UNIVERSITY OF VIRGINIA
DEPARTMENT OF MATHEMATICS

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22903

19 August 80

Dear Peter:

Stefan finally arrived so that I could talk to him for a while. I managed to understand a bit about what you all are assuming about your cohomology theory, and so am able to completely calculate the $RO(G)$ graded coefficient groups with Burnside ring coefficients when G is cyclic of prime order. I gather that nobody had done so.

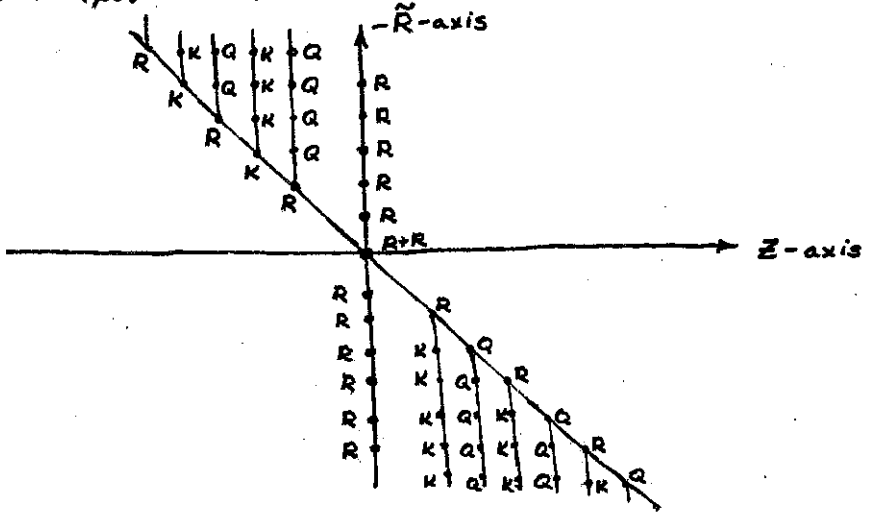
The case of $G = \mathbb{Z}/2\mathbb{Z}$ is the easiest to write down. There are two points $S^0 = (\mathbb{Z}_2, -1)$ and pt, and letting R be a ring, I assume

$$H^0(S^0) \cong R \qquad H^0(pt) = R + R$$

with a base of $H^0(pt)$ being 1, σ and $\sigma^2 = 2\sigma$. The homomorphism $H^0(pt) \longrightarrow H^0(S^0)$ sends 1 to 1 and σ to 2. The transfer sends 1 to σ. The coefficients are ordinary: $H^i(S^0) = H^i(pt) = 0$ for $i \in \mathbb{Z}$, $i \neq 0$. There is a 1-dimensional non-trivial representation for $\mathbb{Z}_2$, denoted $\tilde{R}$.

Then $H^*(S^0) \cong R[t, t^{-1}]$ where $t \in H^{\tilde{R}-1}(S^0)$, $t^{-1} \in H^{1-\tilde{R}}(S^0)$.

Also $H^*(pt)$ looks like



(over)

where $K = \ker\{2: R \longrightarrow R\}$ and $Q = R/2R$.

Specifically

$$H^\alpha(pt) = \begin{cases} R+R & \text{if } \alpha = 0 \\ R & \text{if } \alpha = n\tilde{R} \text{ or } 2n(\tilde{R}-1),\ n = \pm1, \pm2, \pm3, \ldots \\ K & \text{if } \alpha = n\tilde{R}+m \text{ with} \\ & \quad n > 1 \text{ odd and } m \geq n \\ & \quad \text{or } n < 0 \text{ even and } m < n \\ Q & \text{if } \alpha = n\tilde{R}+m \text{ with} \\ & \quad n > 0 \text{ even and } m > n \\ & \quad \text{or } n < -1 \text{ odd and } m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Of course, one can write down the multiplication, the homomorphism $H^*(pt) \longrightarrow H^*(S^0)$ and the transfer.

The case $G = \mathbb{Z}_p$ is quite similar – the groups are $R+R$, R, $Q = R/pR$ and $K = \ker\{p: R \to R\}$. The proof is significantly different for p odd, however.

Of course, knowing the coefficients one can proceed to calculate the cohomology for many spaces, using naive methods.

I'm going to try to write this junk down this week while Stefan is away.

Sincerely,

Bob Stong

P.S. Some of my calculations are worrying Stefan. Some things he hoped to prove aren't true, it appears.

4 Sept 80

Dear Peter:

I apologize for being unclear. Actually, for Z_2 and Z_p (p odd) I can describe $H^*_G(G/H\,;\,\text{Burnside}\otimes R)$ for all H as algebra-module over the Burnside ring and also the homomorphisms induced by maps $G/K \longrightarrow G/H$. I didn't put down $H^*_{Z_2}(Z_2)$ because it was so easy that I never thought about it.

More generally $H^*_G(G\,;\,\text{Burnside}\otimes R) \cong R$ polynomial ring on generators t_ρ and t_ρ^{-1} in $H^{\rho-d}_G(G)$ and $H^{d-\rho}_G(G)$ $(t_\rho \cdot t_\rho^{-1} = 1)$ for ρ a non-trivial irreducible representation of dimension d. Further, if $\varphi_g : G \longrightarrow G : g' \longrightarrow g'g$ then $\varphi_g^*(t_\rho) = (\pm 1)t_\rho$ where the ± 1 is the determinant of g as it acts in the representation ρ. (This holds for all finite G, and indeed is true for all G – if I understand your infinite group case).

Since $H^\alpha_G(G/H) \cong H^{\alpha|H}_H(\text{point})$, in some sense you actually only need to compute the groups for points, but that isomorphism loses a lot of the additional structure. For example, it tells you that $H^\alpha_G(G) = R$ if $\alpha \in \ker(RO(G) \xrightarrow{\dim} Z)$ and is zero otherwise, but doesn't tell you the ring structure and the automorphisms.

Oh. The Burnside ring acts in $H^*_G(G)$ in the obvious way A finite G set with m elements is just multiplication by m – and by the way $t_\rho t_{\rho'} = t_{\rho'} t_\rho$ – strict commutativity holds.

