ON THE $RO(G)$-GRADED EQUIVARIANT ORDINARY COHOMOLOGY OF GENERALIZED G-CELL COMPLEXES FOR $G = \mathbb{Z}/p$

by

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Abstract

It is well known that the cohomology of a finite CW-complex with cells only in even dimensions is free. The equivariant analog of this result for generalized G-cell complexes is, however, not obvious, since $RO(G)$-graded cohomology cannot be computed using cellular chains. We consider $G = \mathbb{Z}/p$ and study G-spaces that can be built as cell complexes using the unit disks of finite dimensional G-representations as cells. Our main result is that, if X is a G-complex containing only even dimensional representation cells and satisfying certain finite type assumptions, then the $RO(G)$-graded equivariant ordinary cohomology is free as a graded module over the cohomology of a point. This extends a result due to Gaunce Lewis about equivariant complex projective spaces with linear $\mathbb{Z}/p$ actions. Our new result applies more generally to equivariant complex Grassmannians with linear $\mathbb{Z}/p$ actions.

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Chapter 1

Introduction

Our main result (Theorem 3.4) is a generalization of a theorem due to Gaunce Lewis (Theorem 2.6 of [9]). The following rough statement of our main result highlights the difference between our theorem and that of Lewis.

Theorem. *Let $G = \mathbb{Z}/p$, for an odd prime p, and let X be a generalized G-cell complex containing only even dimensional representation cells. Suppose further that, for each m, X has only finitely many cells of the form DV with $|V^G| = m$. Then, the equivariant ordinary $RO(G)$-graded cohomology $\mathbf{H}^*_G(X)$ is free as a module over the cohomology of a point $\mathbf{H}^*_G(*)$. Moreover, the dimensions of the generators are even, and there is a one-to-one correspondence between the generators and the represenation cells used to build X.*

In Lewis' result, the cells of X must be attached in a certain order, and the generators for $\mathbf{H}^*_G(X)$ are then in the same dimensions as the cells of X. Our result removes the restriction on the ordering of the cells of X, but at the cost of being unable to specify the exact dimensions (in the $RO(G)$-graded sense) of the generators for $\mathbf{H}^*_G(X)$. A sense of the relative strength of the two theorems can be obtained by

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considering what they tell us about complex Grassmannians with linear $\mathbb{Z}/p$ actions. Lewis' theorem applies to complex projective spaces with linear $\mathbb{Z}/p$ actions and describes their cohomology completely. However, it does not apply to general complex Grassmannians. Our result applies to arbitrary complex Grassmannians with linear $\mathbb{Z}/p$ actions and gives that the cohomology of each of these spaces is free as a module over the cohomology of a point. However, even for complex projective spaces, our result does not specify the dimensions of the cohomology generators.

The ultimate source of the difference between the two theorems is that Lewis' restriction on the ordering of the cells guarantees trivial boundary maps in the long exact sequences associated with the cell attachments. Our theorem, with no such restrictions on the ordering of the cells, applies even when nontrivial boundary maps occur in these long exact sequences. The effect of such a nontrivial boundary map is not to destroy the freeness of the cohomology, but rather to shift the dimensions of some of the generators away from the dimensions of the associated cells of X. This dimension-shifting complicates the passage to limits required in the computation of the cohomology of an infinite cell complex, and therefore complicates the proof of our theorem.

The fact that Lewis' restrictions on the ordering of the cells guarantee trivial boundary maps depends on special properties of $\mathbf{H}^*_G(*)$ that hold only for $G = \mathbb{Z}/p$. Consequently, his result cannot be generalized to other groups. Since our theorem allows nonzero boundary maps, it is possible that it can be generalized to groups other than $\mathbb{Z}/p$. However, this possibility is not explored here.

Throughout this thesis, G denotes the group $\mathbb{Z}/p$ for some odd prime p. Chapters 3, 4, and 5 contain the major results. The four appendices contain various technical lemmas used in their proofs. The majority of the results in the appendices are ones whose analogs for non-equivariant cohomology are well-known.

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In Chapter 2, the basic algebraic background required for understanding Mackey functor-valued equivariant ordinary $RO(G)$-graded cohomology is given. Chapter 3 contains our main theorem along with some of its important relatives. The proof of this main theorem is broken into several steps; the technical details needed to carry out those steps are supplied in the various sections and subsections of this chapter. One major shortcoming of our result is that it does not specify the exact dimensions of the generators of the spaces to which it applies. This difficulty is somewhat mitigated by the fact that there is a certain amount of flexibility in the dimensions (in the $RO(G)$-graded sense) of these cohomology generators. This flexibility is explored in Chapter 4, in which we describe conditions under which two free modules over the cohomology of a point are isomorphic even though their generators are in different dimensions. The application of our main theorem to equivariant complex Grassmannians is presented in Chapter 5.

Appendix A contains some basic results about the category of Mackey functors, including a catalog of the solutions to certain commonly occuring extension problems and a description of the kernels of some standard maps. In Appendix B, the $RO(G)$-graded equivariant ordinary cohomology of a point is computed for arbitrary Mackey functor coefficients, and some consequences of these computations are given. Certain standard maps between modules over the cohomology of a point are used throughout the thesis. The properties of these maps are discussed in detail in Appendix C. The dimension-shifting caused by nonzero boundary maps in the long exact sequences associated with the cell attachments creates special problems in the passage to limits required to compute the cohomology of an infinite cell complex. A technical result needed to resolve these limit problems is contained in Appendix D.

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Chapter 2

Preliminaries

2.1 The Category of Mackey Functors

Mackey functors for a finite group were first introduced by Green [5]; a more abstract approach was also given shortly thereafter by Dress [3]. Subsequently, several other approaches have been given, a survey of which can be found in [16]. Here we mainly use the original approach of Green. This approach is usually refered to as the elementary approach and is particularly convenient for the group $G = \mathbb{Z}/p$.

In section 1 of [9], a tutorial is given on Mackey functors for $\mathbb{Z}/p$. Here, $Mack$ is used to denote the category of all such Mackey functors. Otherwise we adopt the notation used in [9].

A Mackey functor M for $G = \mathbb{Z}/p$ is determined by two $\mathbb{Z}[G]$-modules, $M(G/G)$ and $M(G/e)$, where $M(G/G)$ has trivial G-action, and by two G-maps

$$\rho : M(G/G) \longrightarrow M(G/e) \text{ and } \tau : M(G/e) \longrightarrow M(G/G),$$

whose composite $\rho \circ \tau$ is required to be the trace of the G-action on $M(G/e)$, namely $x \mapsto \sum_{g \in G} gx$. The maps ρ and τ are called the restriction and transfer, respectively.

