g)$ is at least the smaller of $n!$ and the lower bound in Theorem 2, which depends on g and n.

More

Conference Discord: June **23rd** - 30th

Email: `cindy@math.uchicago.edu`

Slides: `math.uchicago.edu/~cindy/2023-ncngt.pdf`