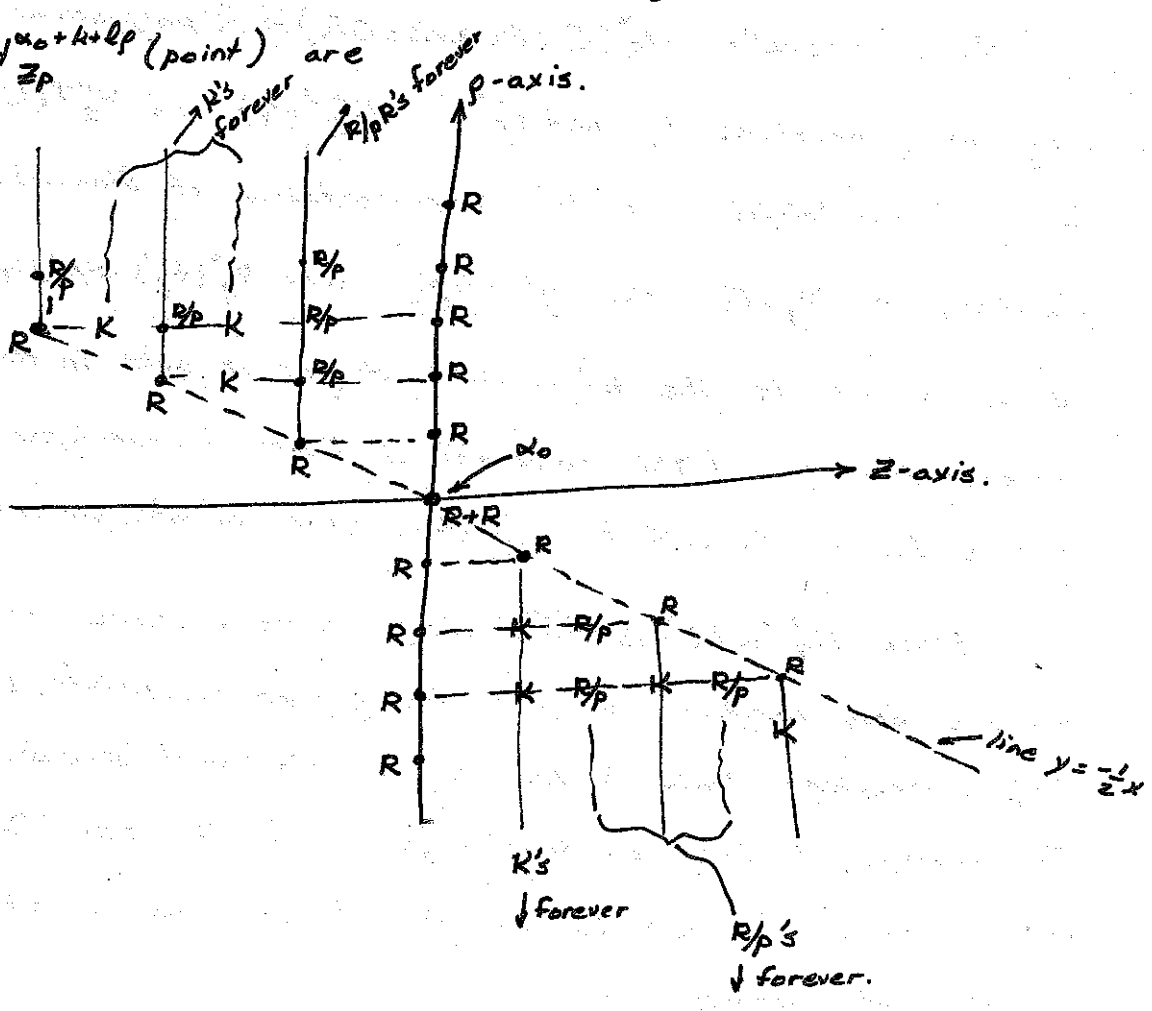
(over)

Normally $t_\rho t_{\rho'} = u t_{\rho'} t_\rho$ for some unit $u \in B(G)$, but the unit here actually becomes 1. because of the way $B(G)$ acts.

that means I reserve the right to be wrong

I actually found a decent description of $H^*_{\mathbb{Z}_p}(\text{point})$ but haven't written down all details yet. Let $\rho_1, \dots, \rho_{\frac{p-1}{2}}$ be the nontrivial 2 dimensional irreducible reps of $\mathbb{Z}_p$ and ρ one of them (fixed). For $\alpha \in RO(\mathbb{Z}_p)$, write $\alpha = \alpha_0 + k + l\rho$ where $\alpha_0 = a_1\rho_1 + \dots + a_{\frac{p-1}{2}}\rho_{\frac{p-1}{2}}$ and $\sum a_i = 0$

Then in the $\mathbb{Z}$-ρ plane with origin at α_0, the groups $H^{\alpha_0+k+l\rho}_{\mathbb{Z}_p}(\text{point})$ are



where $K = \ker\{R \xrightarrow{p} R\}$. The esoteric point is that the $R+R = H^{\alpha_0}_{\mathbb{Z}_p}(\text{point})$ is <u>not</u> the Burnside ring in general — it requires two generators as $B(G)\otimes R$ module. If p is invertible in R it is the Burnside ring, or if $\alpha_0 = 0$ it is.

Unfortunately, I've not really made much progress in the last week or so. I've tried to extend the computation to other G, but largely in seeking general results rather than specific cases. I have a feeling that abelian groups might be accessible, but nonabelian are definitely complicated.

You suggested the calculations as possible theses. There are a couple of points.

1. I don't think you realize how trivial the things I've done actually are. There is nothing deep involved, and I'm not sure this is worthy of a degree.

2. All of my students actually have problems at the moment. They aren't working as hard as they should, but they have stuff to do. If you have some students that you want to put on these things that would be fine. I will try to get my ideas written down and sent to you as soon as I can so they could see how I start.

Let me try to give a rough outline of what I do.

To compute $H_G^*(G)$, you note that $\Sigma^l G \cong \Sigma^d G$ as G-spaces. Then $H_G^\alpha(G) \xrightarrow[\text{suspend}]{\cong} \tilde{H}_G^{\alpha+l}(\Sigma^l G) = \tilde{H}_G^{\alpha+l}(\Sigma^d G) \xrightarrow[\text{suspend}]{\cong} H_G^{\alpha+l-d}(G)$ an isomorphism of $H_G^*(G)$-modules – hence given by multiplication by an invertible class $t_{l-d} \in H_G^{l-d}(G)$. Everything except φ_g^* is then simple. For φ_g^*, you suspend $\varphi_g : G \to G$ to $\Sigma^l \varphi_g : \Sigma^l G \to \Sigma^l G$ and use this to map $\Sigma^d G \to \Sigma^d G$, i.e. identify $\Sigma^l G$ with $\Sigma^d G$. The map is <u>not</u> $\Sigma^d \varphi_g$ – the suspension

sphere at $1 \in G$ has been mapped by $\rho(g^{-1})$ which is of degree $\det \rho(g^{-1})$, and being a sphere with trivial action you see the map.

To do $\mathbb{Z}_2$, let ρ be the nontrivial one dimensional representation. and look at the cofibration (disc-sphere in ρ)

$$\begin{array}{ccccc} S(\rho) & \longrightarrow & D(\rho) & \longrightarrow & D(\rho)/S(\rho) \\ \wr & & \wr & & \wr \\ \mathbb{Z}_2 & \longrightarrow & \text{point} & \longrightarrow & \Sigma^\rho pt. \end{array}$$

You know $H^*_{\mathbb{Z}_2}(\mathbb{Z}_2)$ and you inductively compute the exact sequence. Straightforward calculation hits a couple of questions but when you hit them Waner's assumption will settle it. The coboundary is given by a map

$$\Sigma^\rho pt \longrightarrow \Sigma^1 \mathbb{Z}_2$$

and recalling $\Sigma^1 \mathbb{Z}_2 \cong \Sigma^\rho \mathbb{Z}_2$, the map $\Sigma^\rho pt \longrightarrow \Sigma^\rho \mathbb{Z}_2$ induces the transfer (by Waner) — it is "his" map on the nose.

To do $\mathbb{Z}_p$, choose a ρ to begin with. There is a cofibration

$$\mathbb{Z}_p \longrightarrow S(\rho) \longrightarrow \Sigma^1 \mathbb{Z}_p$$

and the coboundary $H^*_{\mathbb{Z}_p}(\mathbb{Z}_p) \longrightarrow H^*_{\mathbb{Z}_p}(\mathbb{Z}_p)$ is geometrically just $1 - \Phi_T^*$ (T generator of $\mathbb{Z}_p$). That gives you $H^*_{\mathbb{Z}_p}(S(\rho))$.