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As in [9], M is displayed with a diagram

$$\begin{array}{c} M \\ \\ M(G/G) \\ {\scriptstyle\rho}\big\downarrow\big\uparrow{\scriptstyle\tau} \\ M(G/e) \\ \circlearrowleft \\ {\scriptstyle\theta} \end{array}$$

where θ denotes the G-action. When θ is trivial, it is deleted from the diagram. A map f between two Mackey functors M and N consists of two G-maps,

$$f_G : M(G/G) \longrightarrow N(G/G) \text{ and } f_e : M(G/e) \longrightarrow N(G/e),$$

which commute with the restriction and transfer maps in the obvious way. The subscripts are sometimes deleted from the notation when the level of the map is clear in context.

The category $Mack$ is an abelian category. Kernels, cokernels, and so forth are defined levelwise. For easy reference, we recall from [9] the particular Mackey functors that are of interest to us. In the following diagrams, C denotes an abelian group, and d denotes an integer prime to p.

$A[d]$	L	R	$\langle C \rangle$	A_G
$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	C	$\mathbb{Z}$
$(d,p)\big\downarrow\big\uparrow(0,1)$	$p\big\downarrow\big\uparrow 1$	$1\big\downarrow\big\uparrow p$	$0\big\downarrow\big\uparrow 0$	$\Delta\big\downarrow\big\uparrow\nabla$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}^p$
				$\circlearrowleft$
				π

For each of $A[d]$, L, and R, ι denotes the generator at G/e. The twisted Burnside ring Mackey functor $A[d]$ is displayed using the standard $\{\mu, \tau\}$ basis at G/G where

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$\rho(\mu) = d\iota$, $\rho(\tau) = p\iota$, and $\tau(\iota) = \tau$. In fact, the last equality is the reason for naming the second generator τ. Despite this double use of τ, its meaning should be clear in context, and we repeat this convention for L. There, τ denotes the generator at G/G such that $\tau(\iota) = \tau$ and $\rho(\tau) = p\iota$. For R, the standard generator at G/G is denoted by ξ and satisfies $\rho(\xi) = \iota$ and $\tau(\iota) = p\xi$. The displayed Mackey functor $\langle C \rangle$ shows an obvious way in which Ab can be embedded in $Mack$. For A_G, restriction is the diagonal map, transfer is the folding map, and the G-action π is the permutation action that cyclically permutes the coordinates of $\mathbb{Z}^p$.

The Mackey functors of the form $A[d]$ are projective and play an especially important role in our work. Given integers a and b such that $ad + bp = 1$, the elements $\sigma = a\mu + b\tau$ and $\kappa = p\mu - d\tau$ form an alternative basis for $A[d](G/G)$. This $\{\sigma, \kappa\}$ basis is often useful because σ restricts to the generator at G/e and κ generates the kernel of ρ. The equations $\mu = d\sigma + b\kappa$ and $\tau = p\sigma - a\kappa$ can be used to convert back to the standard basis. In Examples 1.1(b) of [9], Lewis shows that the congruence $d_1 \equiv \pm d_2$ modulo p implies $A[d_1] \cong A[d_2]$. In fact, the converse is also true.

Lemma 2.1. *Let d_1 and d_2 be integers prime to p. Then,*
$A[d_1] \cong A[d_2]$ if and only if $d_1 \equiv \pm d_2$ modulo p .

Proof. By our comments prior to the lemma, it suffices to prove that $A[d_1] \cong A[d_2]$ implies $d_1 \equiv \pm d_2$ modulo p. Suppose that $f : A[d_1] \longrightarrow A[d_2]$ is an isomorphism. Let $\{\mu_1, \tau_1, \iota_1\}$ and $\{\mu_2, \tau_2, \iota_2\}$ be standard bases for $A[d_1]$ and $A[d_2]$, respectively. We may assume that $f_e(\iota_1) = \iota_2$, and hence $f_G(\tau_1) = \tau_2$. Write $f_G(\mu_1) = x\mu_2 + y\tau_2$. The map f_G is described by the 2×2 matrix $\left[\begin{smallmatrix} x & 0 \\ y & 1 \end{smallmatrix}\right]$. Since f_G is an isomorphism, it follows that $x = \pm 1$. The equation

$$d_1\iota_1 = f_e(\rho(\mu_1)) = \rho(f_g(\mu_1)) = (d_2x + py)\iota_1$$

therefore gives the desired congruence. □

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The Burnside ring Mackey functor A is just $A[1]$. This Mackey functor plays a role in *Mack* similar to the role played by $\mathbb{Z}$ in *Ab*.

The Mackey functor A_G is a particular instance of a construction due to Dress [3]. In general, any Mackey functor M determines a Mackey functor M_G, and the restriction ρ and transfer τ for M determine maps $\bar{\rho} : M_G \longrightarrow M$ and $\bar{\tau} : M \longrightarrow M_G$. The following diagram displays the values of M_G, $\bar{\rho}$, and $\bar{\tau}$, for the case in which $G = \mathbb{Z}/p$.

$$M_G \xrightarrow{\bar{\tau}} M \xrightarrow{\bar{\rho}} M_G$$

$$\begin{array}{ccccc} M(G/e) & \xrightarrow{\tau} & M(G/G) & \xrightarrow{\rho} & M(G/e) \\ {\scriptstyle\Delta}\downarrow\uparrow{\scriptstyle\nabla} & & {\scriptstyle\rho}\downarrow\uparrow{\scriptstyle\tau} & & {\scriptstyle\Delta}\downarrow\uparrow{\scriptstyle\nabla} \\ M(G/e)^p & \xrightarrow{\tilde{\nabla}} & M(G/e) & \xrightarrow{\tilde{\Delta}} & M(G/e)^p \\ \circlearrowright_{\pi} & & \circlearrowright_{\theta} & & \circlearrowright_{\pi} \end{array}$$

By fixing a generator a in G, the permutation action π on $M(G/e)^p$ can be described by $a^m \cdot (x_0, x_1, \ldots, x_{p-1}) = (x_{-m}, x_{1-m}, \ldots, x_{p-1-m})$, where subscripts are read modulo p. The displayed maps $\tilde{\nabla}$ and $\tilde{\Delta}$ are then given by $\tilde{\nabla}((x_0, x_1, \ldots, x_{p-1})) = \Sigma_{i=0}^{p-1} a^i x_i$ and $\tilde{\Delta}(x) = (x, a^{-1}x, \ldots, a^{-(p-1)}x)$. The given action θ on $M(G/e)$ also provides, for each g in G, a map $\bar{g} : M_G \longrightarrow M_G$. The map

$$\bar{g}_G : M_G(G/G) = M(G/e) \longrightarrow M(G/e) = M_G(G/G)$$

is given by multiplication by g, and the map

$$\bar{g}_e : M_G(G/e) = M(G/e)^p \longrightarrow M(G/e)^p = M_G(G/e)$$

is just $\bar{g}_G$ on each summand. Observe that $\bar{\rho} \circ \bar{\tau} = \Sigma_{g \in G} \bar{g}$.

As described in [8], the category *Mack* comes with a symmetric monoidal product $\square$, called the box product. This product plays the same role in *Mack* as the tensor

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product plays in Ab. In particular, for any fixed Mackey functor M, the functor $?\square M$ is right exact. When $M = A$, this functor is naturally equivalent to the identity.