Now look at the cofibration

$$\begin{array}{ccccc} S(\rho) & \longrightarrow & D(\rho) & \longrightarrow & D(\rho)/S(\rho) \\ & & \wr & & \wr \\ & & pt & & \Sigma^\rho pt \end{array}$$

Again you hit questions, <u>but</u> the coboundary gives

$$\Sigma^\rho pt \longrightarrow \Sigma^1 S(\rho) \xrightarrow[\Sigma \text{ above}]{} \Sigma^1(\Sigma^1 \mathbb{Z}_p) \xrightarrow{=} \Sigma^2 \mathbb{Z}_p \cong \Sigma^\rho \mathbb{Z}_p$$

and this is Waner's transfer

This gives you $H^{\alpha}_{\mathbb{Z}_p}(\text{point})$ for $\alpha = kp+j$ — but ρ was any rep. So in $R^{\frac{p+1}{2}}$ you know the groups that lie along the 2-dimensional subplanes — and need to fill in bigger subspaces. Of course, for $\mathbb{Z}_3$ you are done.

So you look at $\mathbb{Z}_5$, with ρ_1, ρ_2, and that makes the point $\alpha_0 = \rho_2 - \rho_1$ stand out. Hence you see that the α_0's are crucial.

So take $\alpha_0 = V - W$ and write $W = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \cdots \oplus \mathbb{C}_r$, $V = \mathbb{C}_1' \oplus \mathbb{C}_2' + \cdots \oplus \mathbb{C}_r'$ and take an equivariant map $W \longrightarrow V$ given by $z \longrightarrow z^j$ in each complex summand. That gives you a map

$$\Sigma^W pt \longrightarrow \Sigma^V point$$

hence $H^{\alpha}_{\mathbb{Z}_p}(pt) \longrightarrow \tilde{H}^{\alpha+V}_{\mathbb{Z}_p}(\Sigma^V pt) \longrightarrow \tilde{H}^{\alpha+V}_{\mathbb{Z}_p}(\Sigma^W pt) \longrightarrow H^{\alpha+V-W}_{\mathbb{Z}_p}(pt)$

which is a homo of $H^*_{\mathbb{Z}_p}(pt)$-modules — hence multiplication by a class in $H^{V-W}_{\mathbb{Z}_p}(pt) = H^{\alpha_0}_{\mathbb{Z}_p}(pt)$. In exactly the same way there is a class in $H^{W-V}_{\mathbb{Z}_p}(pt)$ — and $a_{V-W} \cdot a_{W-V}$ is in $H^0_{\mathbb{Z}_p}(pt) \cong B(\mathbb{Z}_p) \otimes R$ is $1 + x(\mathbb{Z}_p)$ and is basically one (mod p). It is close enough to invertible to analyze the groups. (actually $x \neq 0$ (p).)

I note that multiplication by a_{W-V} sends

$$H^{\alpha_0}_{\mathbb{Z}_p}(pt) \longrightarrow H^0_{\mathbb{Z}_p}(pt) \cong B(\mathbb{Z}_p) \otimes R = R + R(\mathbb{Z}_p)$$

is monic and one can figure out the image which is $R+R$ (it is generated by p, $p(\mathbb{Z}_p)$, and $1 + x(\mathbb{Z}_p)$ with $x \neq 0$ mod p.) That submodule of $B(\mathbb{Z}_p) \otimes R$ is <u>not</u> isomorphic to $B(\mathbb{Z}_p) \otimes R$ — it can't be generated by one element.

(over).

Why do I think I can do abelian groups? Maybe it is optimistic, but I think one has the full bag of tricks needed. Life would be nicer if you restricted to a field of characteristic prime to $|G|$ as R. Then the analogues of the $H^{\alpha\alpha}$'s are generated by invertibles and you ought to be able to write down a nice closed form.

For G abelian of odd order, R = field with $|G|$ invertible, I'd swear it is not hard. (Added on 8 Sept).

You asked about S^1. It is awful! I calculated a piece of it yesterday and it is fantastically large. (Of course, $RO(S^1)$ is big, but even the individual groups are large — if S^1 acts on $\mathbb{C}$ by $z \cdot w = z^2 w$ $(z \in S^1, w \in \mathbb{C})$ is the rep ρ^2, the groups $H^{n\rho^2}_{S^1}(pt) \cong$ a number of copies of R, and that number increases steadily with n. It looks bad, and one should compute all $G = \mathbb{Z}_m$ first. Ugly, ugly.

The problem is that the Burnside ring of S^1 is too damned small — in comparison with that of $\mathbb{Z}_m$.

Bob

Note: This is now 8 Sept. My car died and all kinds of mess, so this didn't get into the mail.

STONY

16 October 80

Dear Peter:

For several days now Stefan and I have been hassling over a result, and finally, I have what I think is a proof.

Let G be a finite group of <u>odd</u> order. Let F be a field of characteristic zero or prime to the order of G. Then for any G-space X,

$$H^*_G(X; \text{Burnside} \otimes F) \cong \bigoplus_{(H)} e_H \cdot H^*(X^{(H)}/G; F) \otimes_F F[{}_H t_\rho, {}_H t_\rho^{-1}]$$

where the sum is over conjugacy classes of subgroups H; e_H is a minimal idempotent in $A(G) \otimes F$ (explicitly e_H is the finite set for which the number of points fixed by K is 1 if $K=H$ and 0 if $K \neq H$); $X^{(H)}$ is the set of points of X fixed by some conjugate of H; and $F[{}_H t_\rho, {}_H t_\rho^{-1}]$ denotes a "polynomial ring" on generators ${}_H t_\rho$ and their inverses, ρ being any non-trivial irreducible real representation of G (i.e. one generator for each such ρ), and ${}_H t_\rho$ has degree $\rho - |\rho^H| \in RO(G)$, where $|\rho^H|$ is the dimension of the fixed point set of H in ρ.

The result is valid only for groups of odd order. For even order G the analogues are far more complicated. For $G = \mathbb{Z}_2$ and F of characteristic not equal to 2,

$$H^*_{\mathbb{Z}_2}(X; \text{Burnside} \otimes F)$$

$$\cong e_{\mathbb{Z}_2} H^*(X^{\mathbb{Z}_2}; F) \otimes_F F[{}_{\mathbb{Z}_2}t_\rho^{-1}, {}_{\mathbb{Z}_2}t_\rho]$$

$$\oplus e_1 \Big\{ H^*(X/\mathbb{Z}_2; F) \otimes_F F[{}_1t_\rho^2, {}_1t_\rho^{-2}]$$

$$+ {}_1t_\rho \cdot H^*(X/\mathbb{Z}_2, X^{\mathbb{Z}_2}; F) \otimes_F F[{}_1t_\rho^2, {}_1t_\rho^{-2}] \Big\}$$

where that latter means that even powers of ${}_1t_\rho$ have coefficients from $H^*(X/\mathbb{Z}_2)$ but the odd powers have coefficients from $X/\mathbb{Z}_2$ mod the fixed point set of $\mathbb{Z}_2$.

The situation for infinite groups will be terrible. I worked on $G = S^1$ and F having characteristic zero and the behavior is radically different. The most crucial thing which breaks down is that "Burnside" $A(S^1) \cong \mathbb{Z}$ is not at all similar in structure to $A(H)$ for $H \subset S^1$. For finite groups $A(G) \otimes F \longrightarrow A(H) \otimes F$ is epic and the idempotents split $H^*_G(\ ; A \otimes F)$ up into a sum of theories with "almost" trivial coefficient system. No such splitting occurs for $G = S^1$

Sincerely,

Bob Stong

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13 Nov 80

Dear Peter:

In the decomposition

$$H^*_G(X, A\otimes F) \cong \bigoplus_{(H)} e_H \cdot H^*(\cdots)\otimes_F F[{}_H t_\rho, {}_H t_\rho^{-1}]$$

the e_H's could be thought of as placeholders but they actually are a tiny bit more. You can consider them as classes in the cohomology with $1 = \sum_{(H)} e_H$, $e_H \cdot e_K = \begin{cases} e_H & H = K \\ 0 & H \neq K \end{cases}$

and $e_H \cdot {}_H t_\rho = {}_H t_\rho$, etc. If one wanted to fully describe the multiplication you would need them — in some form.

They <u>are</u> the projections as well — i.e. multiplication by e_H is the projection onto ~~a~~ the summand.

Bob.

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Bob.

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11 Dec 80

CHARLOTTESVILLE, VA.
22903

To Various Friends:

It is with no pleasure that I must admit that my calculation of $H^*_G(X, A \otimes F)$ for G of odd order and F of characteristic prime to $|G|$ or zero is, in fact, <u>incorrect</u>.

Stefan Waner encountered a discrepancy and my argument is not correct except for G abelian — where the proof does seem valid. The <u>correct</u> result for G odd <u>seems</u> to be

$$H^*_G(X, A \otimes F) = \sum_{(H)} e_N H^*(X^H/NH; F) \otimes_F F[{}_N t_\rho, {}_N t_\rho^{-1} / \deg {}_N t_\rho = \rho - |\rho^H|]$$

<u>but</u> it is not certain that the details in the proof can be carried off in the nonabelian case. My previous claim used $X^{(H)}/G$ rather than X^H/NH and was definitely incorrect.

Sorry About That,

Bob Stong

Peter:

I think this is in fact what McClure's results say.

Bob

Note: Partial Write Up. More Can be said.

Ordinary Cohomology Of Points

R.E. Stong

§ 1. Introduction

The purpose of this note is the calculation of the cohomology of points and other elementary spaces using ordinary cohomology. Unfortunately, this is not as trivial as it sounds; the spaces are spaces with a group action and the cohomology theory is ordinary equivariant cohomology graded by the real representation ring and with coefficients in the Burnside ring.

In order that any of this be meaningful, it is necessary to completely describe the cohomology theory being considered. To do so, let G be a finite group and R a commutative ring with unit.

Bredon [ ] was the first to make a detailed study of equivariant cohomology theories. He considered theories $H_G^*(-;T)$ on the category of G-pairs, indexed by the integers $\mathbb{Z}$. These theories have a suspension isomorphism only for suspensions with trivial G-action in the suspension coordinates. The coefficient system for such a theory is the contravariant functor it defines on the category of orbit spaces G/H and equivariant maps between them. Bredon calls such a theory "classical" if for all H, $H_G^n(G/H;T)=0$ for $n\neq 0$.

For example, there is a Bredon "classical" cohomology theory with

$$H_G^n(G/H;B\otimes R)=\begin{cases}0 & \text{if } n\neq 0\\ B(H)\otimes R & \text{if } n=0\end{cases}$$

where $B(H)$ is the Burnside ring of H. An equivariant map $\varphi: G/H \longrightarrow G/K$ sends H to a coset gK (and $g'H$ to $g'gK$) and in order that this be well-defined one must have $g^{-1}Hg \subset K$. The homomorphism $\varphi^*: B(K)\otimes R \longrightarrow B(H)\otimes R$ sends a finite set F on which K acts to the same finite set F on which $h \in H$ acts as $g^{-1}hg \in K$.

It is immediate that there are suspensions of G spaces for all real representations of G, i.e. allowing non-trivial action in the suspension coordinate, and if one wishes suspension isomorphisms for all of these suspensions, then the theory should be indexed by the real representation ring of G, $RO(G)$, rather than just by integers or trivial representations. The suspension should be an isomorphism $H_G^\alpha(X;T) \cong \tilde{H}_G^{\alpha+V}(\Sigma^V X;T)$.

Recently, Waner [ ] has shown that a Bredon cohomology theory extends to an $RO(G)$ graded theory having all suspension isomorphisms if and only if the Bredon coefficient system extends to a Mackey functor. Specifically, for an $RO(G)$-graded theory $H_G^*(\ ;T)$ having all suspension isomorphisms and equivariant map $\varphi: G/H \longrightarrow G/K$, there is a representation V of G so that the map $\varphi \times 0: G/H \longrightarrow G/K \times V$ can be deformed to an embedding, and hence on a neighborhood of G/H, is given by $G/H \times D(V) \hookrightarrow G/K \times D(V)$. The Pontrjagin-Thom construction gives a collapsing map $\Sigma^V(G/K) \xrightarrow{\varphi'} \Sigma^V(G/H)$, inducing

$$\varphi_*: H_G^\alpha(G/H) \xrightarrow{\cong} \tilde{H}_G^{\alpha+V}(\Sigma^V(G/H)) \xrightarrow{(\varphi')^*} \tilde{H}_G^{\alpha+V}(\Sigma^V(G/K)) \xrightarrow{\cong} H_G^\alpha(G/K).$$

These covariant homomorphisms induced by φ are the Mackey functor "structure."

For the Bredon theory described earlier the homomorphism $\varphi_*: B(H)\otimes R \longrightarrow B(K)\otimes R$ assigning to a finite set F with H action the finite set $K\times_{g^{-1}Hg} F$ obtained by extending the obvious action of $g^{-1}Hg$ on F to K, defines a Mackey functor structure.

Thus, there is an $RO(G)$-graded cohomology theory $H^*_G(__; B\otimes R)$ with the given Bredon coefficient system and Mackey functor structure, which is called <u>ordinary equivariant cohomology with coefficients in the Burnside ring</u> (tensored with R).

From the description given, it is quite clear that one does not even know the cohomology groups $H^\alpha_G(G/H; B\otimes R)$ for the "points" G/H except in the special case when α is an integer. Hence, a goal of this paper is to determine the cohomology for points or more precisely the full coefficient system. To do so, heavy use will be made of

<u>Fact</u>. (J. Peter May) Ordinary equivariant cohomology with coefficients in the Burnside ring tensored with R is a multiplicative cohomology theory.

Thus, it is assumed that $H^*_G(X; B\otimes R)$ has the usual multiplicative properties, and in particular is an associative, commutative ring with unit. To describe the commutativity rule, for $a\in H^\alpha_G(X; B\otimes R)$, $b\in H^\beta_G(X; B\otimes R)$, one has

$$b\cup a = u\cdot(a\cup b)\in H^{\alpha+\beta}_G(X; B\otimes R)$$

where u is a unit in the Burnside ring described by

the fact that the number of points of u fixed by $H \subset G$ is $(-1)^r$ where $r = \dim(\alpha^H) \cdot \dim(\beta^H)$, with $\dim(\alpha^H)$ being the dimension of the fixed point subspace of H in the "virtual" representation α.