A formula for the box product of two $\mathbb{Z}/p$-Mackey functors is given in [9]. Given two Mackey functors M and N, their box product $M \square N$ is described by the diagram

$$M \square N \tag{2.1}$$

$$[(M(G/G) \otimes N(G/G)) \oplus (M(G/e) \otimes N(G/e))]/\approx$$
$$(\rho_M \otimes \rho_N, p) \downarrow \uparrow i_2$$
$$M(G/e) \otimes N(G/e)$$
$$\circlearrowright$$
$$\theta_M \otimes \theta_N$$

The equivalence relation $\approx$ is determined by the Frobenius relations

$$m_G \otimes \tau n_e \approx \rho m_G \otimes n_e \text{ and } \tau m_e \otimes n_G \approx m_e \otimes \rho n_G,$$

where $m_H \in M(G/H)$ and $n_H \in N(G/H)$ for $H = e, G$. Note that the displayed restriction and transfer maps are well defined with respect to the quotient modulo $\approx$.

By using the description of $M \square N$ given in diagram 2.1, Proposition 1.3 of [9] characterizes a map $f : M \square N \longrightarrow P$ out of a box product. The map f determines and is determined by a pair of G-maps $F_G : M(G/G) \otimes N(G/G) \longrightarrow P(G/G)$ and $F_e : M(G/e) \otimes N(G/e) \longrightarrow P(G/e)$ that commute with restriction in the obvious way and respect the Frobenius relations. This characterization is useful for understanding multiplicative structures given by maps out of box products.

The box product can also be used to define the notion of a ring in $Mack$. Specifically, a Mackey functor ring is a Mackey functor S, together with structure maps $\eta : A \longrightarrow S$ and $\phi : S \square S \longrightarrow S$ which specify the unit and multiplication respectively, and which satisfy the usual coherence diagrams. Equivalently, a ring is a Mackey functor S such that ρ is a ring map between the rings $S(G/G)$ and $S(G/e)$, and τ

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is an $S(G/G)$-module map via ρ. From this, A and R are easily seen to be Mackey functor rings.

In general, box products are difficult to compute, but only a few simple cases are needed in our work. For easy reference, the particular box products of interest to us are recorded in Table 2.1. All of the results displayed there can easily be extracted from Examples 1.2 of [9].

$\square$	$A[d_2]$	L	R	$\langle D \rangle$
$A[d_1]$	$A[d_1 d_2]$	L	R	$\langle D \rangle$
L	L	L	L	0
R	R	L	R	$\langle D/pD \rangle$
$\langle C \rangle$	$\langle C \rangle$	0	$\langle C/pC \rangle$	$\langle C \otimes D \rangle$

Table 2.1: Box Products

Another nice property of the Mackey functors L, R, and $\langle C \rangle$ is that maps between them are easy to understand. For the most part this is a consequence of the following isomorphisms.

Proposition 2.2. *For any Mackey functor M,*

(a) $Mack(L, M) \cong Hom_G(L(G/e), M(G/e))$ *and*

(b) $Mack(M, R) \cong Hom_G(M(G/e), R(G/e))$.

Since $L(G/e)$ has trivial G-action, for any Mackey functor M,

$$Hom_G(L(G/e), M(G/e)) \cong Ab(L(G/e), M(G/e)^G).$$

Moreover, for a Mackey functor M with trivial G-action, Proposition 2.2 can be

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written as

$$Mack(L, M) \cong Ab(L(G/e), M(G/e)).$$

Of course, analogous comments apply to the isomorphism involving R.

Proposition 2.2 tells us that maps out of L or into R are completely determined by their values at G/e. In particular, $Mack(L, \langle C \rangle) = Mack(\langle C \rangle, R) = 0$. Many other useful consequences of Proposition 2.2 are described in Appendix A.

2.2 Generalized G-cell Complexes

Given a finite dimensional real G-representation V, let DV and SV denote the unit disk and unit sphere of V, respectively. The one-point compactification of V is denoted by S^V; the basepoint of S^V is taken to be the point at infinity. A *generalized G-cell complex* is a G-space X together with a sequence of G-subspaces X_n for $n \geq 0$ such that X_0 is a disjoint union of orbits, X_{n+1} is obtained by attaching cells to X_n via G-maps, and $X = \bigcup_{n \geq 0} X_n$ has the colimit topology. Moreover, the cells of X are required to be one of the following two types: a *representation cell* of the form DV for a finite dimensional G-representation V, or an *orbit cell* of the form $G \times D^m$ for some positive integer m. For a representation cell DV, the attaching map is a G-map out of SV. For an orbit cell $G \times D^m$, the attaching map is a G-map out of $G \times S^m$. In contrast, ordinary G-cell complexes are built using cells of the form $G/H \times D^m$ for H a subgroup of G. It can be shown that generalized G-cell complexes have the G-homotopy type of ordinary G-cell complexes.

Our main interest is in spaces that can be built with only even dimensional representation cells. In particular, equivariant complex Grassmannians are considered in Chapter 5.

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2.3 Cohomology and $RO(G)$-graded Modules

The cohomology that we study has both coefficients and values in $Mack$. Specifically, we compute $RO(G)$-graded equivariant ordinary cohomology with coefficients in some Mackey functor M. This cohomology theory is described in [12, 13, 14].

$RO(G)$ denotes the Grothendieck ring of isometric isomorphism classes of finite dimensional real G-representations. Similarly, $RSO(G)$ denotes the Grothendieck group of SO-isomorphism classes of finite dimensional real G-representations. Of course, $RO(G)$ is a quotient of $RSO(G)$ via the map $\pi : RSO(G) \longrightarrow RO(G)$ which identifies each irreducible representation with its complex conjugate. Since $G = \mathbb{Z}/p$, it is easy to see that $RSO(G)$ is additively generated by 1, the trivial real representation, and $\lambda_1, \ldots, \lambda_{p-1}$, the (complex) representations defined by $\lambda_k(m) = e^{2\pi km i/p}$, for m in $\mathbb{Z}/p$. Of course, for each k, λ_k and λ_{p-k} are equivalent in $RO(G)$ via complex conjugation. Ordinarily we work with $RO(G)$ and only refer to $RSO(G)$ at points where it is necessary to pay close attention to orientations.

For a virtual representation $\alpha = V - W$ in $RO(G)$, the fixed dimension $|\alpha^G| = |V^G| - |W^G|$ and the total dimension $|\alpha| = |V| - |W|$ are either both even or both odd. Consequently, every α in $RO(G)$ may be described as being either *even* or *odd*. Also observe that, if V is an actual representation, then $|V| \geq |V^G| \geq 0$. Consequently, a virtual representation α is said to be *representation-like* if $|\alpha| \geq |\alpha^G| \geq 0$. In our work, it is most often the case that only the dimensions $(|\alpha^G|, |\alpha|)$ of a virtual representation α matter. One exception is when $|\alpha^G| = |\alpha| = 0$. As in [9], the additive subgroup of virtual representations α with $|\alpha^G| = |\alpha| = 0$ is denoted by $RO_0(G)$. When we are working in $RSO(G)$, we write $RSO_0(G)$ for the obvious analog of $RO_0(G)$.

Let M be a Mackey functor, X a G-space, and α a virtual representation in $RO(G)$. The Mackey functor value of the unreduced equivariant cohomology of X

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in dimension α with M-coefficients is denoted by $\mathbf{H}^{\alpha}_{G}(X;M)$. Similarly, the abelian group value is denoted by $H^{\alpha}_{G}(X;M)$. When $\alpha = n$, the trivial representation $\mathbb{R}^n$, $H^{n}_{G}(X;M)$ is Bredon cohomology [1, 12]. Whenever the coefficients are absent in our notation, Burnside ring coefficients are understood. Note that here our notation is slightly different from that used in [9], where the reduced cohomology $\tilde{\mathbf{H}}^{\alpha}_{G}(X;M)$ is considered. Of course, $\mathbf{H}^{\alpha}_{G}(X;M) = \tilde{\mathbf{H}}^{\alpha}_{G}(X_+;M)$.

For our purposes, it suffices to note that each functor $\tilde{\mathbf{H}}^{\alpha}_{G}(?;M)$ is a contravariant functor from the homotopy category of based G-spaces $hG\mathcal{T}$ to $Mack$, which satisfies the usual axioms:

(i) Additivity: Wedge sums in $hG\mathcal{T}$ are carried to direct products in $Mack$.

(ii) Exactness: Cofibre sequences in $hG\mathcal{T}$ are carried to long exact sequences in $Mack$.

(iii) Suspension: $\tilde{\mathbf{H}}^{\alpha}_{G}(X;M) \cong \tilde{\mathbf{H}}^{\alpha+V}_{G}(\Sigma^V X;M)$, for any representation V and any α in $RO(G)$.

(iv) Dimension: $\tilde{\mathbf{H}}^{0}_{G}(S^0;M) = M$, and $\tilde{\mathbf{H}}^{n}_{G}(S^0;M) = 0$, for $n \in \mathbb{Z}$ with $n \neq 0$.

Many of the properties of $\mathbf{H}^{\alpha}_{G}(?;M)$ are most easily seen from the fact that $\mathbf{H}^{\alpha}_{G}(?;M)$ can be represented by an Eilenberg-Mac Lane Spectrum $K(M,0)$. That is,

$$\mathbf{H}^{\alpha}_{G}(X;M)(G/H) = [\Sigma^{\infty}X_+ \wedge G/H_+, \Sigma^{\alpha}K(M,0)]_G$$

for each subgroup H of G.

In terms of the group-valued cohomology, the Mackey functor value $\mathbf{H}^{\alpha}_{G}(X;M)$

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can be displayed by the diagram

$$\mathbf{H}_G^\alpha(X;M)$$

$$\begin{array}{c} H_G^\alpha(X;M) \\ {\scriptstyle \bar{\rho}_*}\big\downarrow \quad \big\uparrow{\scriptstyle \bar{\tau}_*} \\ H_G^\alpha(X;M_G) \\ \circlearrowright \\ {\scriptstyle \bar{g}_*} \end{array}$$

The restriction, transfer, and G-action are determined by the maps $\bar{\rho}$, $\bar{\tau}$, and $\bar{g}$, for each g in G, acting on the coefficients. Note that $H_G^\alpha(X;M_G)$ is isomorphic to the nonequivariant cohomology group $H^{|\alpha|}(X;M(G/e))$.

2.3.1 Cohomology of a Point

The results of the computation of $\mathbf{H}_G^*(*)$ in [9] are recorded in the following lemma.

Lemma 2.3. *Let $\alpha \in RO(G)$. Then*

$$\mathbf{H}_G^\alpha(*) = \begin{cases} A[d_\alpha] & \text{if } |\alpha| = |\alpha^G| = 0 \\ R & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0 \\ L & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| > 0 \\ \langle \mathbb{Z} \rangle & \text{if } |\alpha| \neq 0 \text{ and } |\alpha^G| = 0 \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^G| > 1, \text{ and } \alpha \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

The values d_α in Lemma 2.3 are given by a function $d_?$ from $RSO_0(G)$ to $\mathbb{Z}$ described in [9]. For our purposes, it suffices that $d_?$ induces a homomorphism from

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$RSO_0(G)$ to $(\mathbb{Z}/p)^*$, the multiplicative group of units in $\mathbb{Z}/p$. This homomorphism is completely determined by the formula $d_{\lambda_k-\lambda_l} \equiv k$ modulo p. Passage to quotients gives an induced homomorphism $RO(G) \longrightarrow (\mathbb{Z}/p)^*/\{\pm 1\}$, also denoted by $d_?$, making the diagram

$$\begin{array}{ccc} RSO(G) & \longrightarrow & (\mathbb{Z}/p)^* \\ \downarrow & & \downarrow \\ RO(G) & \longrightarrow & (\mathbb{Z}/p)^*/\{\pm 1\} \end{array}$$

commute. Note that Lemma 2.1 tells us that all three interpretations of the value d_α (an integer, a congruence class, and a congruence class modulo ± 1) unambiguously specify the isomorphism type of the Mackey functor $\mathbf{H}^\alpha_G(*) = A[d_\alpha]$.

For $\alpha \notin RO_0(G)$, the values of $\mathbf{H}^\alpha_G(*)$ depend only on $|\alpha^G|$ and $|\alpha|$, and hence we adopt the convention from [9] of plotting all of the values $\mathbf{H}^\alpha_G(*)$ as in Table 2.2.

			$\vert\alpha\vert$				
			$\wedge$				
	$\vdots$	$\vdots$	$\vdots$				
$\cdots$	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}\rangle$				
$\cdots$	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}\rangle$				
$\cdots$	R	R	$A[d_\alpha]$	L	L	L	$\cdots > \vert\alpha^G\vert$
				0	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}/p\rangle$	$\cdots$
			$\langle\mathbb{Z}\rangle$				
				0	$\langle\mathbb{Z}/p\rangle$	$\langle\mathbb{Z}/p\rangle$	$\cdots$
			$\langle\mathbb{Z}\rangle$				
			$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 2.2: $\mathbf{H}^*_G(*)$

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By following the line of argument in Appendix A in [9], it is possible to compute $\mathbf{H}^*_G(*; N)$ for any Mackey functor N. Specifically we are interested in $N = R, L$, and $\langle \mathbb{Z} \rangle$. Tables and computations of these values are given in Appendix B.

The tables of the cohomology of a point here and in Appendix B explain the reason for our special interest in the few Mackey functors appearing in Table 2.1. The reason for our interest in the Mackey functor A_G is explained by the following result from [9].