Note: For any $RO(G)$ graded theory with suspensions, $B(G)$ acts on the theory. For $\varphi: G/H \longrightarrow G/G = pt$, the composite $\Sigma^V pt \xrightarrow{\Sigma^V \varphi} \Sigma^V G/H \xrightarrow{\varphi'} \Sigma^V pt$ is a stable map of pt (gives a map $\Sigma^V X \longrightarrow \Sigma^V X$ for all X) and induces the action by G/H in $B(G)$, which is multiplication by some class in the 0-dimensional cohomology of pt. For the ordinary theory, this class is $\varphi_* \varphi^*(1)$ and $B(G) \otimes R \longrightarrow B(H) \otimes R \longrightarrow B(G) \otimes R$ sends F to $G \times_H (F|_H)$ which is isomorphic to $G/H \times F$ with diagonal G action. Thus the action of $u \in B(G)$ on $H^\alpha_G(X; B \otimes R)$ is simply multiplication by $u \in B(G) \otimes R = H^0_G(pt; B \otimes G)$.

§2. Equivariant Cohomology of G

Proposition. For any finite group G, the cohomology $H^*_G(G; B\otimes R)$ is the "polynomial" ring over R on classes $t_\rho \in H^{\rho-d}_G(G; B\otimes R)$ and $t_\rho^{-1} \in H^{d-\rho}_G(G; B\otimes R)$, with $t_\rho \cdot t_\rho^{-1} = 1$, where ρ is an irreducible non-trivial representation of G of dimension d. If $\varphi_g : G \to G$ is the equivariant map $\varphi_g(g') = g'g$, then $\varphi_g^*(t_\rho) = \pm t_\rho$ where the sign, ± 1, is the determinant of the action of g in the representation ρ.

Proof: For any G space X, the product $G\times X$ with diagonal G action is the same as $G\times X$ with trivial action on X. In particular, one has an isomorphism $\Sigma^V G \cong \Sigma^d G$, where $d = \dim V = |V|$ for any representation V.

Consider the composite

$$H^\alpha_G(G) \xrightarrow[\cong]{\text{susp}} \tilde H^{\alpha+V}_G(\Sigma^V G) = \tilde H^{\alpha+V}_G(\Sigma^d G) \xrightarrow[\cong]{(\text{susp})^{-1}} H^{\alpha+V-d}_G(G).$$

The suspension sends $x \in H^\alpha_G(G)$ to $x\cdot\theta_V$, where $\theta_V \in \tilde H^V_G(\Sigma^V G)$ is the suspension class, and $x\cdot\theta_V = y\cdot\theta_d$ for some y, and the composite sends x to y. This composite is a homomorphism of $H^*_G(G)$-modules and is given by multiplication by an invertible element $t_V \in H^{V-d}_G(G)$, $\theta_V = t_V\cdot\theta_d$, with multiplication by t_V^{-1} being the inverse of the composite.

In particular, $H^{V-W}_G(G) = H^{|V|-|W|}_G(G)$ is zero if $|V|\neq|W|$ and is R if $|V|=|W|$, with generator $t_V\cdot t_W^{-1}$.

To see that $H^*_G(G)$ is "polynomial", note first that the Burnside ring $B(G)$ acts in $H^*_G(G)$ so that the finite G-set F acts as multiplication by the number of elements in F. Then

$t_V \cdot t_W = u \cdot t_W \cdot t_V$ where u is a certain unit in $B(G)$. Assuming that $u = \Sigma u_H (G/H)$, multiplication by u is, in $H^*_G(G)$, the same as multiplication by $\Sigma u_H |G/H| =$ number of points in u fixed by the unit subgroup of G. By the characterization of u that number is $(-1)^r$ where $r = (\dim V^{\frac{1}{2}} |V|) \cdot (\dim W^{\frac{1}{2}} |W|) = 0$, so $t_V \cdot t_W = t_W \cdot t_V$. Thus $H^*_G(G)$ is a strictly commutative ring, and so is "polynomial" on the asserted generators, with only the relations $t_\rho t_\xi^{-1} = 1$.

To determine $(\varphi_g)^*$, one first suspends φ_g to

$$\Sigma^\ell \varphi_g : \Sigma^\ell(G) \longrightarrow \Sigma^\ell(G) \quad \text{for which} \quad (\Sigma^\ell \varphi_g)^*(x \theta_\rho) = \varphi_g^*(x) \cdot \theta_\rho$$

and, using the isomorphism $\Sigma^\ell(G) \cong \Sigma^d(G)$, $d = \dim \rho$, considers $\Sigma^\ell \varphi_g$ as a map of $\Sigma^d(G)$ into itself, where it is <u>not</u> the same as $\Sigma^d \varphi_g$. Specifically one identifies $G \times D(\rho)$ with $G \times D(d)$ by

$$f: G \times D(\rho) \longrightarrow G \times D(d) : (g, x) \longrightarrow (g, \rho(g^{-1})x)$$

so that $(g'g, \rho(g')x)$ goes to $(g'g, \rho(g'g)^{-1}\rho(g')x) = (g'g, \rho(g^{-1})x)$, and the map of interest is

$$G \times D(d) \xrightarrow{f^{-1}} G \times D(\rho) \xrightarrow{\varphi_g \times 1} G \times D(\rho) \xrightarrow{f} G \times D(d)$$

with

$$\begin{aligned} f \circ (\varphi_g \times 1) \circ f^{-1}(g', x) &= f \circ (\varphi_g \times 1)(g', \rho(g')x) \\ &= f(g'g, \rho(g')x) \\ &= (g'g, \rho(g^{-1})x), \end{aligned}$$

i.e. $\Sigma^\ell \varphi_g$ thought of as a map $\Sigma^d G \longrightarrow \Sigma^d G$ is given by φ_g and the linear map $\rho(g^{-1})$ in the suspension coordinate, which has degree equal to the determinant of $\rho(g^{-1})$ or $\rho(g)$; i.e. is $\det \rho(g) \cdot \Sigma^d \varphi_g$.

Thus $(\Sigma^\rho \varphi_g)^*(\theta_d) = \det\rho(g)\theta_d$. From $\theta_\rho = t_\rho \theta_d$, one must have $\varphi_g^*(t_\rho) = \det\rho(g) t_\rho$. □

? . Note: Waner [ ] observes the fact that $H_G^\alpha(G/H) \cong H_H^{\alpha|H}(\text{point})$ which gives, in this case $H_G^\alpha(G) = H_1^{|\alpha|}(\text{point}) = H^{|\alpha|}(\text{pt}) = 0$ or R as $|\alpha| \neq 0$ or $|\alpha| = 0$. All that was involved was to make this isomorphism more explicit.

§3. The Case $G = \mathbb{Z}_2$

Throughout this section, G will be the cyclic group of order 2, $\mathbb{Z}_2$, and ρ will denote the unique nontrivial irreducible representation. The Burnside ring is $\mathbb{Z} \oplus \mathbb{Z}\sigma$ with generators 1, the class of a point, and σ, the class of $\mathbb{Z}_2$, and $\sigma^2 = 2\sigma$. The representation ring $RO(\mathbb{Z}_2)$ is $\mathbb{Z} \oplus \mathbb{Z}\rho$, with 1 being the trivial 1-dimensional representation.

From the previous section, one has $H^*_{\mathbb{Z}_2}(\mathbb{Z}_2) \cong R[t, t^{-1}]$ where $t = t_\rho \in H^{\rho-1}_{\mathbb{Z}_2}(\mathbb{Z}_2)$.

One has a cofibration of sphere and disc in ρ

$$\begin{array}{ccccc} S(\rho) & \xrightarrow{i} & D(\rho) & \xrightarrow{j} & D(\rho)/S(\rho) \\ \| & & \| & & \| \\ \mathbb{Z}_2 & & [-1,1] & & \Sigma^\rho(pt) \\ & & \simeq & & \\ & & pt & & \end{array}$$

giving an exact sequence in cohomology

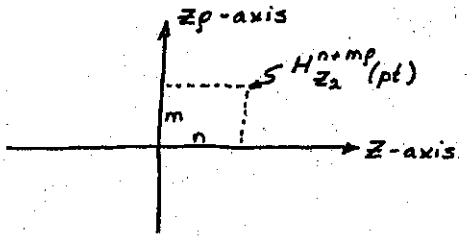
$$H^*_{\mathbb{Z}_2}(\mathbb{Z}_2) \xleftarrow{i^*} H^*_{\mathbb{Z}_2}(pt) \xleftarrow{j^*} \tilde{H}^*_{\mathbb{Z}_2}(\Sigma^\rho pt), \qquad \delta : \tilde{H}^*_{\mathbb{Z}_2}(\Sigma^\rho pt) \to H^*_{\mathbb{Z}_2}(\mathbb{Z}_2)$$

or the corresponding Gysin sequence

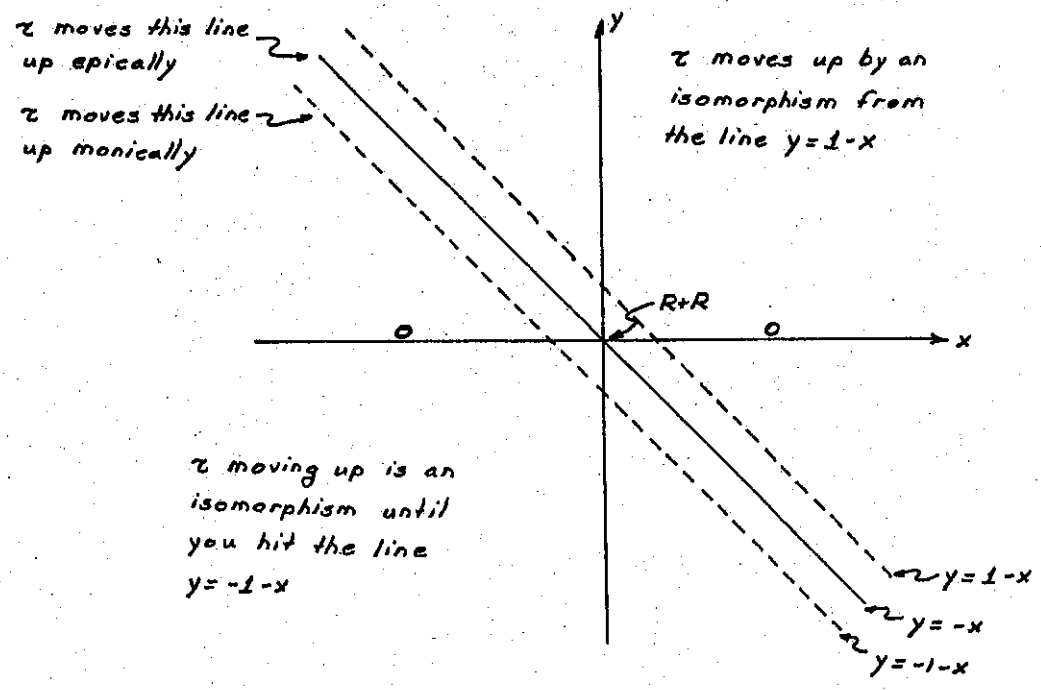
$$H^*_{\mathbb{Z}_2}(\mathbb{Z}_2) \xleftarrow{\varphi} H^*_{\mathbb{Z}_2}(pt) \xleftarrow{\cup \tau} H^*_{\mathbb{Z}_2}(pt), \qquad \psi : H^*_{\mathbb{Z}_2}(pt) \to H^*_{\mathbb{Z}_2}(\mathbb{Z}_2)$$

in which $\cup\tau$ is right multiplication by the class $\tau = j^* \theta_\rho \in H^\rho_{\mathbb{Z}_2}(pt)$, and has degree ρ, φ has degree 0, and ψ has degree $1-\rho$. The goal is to use this sequence to compute $H^*_{\mathbb{Z}_2}(pt)$ inductively.

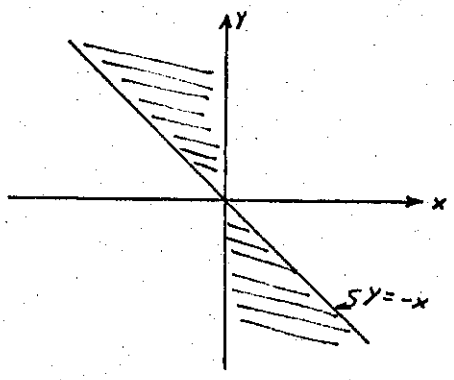
It is convenient to draw diagrams in order to keep track of the groups $H^\alpha_{\mathbb{Z}_2}(pt)$. In the plane, the group $H^\alpha_{\mathbb{Z}_2}(pt)$, $\alpha = n + m\rho$ will be located at the point (n, m).



In particular, $H^*_{\mathbb{Z}_2}(\mathbb{Z}_2)$ is known to be zero except along the line $y=-x$. Since φ has degree zero, $\cup\tau$ is epic except along the line $y=-x$, i.e. maps onto the groups not on that line. Further, ψ has degree $1-p$ and so $\cup\tau$ is monic except along the line $y=-x$. Since one has $H^i_{\mathbb{Z}_2}(pt)=0$ for $i\in\mathbb{Z}$, $i\neq 0$, one has



and so the only nonzero groups lie in the shaded regions



and one need only calculate the groups along the lines $y=-x$, $y=1-x$, and $y=-1-x$. Thus, for $k\in\mathbb{Z}$, one needs to understand the sequences

$$0 \to H^{k(p-1)-p}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{k(p-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} H^{k(p-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\psi} H^{(k-1)(p-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{(k-1)(p-1)+p}_{\mathbb{Z}_2}(pt) \to 0$$

First, one has the case $k=0$

$$0 \longrightarrow H^{-p}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^0_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} H^0_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\psi} H^{1-p}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^1_{\mathbb{Z}_2}(pt) \longrightarrow 0$$

$$\qquad\qquad\quad \| \qquad\qquad\quad \| \qquad\qquad\qquad\qquad\qquad\quad \|$$

$$\qquad\qquad R+R \xrightarrow{\varphi^*} R \qquad\qquad\qquad\qquad\qquad\quad 0$$

where $\varphi = \varphi^* : B(\mathbb{Z}_2)\otimes R \longrightarrow B(1)\otimes R = R$ is the homomorphism induced by $\varphi: \mathbb{Z}_2 \longrightarrow pt$ with $\varphi^*(1)=1$, $\varphi^*(\sigma)=2$. Thus $H^{1-p}_{\mathbb{Z}_2}(pt)=0$, $H^{-p}_{\mathbb{Z}_2}(pt)\cong R$ on a generator g_{-1} such that $g_{-1}\tau = 2-\sigma$. Further $(\sigma g_{-1})\tau = \sigma(g_{-1}\tau) = \sigma(2-\sigma) = 2\sigma-\sigma^2 = 0$ so $\sigma g_{-1}=0$.