Lemma 2.4. *Let $\alpha \in RO(G)$. Then*

$$\mathbf{H}^\alpha_G(G) = \begin{cases} A_G & \text{if } |\alpha| = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2 $RO(G)$-Graded Modules and Maps

For a Mackey functor M, $T(M)$ denotes $\mathbf{H}^*_G(*; M)$; when $M = A$, just T is used. Similary, $T_G(M)$ denotes $\mathbf{H}^*_G(G; M)$. For a ring S, the cup product makes $T(S)$ an $RO(G)$-graded ring. If X is a G-space and M is an S-module, then $\mathbf{H}^*_G(X; M)$ is an $RO(G)$-graded module over $T(S)$. This module structure is central in our work, and the category of all $RO(G)$-graded modules over $T(S)$ is denoted by $T(S)$-Mod. The maps in $T(S)$-Mod are taken to be the graded maps of degree zero.

Given a module M in $T(S)$-Mod and some α in $RO(G)$, M^α denotes the Mackey functor value of M in dimension α, and $\Sigma^\alpha M$ denotes the shifted module obtained by defining $(\Sigma^\alpha M)^\beta = M^{\alpha+\beta}$ for all β in $RO(G)$. Similarly, for a map f in $T(S)$-Mod, f^α denotes the Mackey functor map in dimension α, and $\Sigma^\alpha f$ denotes a shifted map. An *shift endomorphism* of a module M is any map of the form $\Sigma^\alpha M \longrightarrow \Sigma^\beta M$.

A module M in $T(S)$-Mod comes with a collection of multiplication maps

$$\psi^\beta_\alpha : T(S)^\alpha \square M^\beta \longrightarrow M^{\alpha+\beta}$$

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which must satisfy the usual coherence diagrams. When the dimensions α and β are clear in context, the map ψ_α^β is simply denoted by ψ. A map f between two modules N and M consists of a collection of Mackey functor maps $f^\alpha : N^\alpha \longrightarrow M^\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} T(S)^\alpha \square N^\beta & \xrightarrow{1\square f^\beta} & T(S)^\alpha \square M^\beta \\ \downarrow{\scriptstyle \psi_N} & & \downarrow{\scriptstyle \psi_M} \\ N^{\alpha+\beta} & \xrightarrow{f^{\alpha+\beta}} & M^{\alpha+\beta}. \end{array} \tag{2.2}$$

Of course, this is the expected diagram based on our analogy between the box product and the usual tensor product. Some useful consequences of diagram (2.2) are presented in Appendix C.

Our work in the category of $T(S)$-Modules is primarily concerned with projective modules and more specifically free modules. Projective modules are defined by the expected universal property in the category $T(S)$-Mod. As explained in [9], $T_G(S)$ is a projective module. In fact, a sum of modules of the form $\Sigma^{-\alpha}T_G(S)$ for α in $RO(G)$ are the only examples of non-free projective $T(S)$-modules that we consider. Such modules are called *standard projectives*. Free $T(S)$-modules are defined as direct sums of modules of the form $\Sigma^{-\alpha}T(S)$ for α in $RO(G)$. We will make frequent use of the following Proposition that characterizes maps out of a free module.

Proposition 2.5. *For any M in $T(S)$-Mod and any α in $RO(G)$,*
$T(S)$-$Mod(\Sigma^{-\alpha}T(S), M) \cong Mack(S, M^\alpha)$.

More specifically, a module map $f : \Sigma^{-\alpha}T(S) \longrightarrow M$ is completely determined by a map of Mackey functors $f^\alpha : S \longrightarrow M^\alpha$ via the diagram

$$\begin{array}{ccccc} T(S)^{\beta-\alpha} \square A & \xrightarrow{1\square\eta} & T(S)^{\beta-\alpha} \square S & \xrightarrow{1\square f^\alpha} & T(S)^{\beta-\alpha} \square M^\alpha \\ \uparrow{\scriptstyle \cong} & & & & \downarrow{\scriptstyle \psi} \\ (\Sigma^{-\alpha}T(S))^\beta & & \xrightarrow[f^\beta]{} & & M^\beta \end{array} \tag{2.3}$$

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which specifies f^β for all β. Thus, a map out of $\Sigma^{-\alpha}T$ is specified by a map out of the Burnside ring. For easy reference, the following proposition recalls Proposition 4.12 of [9], which gives an analog of Proposition 2.5 for standard projective modules.

Proposition 2.6. *Let M be a T-module, and let α be in $RO(G)$. A T-module map $f : \Sigma^{-\alpha}T_G \longrightarrow M$ is completely determined by the image of*

$$(1, 0, \ldots, 0) \in \mathbb{Z}^p = (\Sigma^{-\alpha}T_G)^\alpha(G/e)$$

in $M^\alpha(G/e)$.

A free module like $F = \oplus_{i=1}^n \Sigma^{-\alpha_i}T$ is described as a free module with generators in dimensions $\alpha_1, \ldots, \alpha_n$. Loosely speaking, these are the dimensions where the Burnside rings appear in F. More precisely, $(\Sigma^{-\alpha_i}T)^{\alpha_i} = A$. Further, it is convenient to describe F by a plot of its generators. For this, the Cartesian plane is used, and the symbol $\bullet$ is marked at coordinates $(|\alpha^G|, |\alpha|)$ for each generator α. However, if two or more generators share the same coordinates, we put the number n of generators in that shared location. For example, the plot of $F = T \oplus \Sigma^{-\lambda_1}T \oplus \Sigma^{-\lambda_2}T \oplus \Sigma^{-(2+\lambda_1)}T$ is given in Table 2.3.

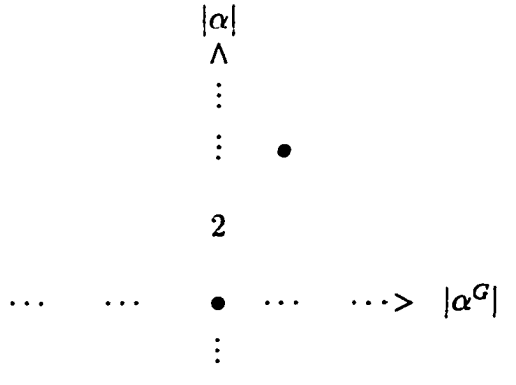


Table 2.3: Plot of $T \oplus \Sigma^{-\lambda_1}T \oplus \Sigma^{-\lambda_2}T \oplus \Sigma^{-(2+\lambda_1)}T$

Note that, by giving only a plot, information is lost about the exact dimensions of the generators. However, as we will see in the proof of Theorem 3.4, these plots

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are very useful for organizing computations. Theorem 4.4 describes more precisely how much information about the generators is retained by merely a plot of their coordinates.