For $k=1$, one has

$$0 \longrightarrow H^{-1}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{p-1}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} H^{p-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\psi} H^0_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^p_{\mathbb{Z}_2}(pt) \longrightarrow 0$$

$$\qquad\quad \| \qquad\qquad\qquad\quad \| \qquad\qquad\quad \|$$

$$\qquad\quad 0 \qquad\qquad\qquad\quad R \qquad\qquad\quad R+R$$

and in order to understand this sequence one needs to know the homomorphism ψ, which began life as the coboundary $\delta: H^{p-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) \longrightarrow \tilde{H}^p_{\mathbb{Z}_2}(\Sigma^p pt)$. The coboundary for a pair (X,A) is induced by the map $X/A = X \cup \operatorname{cone} A \longrightarrow (X \cup \operatorname{cone} A)\cup \operatorname{cone} X$, the latter being homotopy equivalent to the suspension of A, i.e. δ is realized as $\tilde{H}^p_{\mathbb{Z}_2}(\Sigma^1\mathbb{Z}_2) \longrightarrow \tilde{H}^p_{\mathbb{Z}_2}(\Sigma^p pt)$ from a map $\Sigma^p pt \xrightarrow{f} \Sigma^1\mathbb{Z}_2$. If you consider $\Sigma^p pt$ as the circle S^1 with conjugation, with base point 1, pt being -1, the map simply collapses $\{\pm 1\}$ to the base point. This is exactly the map you obtain by collapsing the complement of a tubular neighborhood of $\mathbb{Z}_2 = \{\pm i\}$ to the base point, i.e. identifying $\Sigma^1\mathbb{Z}_2 \cong \Sigma^p\mathbb{Z}_2$ gives f as being the map $\varphi': \Sigma^p pt \longrightarrow \Sigma^p\mathbb{Z}_2$. Thus the composite

$$H^0_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow[\cong]{\cup t} H^{p-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\psi} H^0_{\mathbb{Z}_2}(pt)$$

is precisely $\varphi_a : R \longrightarrow R+R : \varphi_a(1) = \sigma$. Thus $H^{p-1}_{\mathbb{Z}_2}(pt)=0$ and $H^p_{\mathbb{Z}_2}(pt)\cong R$ generated by τ with $\sigma\tau = 0$.

For $k=2$, one has

$$0 \longrightarrow \underset{0}{H^{\rho-2}_{\mathbb{Z}_2}(pt)} \longrightarrow H^{2(\rho-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} \underset{R}{H^{2(\rho-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2)} \xrightarrow{\psi} \underset{0}{H^{\rho-1}_{\mathbb{Z}_2}(pt)} \xrightarrow{\cup\tau} H^{2\rho-1}_{\mathbb{Z}_2}(pt) \longrightarrow 0$$

from which $H^{2\rho-1}_{\mathbb{Z}_2}(pt) = 0$ and $H^{2(\rho-1)}_{\mathbb{Z}_2}(pt) \cong R$ on a generator h with $\varphi(h) = t^2$. Further $\varphi(\sigma h) = \varphi(\sigma)\varphi(h) = 2\varphi(h) = \varphi(2h)$ so $\sigma h = 2h$.

For $k = 2j > 2$, one now has

$$0 \longrightarrow \underset{0}{H^{(2j-1)\rho-2j}_{\mathbb{Z}_2}(pt)} \xrightarrow{\cup\tau} H^{2j(\rho-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} \underset{R}{H^{2j(\rho-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2)} \xrightarrow{\psi} H^{(2j-1)(\rho-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{2j\rho-(2j-1)}_{\mathbb{Z}_2}(pt) \longrightarrow 0$$

and $\varphi(h^j) = \varphi(h)^j = t^{2j}$ generates $H^{2j(\rho-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2)$, so $H^{2j(\rho-1)}_{\mathbb{Z}_2}(pt) \cong R$ with generator h^j, and multiplication by τ sends $H^{(2j-1)(\rho-1)}_{\mathbb{Z}_2}(pt) \longrightarrow H^{2j\rho-(2j-1)}_{\mathbb{Z}_2}(pt)$ isomorphically.

For $k = 2j+1 > 1$, one has

$$0 \longrightarrow \underset{0}{H^{2j\rho-(2j+1)}_{\mathbb{Z}_2}(pt)} \xrightarrow{\cup\tau} H^{(2j+1)(\rho-1)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} \underset{R}{H^{(2j+1)(\rho-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2)} \xrightarrow{\psi} \underset{R}{H^{2j(\rho-1)}_{\mathbb{Z}_2}(pt)} \xrightarrow{\cup\tau} H^{(2j+1)\rho-2j}_{\mathbb{Z}_2}(pt) \longrightarrow 0$$

and since ψ is a homomorphism of left $H^{*}_{\mathbb{Z}_2}(pt)$ modules the diagram

$$\begin{array}{ccc} R \cong H^{(2j+1)(\rho-1)}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{2j(\rho-1)}_{\mathbb{Z}_2}(pt) \cong R \\ \uparrow \varphi(h^j)\cup & & \uparrow h^j\cup \\ R \cong H^{\rho-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{0}_{\mathbb{Z}_2}(pt) \cong R+R \end{array}$$

commutes, giving $\psi(t^{2j+1}) = \psi(t^{2j}t) = \psi(\varphi(h^j)t) = h^j\psi(t) = \sigma h^j$, but $\sigma h = 2h$, so $\psi(t^{2j+1}) = 2h^j$. Thus

$$H^{(2j+1)(\rho-1)}_{\mathbb{Z}_2}(pt) \cong K = \text{kernel}\{\times 2 : R \longrightarrow R\}$$

and

$$H^{(2j+1)\rho-2j}_{\mathbb{Z}_2}(pt) \cong R/2R.$$

It is also immediate that multiplication by h gives an isomorphism

$$h\cup : H^{(2j+1)(\rho-1)}_{\mathbb{Z}_2}(pt) \cong K \longrightarrow H^{(2j+3)(\rho-1)}_{\mathbb{Z}_2}(pt) \cong K.$$