Since our work is primarily with T-Modules, the generators and relations for T listed in Theorem 4.9 of [9] are used extensively. For example, if $|\alpha| = 0$, then ι_α is a generator for $T^\alpha(G/e) \cong \mathbb{Z}$. At the G/G-level, the standard generators are

μ_α	in	$T^\alpha = A[d_\alpha]$	for $\|\alpha\| = \|\alpha^G\| = 0$;
$\tau_\alpha = \tau(\iota_\alpha)$	in	$T^\alpha = A[d_\alpha]$ or L	for $\|\alpha\| = 0$, $\|\alpha^G\| \geq 0$, and α even;
ξ_α	in	$T^\alpha = R$	for $\|\alpha\| = 0$, $\|\alpha^G\| < 0$, and α even;
ϵ_α	in	$T^\alpha = \langle \mathbb{Z} \rangle$	for $\|\alpha\| > 0$ and $\|\alpha^G\| = 0$;
$\epsilon_{-\alpha}^{-1}\kappa_0$	in	$T^\alpha = \langle \mathbb{Z} \rangle$	for $\|\alpha\| < 0$ and $\alpha^G = 0$; and
ν_α	in	$T^\alpha = \langle \mathbb{Z}/p \rangle$	for $\|\alpha\| < 0$, $\|\alpha^G\| > 1$, and α odd.

In order to use the generators and relations for $T = \mathbf{H}^*_G(*)$ described in Theorem 4.9 of [9], we must work with $RSO(G)$-graded cohomology rather than $RO(G)$-graded cohomology. The point here is that, if α and α' are distinct elements of $RSO(G)$ which are identified in $RO(G)$, then, for any G-space X and Mackey functor M, the cohomology Mackey functors $\mathbf{H}^\alpha_G(X; M)$ and $\mathbf{H}^{\alpha'}_G(X; M)$ are isomorphic. However, the isomorphism is orientation reversing in the same sense that the degree -1 map on S^1 induces an orientation reversing endomorphism of the nonequivariant fundamental group $\pi_1(Y, *)$ of any space Y. In defining nonequivariant singular homology, we implicitly select an orientation on the standard n-simplex. Specifying generators for T involves analogous choices of orientations. To specify a generator in T^α, we must specify an orientation for α and hence regard α as an element of $RSO(G)$. Otherwise, the isomorphisms implicit in our $RO(G)$-grading can scramble the signs specified implicitly in the generators and relations listed in Theorem 4.9 of [9]. Thus we adopt the convention that, whenever we are using specific generators for T^α,

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we are regarding α as an element of $RSO(G)$. Otherwise, we work with $RO(G)$-graded cohomology. For example, Chapter 4 contains results about isomorphisms between free $RO(G)$-graded T-modules (e.g. Lemma 4.1). However, to prove these results by giving explicit formulas for the isomorphisms, we specify orientations for the dimensions of the generators of the free modules.

Some of the standard generators for T are used to define particular module maps called standard shift maps.

Definition 2.7. The following maps are called *standard shift maps*:

(a) If β is in $RO(G)$ with $|\beta| = 0$ and $|\beta^G| < 0$, then there is a map $h : T \longrightarrow \Sigma^\beta T$ defined by $h^0(\mu) = \xi_\beta$. A *horizontal shift map* is a map of the form $\pm\Sigma^\alpha h$.

(b) If γ is in $RO(G)$ with $|\gamma| > 0$ and $|\gamma^G| = 0$, then there is a map $v : T \longrightarrow \Sigma^\gamma T$ defined by $v^0(\mu) = \epsilon_\gamma$. A *vertical shift map* is a map of the form $\pm\Sigma^\alpha v$.

(c) If η is in $RO(G)$ with $|\eta| < 0$, $|\eta^G| > 1$, and η odd, then, for any integer a prime to p, there is a map $\partial : T \longrightarrow \Sigma^\eta T$ defined by $\partial^0(\mu) = a\nu_\eta$. A *diagonal shift map* is a map of the form $\pm\Sigma^\alpha\partial$.

A very easy way to characterize standard shift maps is given in the following lemma.

Lemma 2.8. *Let α_1 and α_2 be elements of $RO(G)$ such that one of the following conditions hold:*

(i) $|\alpha_1 - \alpha_2| = 0$ *and* $|\alpha_1^G - \alpha_2^G| < 0$

(ii) $|\alpha_1 - \alpha_2| > 0$ *and* $|\alpha_1^G - \alpha_2^G| = 0$

(iii) $|\alpha_1 - \alpha_2| < 0$, $|\alpha_1^G - \alpha_2^G| > 1$, *and* $\alpha_1 - \alpha_2$ *is odd.*

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If $s : \Sigma^{-\alpha_1}T \longrightarrow \Sigma^{-\alpha_2}T$ *is a* T*-module map such that* s^{α_1} *is an epimorphism, then* s *is a standard shift map. Moreover,* s *is a horizontal, vertical, or diagonal shift map, respectively, if condition (i), (ii), or (iii) holds, respectively.*

It is important to note that these standard shift maps are simply shift endomorphisms of T given by multiplication by an appropriate generator. Therefore, by using the relations from Theorem 4.9 of [9], the standard shift maps can be described in every dimension. A detailed description of these maps is given in Appendix C.

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Chapter 3

The Main Theorem

In [9], Lewis proves a theorem (Theorem 2.6) which has roughly the form:

Theorem. *Let $G = \mathbb{Z}/p$, for some prime p, and let X be a generalized G-cell complex containing only even dimensional cells and satisfying certain finiteness conditions. Suppose that X is built in such a way that, whenever a representation cell DV is added, all previously added representation cells DW satisfy*

$$|V| > |W| \implies |V^G| \geq |W^G|.$$

Then, the equivariant ordinary $RO(G)$-graded cohomology $\mathbf{H}^*_G(X)$ is a sum $P \oplus F$, where $P = \bigoplus \Sigma^{-m}\mathbf{H}^*_G(G)$ is a projective module over $\mathbf{H}^*_G(*)$ with one generator in dimension m for each orbit cell $G \times D^m$ used to build X, and $F = \bigoplus \Sigma^{-W}\mathbf{H}^*_G(*)$ is a free module over $\mathbf{H}^*_G(*)$ with one generator in dimension W for each representation cell DW used to build X.

The constraint in Lewis' Theorem on the order in which the representation cells are added guarantees trivial boundary maps in the long exact sequences associated with the cell attachments. In this chapter, we prove a generalization of Theorem 2.6

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of [9] which removes this constraint on the ordering of the cells. In order to prove this result, we are forced to consider nontrivial boundary maps.

Theorem 3.1. *Let $G = \mathbb{Z}/p$, for an odd prime p, let $S = A$, R, or $\langle\mathbb{Z}\rangle$, and let X be a generalized G-cell complex containing only even dimensional cells. If the cells of X satisfy the conditions:*

(i) for each integer m, X has only finitely many cells of the form $G \times D^m$, and,

(ii) for each integer m, X has only finitely many cells of the form DV with $|V^G| = m$,

then the equivariant ordinary $RO(G)$-graded cohomology $\mathbf{H}^*_G(X;S)$ is projective as a module over the cohomology of a point $\mathbf{H}^*_G(*;S)$.