For $k = -(2j-1) \leq -1$, i.e. $j \geq 1$, one has

$$0 \to H^{2j-1-2j\rho}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{(2j-1)(1-\rho)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} H^{(2j-1)(1-\rho)}_{\mathbb{Z}_2}(\mathbb{Z}_2) \xrightarrow{\psi} H^{2j(1-\rho)}_{\mathbb{Z}_2}(pt) \xrightarrow{\cup\tau} H^{2j-(2j-1)\rho}_{\mathbb{Z}_2}(pt) \to 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \qquad\qquad\qquad\qquad\qquad\qquad\quad \|$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad R \qquad\qquad\qquad\qquad\qquad\qquad\quad 0$$

and the diagram

$$\begin{array}{ccccc} R \cong & H^{\rho-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{0}_{\mathbb{Z}_2}(pt) \cong R+R & \\ & \uparrow \varphi(h^j)\cup & & \uparrow h^j\cup & \\ R \cong & H^{(2j-1)(1-\rho)}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{2j(1-\rho)}_{\mathbb{Z}_2}(pt) & \to 0 \end{array}$$

commutes with $h^j\psi(t^{-(2j-1)}) = \psi(t) = \sigma$, and since ψ maps onto $H^{2j(1-\rho)}_{\mathbb{Z}_2}(pt)$, $H^{2j(1-\rho)}_{\mathbb{Z}_2}(pt) \cong R$ on the generator $u_{-2j} = \psi(t^{-(2j-1)})$ with ψ being an isomorphism. Further, $\cup\tau : H^{2j-1-2j\rho}_{\mathbb{Z}_2}(pt) \longrightarrow H^{(2j-1)(1-\rho)}_{\mathbb{Z}_2}(pt)$ is an isomorphism, and taking the special case $j=1$, $H^{1-\rho}_{\mathbb{Z}_2}(pt) = 0$ so $H^{1-2\rho}_{\mathbb{Z}_2}(pt) = 0$.

One also has the larger commutative diagram

$$\begin{array}{ccccc} R & & R+R & & R \\ \| & & \| & & \| \\ H^{\rho-1}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{0}_{\mathbb{Z}_2}(pt) & \xrightarrow{\varphi} & H^{0}_{\mathbb{Z}_2}(\mathbb{Z}_2) \\ \uparrow \varphi(h^j)\cup & & \uparrow h^j\cup & & \uparrow \varphi(h^j)\cup \\ H^{(2j-1)(1-\rho)}_{\mathbb{Z}_2}(\mathbb{Z}_2) & \xrightarrow{\psi} & H^{2j(1-\rho)}_{\mathbb{Z}_2}(pt) & \xrightarrow{\varphi} & H^{2j(1-\rho)}_{\mathbb{Z}_2}(\mathbb{Z}_2) \\ \| & & \| & & \| \\ R & & R & & R \end{array}$$

and $(\varphi(h^j)\cup)\circ\varphi\circ\psi(t^{-(2j-1)}) = \varphi\circ\psi(t) = \varphi(\sigma) = 2$ so

$\varphi(u_{-2j}) = \varphi(\psi(t^{-(2j-1)})) = 2t^{-2j}$.

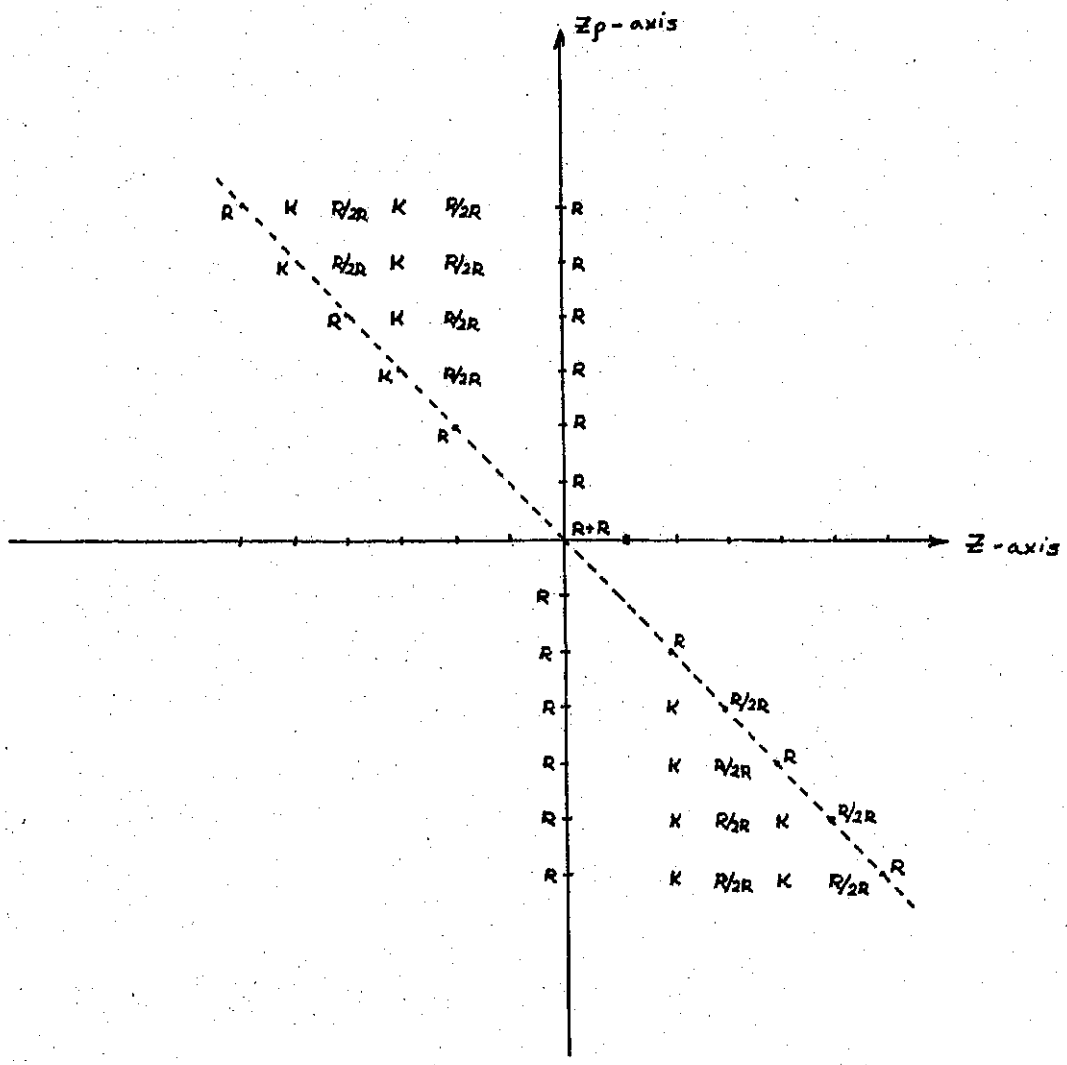
For $k = -2j \leq -2$; i.e. $j \geq 1$, one has

$$0 \to H^{2j-(2j+1)p}_{\mathbb{Z}_2}(pt) \to H^{2j(1-p)}_{\mathbb{Z}_2}(pt) \xrightarrow{\varphi} H^{2j(1-p)}_{\mathbb{Z}_2}(Z_2) \xrightarrow{\psi} H^{(2j+1)(1-p)}_{\mathbb{Z}_2}(pt) \to H^{2j+1-2jp}_{\mathbb{Z}_2}(pt) \to 0$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ R & \xrightarrow{\times 2} & R & & 0 \end{array}$$

from which $H^{2j-(2j+1)p}_{\mathbb{Z}_2}(pt) \cong K = \ker\{\times 2 : R \to R\}$ and $H^{(2j+1)(1-p)}_{\mathbb{Z}_2}(pt) \cong R/2R$.

Thus, the groups $H^{\alpha}_{\mathbb{Z}_2}(pt)$ are completely determined and the diagram of groups is



with the obvious repetitions of the pattern running off to infinity.