Moreover, $\mathbf{H}^*_G(X;S) \cong P_S \oplus F_S$, where

(i) $P_S = \bigoplus \Sigma^{-m}\mathbf{H}^*_G(G;S)$ is a projective module over $\mathbf{H}^*_G(*;S)$ with one generator in dimension m for each orbit cell $G \times D^m$ used to build X, and,

(ii) F_S is a free module over $\mathbf{H}^*_G(*;S)$ with one generator in an even representation-like dimension for each represenation cell used to build X.

Further, the dimensions of the generators can be chosen to be the same for $S = A$, R, and $\langle\mathbb{Z}\rangle$.

Remark 3.2. (a) For $S = A$, R, and $\langle\mathbb{Z}\rangle$, the dimensions of the generators for F_S and the dimensions of the representation cells used to build X need not be the same. In fact, they will be the same if and only if the boundary maps associated with the cell attachments used to build X are all zero. This is the relationship between the conclusion of Theorem 3.1 and the conclusion of Theorem 2.6 of [9].

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(b) For $S = \langle \mathbb{Z} \rangle$, the dimensions of the generators for $F_{\langle \mathbf{Z} \rangle}$ can alternatively be chosen to be the same as the dimesions of the representation cells used to build X. As is shown in Lemma 4.7, any summand $\Sigma^{-\omega}\mathbf{H}^*_G(*; \langle \mathbb{Z} \rangle)$ can be replaced by a summand $\Sigma^{-\omega'}\mathbf{H}^*_G(*; \langle \mathbb{Z} \rangle)$, where $|(\omega')^G| = |\omega^G|$. Also note that $P_{\langle \mathbf{Z} \rangle} = 0$ since $\mathbf{H}^*_G(G; \langle \mathbb{Z} \rangle) = 0$.

The case in which $S = A$ is the case of real interest in Theorem 3.1. However, our proof of this result for that case requires the use of the other coefficients R and $\langle \mathbb{Z} \rangle$. It is possible to define the $RO(G)$-graded box product ? $\square_T$? for T-modules, and there ought to be a Universal Coefficients Theorem for $RO(G)$-graded equivariant ordinary cohomology based on this product. However, such a theorem is not yet available. Our proof of Theorem 3.1 is therefore structured to circumvent this deficiency. Specifically, rather than describing ? $\square_T$? in full generality, we introduce a simpler construction sufficient for our needs.

Let Q be a projective T-module of the form

$$Q = \left(\bigoplus_{m \in \mathcal{M}} \Sigma^{-m} T_G \right) \oplus \left(\bigoplus_{\omega \in \mathcal{W}} \Sigma^{-\omega} T \right),$$

where $\mathcal{M}$ is an indexed set of integers, and $\mathcal{W}$ is an indexed set of elements of $RO(G)$. Note that we allow $\mathcal{M}$ and $\mathcal{W}$ to have elements that occur more than once. Associated to Q there is a functor, also denoted by Q, from *Mack* to *T-Mod*. For each Mackey functor N, define

$$Q(N) = \left(\bigoplus_{m \in \mathcal{M}} \Sigma^{-m} T(N)_G \right) \oplus \left(\bigoplus_{\omega \in \mathcal{W}} \Sigma^{-\omega} T(N) \right), \tag{3.1}$$

and, for each map of Mackey functors $N_1 \xrightarrow{h} N_2$, define the T-module map

$$Q(N_1) \xrightarrow{Q(h)} Q(N_2)$$

to be the obvious sum of suspensions of the maps

$$\mathbf{H}^*_G(G; N_1) \longrightarrow \mathbf{H}^*_G(G; N_2) \quad \text{and} \quad \mathbf{H}^*_G(*; N_1) \longrightarrow \mathbf{H}^*_G(*; N_2)$$

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induced by the map $N_1 \xrightarrow{h} N_2$. The map $Q(N_1) \dashrightarrow Q(N_2)$ is called a *change of coefficients map*. Had we defined ? $\square_T$?, it would be obvious that $Q(N)$ and $Q(N_1) \longrightarrow Q(N_2)$ were just $Q \square_T T(N)$ and $Q \square_T T(N_1) \xrightarrow{id \square T(h)} Q \square_T T(N_2)$, respectively. Also note that, if Q_1 and Q_2 are two projective T-modules of the type given in equation (3.1), then a map $f : Q_1 \longrightarrow Q_2$ induces a map $f(N) : Q_1(N) \longrightarrow Q_2(N)$. On each summand $\Sigma^{-m} T_G(N)$ and for each $\gamma \in RO(G)$, $f(N)^\gamma$ is given by

$$(\Sigma^{-m} T_G(N))^\gamma = T(N)^{\gamma - m} \square A_G \xrightarrow{id \square f^m} T(N)^{\gamma - m} \square Q_2^m \xrightarrow{mult} Q_2(N)^\gamma.$$

On each summand $\Sigma^{-\omega} T(N)$ and for each $\gamma \in RO(G)$, $f(N)^\gamma$ is given by

$$(\Sigma^{-\omega} T(N))^\gamma = T(N)^{\gamma - \omega} \square A \xrightarrow{id \square f^\omega} T(N)^{\gamma - \omega} \square Q_2^\omega \xrightarrow{mult} Q_2(N)^\gamma.$$

Of course, $f(N)$ is just $f \square id : Q_1 \square_T T(N) \longrightarrow Q_2 \square_T T(N)$.

For us, the most important instances of change of coefficients maps are those induced by the canonical short exact sequence of coefficients

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \longrightarrow 0.$$

Propositions B.14 and B.16 together tell us that, if Q is a sum of a standard projective T-module and a free T-module, then this short exact sequence of coefficients induces a short exact sequence of T-modules

$$0 \longrightarrow Q(\langle \mathbb{Z} \rangle) \longrightarrow Q \longrightarrow Q(R) \longrightarrow 0.$$

Of course $Q(\langle \mathbb{Z} \rangle)$ and $Q(R)$ also have the structure of a $T(\langle \mathbb{Z} \rangle)$-module and a $T(R)$-module, respectively. The following lemma is used frequently.

Lemma 3.3. *Let Q be a sum of a standard projective T-module and a free T-module. Assume that*

$$0 \longrightarrow H' \xrightarrow{i} H \xrightarrow{r} H'' \longrightarrow 0$$

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is a short exact sequence of T-modules and H'' is a $T(R)$-module. If $f : Q \longrightarrow H$ is a T-module map, then there are unique maps

$$f' : Q(\langle\mathbb{Z}\rangle) \longrightarrow H' \quad \text{and} \quad f'' : Q(R) \longrightarrow H''$$

such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q(\langle\mathbb{Z}\rangle) & \longrightarrow & Q & \longrightarrow & Q(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & H' & \xrightarrow{i} & H & \xrightarrow{r} & H'' & \longrightarrow & 0 \end{array}$$

commutes. The map f'' is a $T(R)$-module map. Moreover, if H' is a $T(\langle\mathbb{Z}\rangle)$-module, then the map f' is a $T(\langle\mathbb{Z}\rangle)$-module map.

Proof. The existence and uniqueness of the $T(R)$-module map f'' making the right square commute comes from Lemma C.8. The T-module map f' making the left square commute is then the unique induced map between kernels. The fact that f' is a $T(\langle\mathbb{Z}\rangle)$-module map if H' is a $T(\langle\mathbb{Z}\rangle)$-module follows from the surjectivity of the ring homomorphism $A \longrightarrow \langle\mathbb{Z}\rangle$. □

We can now state the variant of Theorem 3.1 that is actually going to be proven.

Theorem 3.4. *Let $G = \mathbb{Z}/p$, for an odd prime p, and let X be a generalized G-cell complex containing only even dimensional cells. If the cells of X satisfy the conditions:*

(i) for each integer m, X has only finitely many cells of the form $G \times D^m$, and,

(ii) for each integer m, X has only finitely many cells of the form DV with $|V^G| = m$,

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then the equivariant ordinary $RO(G)$-graded cohomology $\mathbf{H}^*_G(X)$ is isomorphic to a sum $P \oplus F$, where

(i) $P = \bigoplus_{m \in \mathcal{M}} \Sigma^{-m} \mathbf{H}^*_G(G)$ is a projective module over $\mathbf{H}^*_G(*)$ such that $\mathcal{M}$ is the indexed set $\{m : G \times D^m$ is a cell of $X\}$, and

(ii) $F = \bigoplus_{\omega \in \mathcal{W}} \Sigma^{-\omega} \mathbf{H}^*_G(*)$ is a free module over $\mathbf{H}^*_G(*)$ such that $\mathcal{W}$ is an indexed set of even representation-like elements of $RO(G)$, and $\mathcal{W}$ is in one-to-one correspondence with the set of representation cells used to build X.

Moreover, there are isomorphisms

$$\mathbf{H}^*_G(X; \langle \mathbb{Z} \rangle) \cong P(\langle \mathbb{Z} \rangle) \oplus F(\langle \mathbb{Z} \rangle) \quad \textit{and} \quad \mathbf{H}^*_G(X; R) \cong P(R) \oplus F(R)$$

such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(\langle \mathbb{Z} \rangle) \oplus F(\langle \mathbb{Z} \rangle) & \longrightarrow & P \oplus F & \longrightarrow & P(R) \oplus F(R) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(X; \langle \mathbb{Z} \rangle) & \longrightarrow & \mathbf{H}^*_G(X) & \longrightarrow & \mathbf{H}^*_G(X; R) & \longrightarrow & 0 \end{array}$$

commutes and has exact rows. Here, the maps on the top row are the canonical change of coefficients maps and the maps on the bottom row are the maps in cohomology induced by the canonical coefficient maps.

Proof. The proof is given in Section 3.1. □

In the course of our proof of Theorem 3.4, it is necessary to consider the case in which a single new representation cell is added to a space with projective cohomology. At that point in the proof of Theorem 3.4, the following theorem is used.

Theorem 3.5. *Let B be a G-space, and let*

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(i) $P = \bigoplus_{m\in\mathcal{M}} \Sigma^{-m}\mathbf{H}^*_G(G)$ *be a projective module over* $\mathbf{H}^*_G(*)$ *such that* $\mathcal{M}$ *is an indexed set of nonnegative even integers, and*

(ii) $F = \bigoplus_{\omega\in\mathcal{W}} \Sigma^{-\omega}\mathbf{H}^*_G(*)$ *be a free module over* $\mathbf{H}^*_G(*)$ *such that* $\mathcal{W}$ *is an indexed set of even representation-like elements of* $RO(G)$.

Suppose that, for $S = A$, R *and* $\langle\mathbb{Z}\rangle$, *there are isomorphisms*

$$\mathbf{H}^*_G(B;S) \cong P(S) \oplus F(S),$$

such that the canonical diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(\langle\mathbb{Z}\rangle)\oplus F(\langle\mathbb{Z}\rangle) & \longrightarrow & P\oplus F & \longrightarrow & P(R)\oplus F(R) & \longrightarrow & 0 \\ & & \Psi_{\langle\mathbb{Z}\rangle}\downarrow\cong & & \Psi\downarrow\cong & & \Psi_R\downarrow\cong & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(B) & \longrightarrow & \mathbf{H}^*_G(B;R) & \longrightarrow & 0 \end{array}$$

commutes and has exact rows.

If Y *is a* G*-space formed by attaching an even dimensional representaton cell* DV *to* B, *then there is a free* $\mathbf{H}^*_G(*)$*-module* $F' = \bigoplus_{\omega'\in\mathcal{W}'} \Sigma^{-\omega'}\mathbf{H}^*_G(*)$ *such that*

(i) $\mathcal{W}'$ *is is an indexed set of even representation-like elements of* $RO(G)$;

(ii) $Rank(F') = 1 + Rank(F)$;

(iii) for $S = A$, R *and* $\langle\mathbb{Z}\rangle$, $\mathbf{H}^*_G(Y;S) \cong P(S) \oplus F'(S)$; *and*

(iv) the canonical diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(\langle\mathbb{Z}\rangle)\oplus F'(\langle\mathbb{Z}\rangle) & \longrightarrow & P\oplus F' & \longrightarrow & P(R)\oplus F'(R) & \longrightarrow & 0 \\ & & \Psi'_{\langle\mathbb{Z}\rangle}\downarrow\cong & & \Psi'\downarrow\cong & & \Psi'_R\downarrow\cong & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(Y;\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(Y) & \longrightarrow & \mathbf{H}^*_G(Y;R) & \longrightarrow & 0 \end{array}$$

commutes and has exact rows.

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Moreover, if $\mathcal{W}$ is decomposed as $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$, where $\mathcal{W}_1$ consists of all of the elements ω of $\mathcal{W}$ such that $|\omega^G| < |V^G|$, then $\mathcal{W}'$ decomposes as $\mathcal{W}_1 \cup \mathcal{W}_2'$, where $\mathcal{W}_2'$ is an indexed set all of whose elements have fixed dimension at least $|V^G|$. If we let $F_1 = \bigoplus_{\omega \in \mathcal{W}_1} \Sigma^{-\omega} \mathbf{H}_G^*(*)$, $F_2 = \bigoplus_{\omega \in \mathcal{W}_2} \Sigma^{-\omega} \mathbf{H}_G^*(*)$, and $F_2' = \bigoplus_{\omega' \in \mathcal{W}_2'} \Sigma^{-\omega'} \mathbf{H}_G^*(*)$, then, for $S = A$, R, and $\langle \mathbb{Z} \rangle$, the diagram

$$\begin{array}{ccccccc} P(S) \oplus F_1(S) \oplus F_2'(S) & \xrightarrow[\cong]{\Psi'} & \mathbf{H}_G^*(Y;S) & \xrightarrow{i^*} & \mathbf{H}_G^*(B;S) & \xrightarrow[\cong]{\Psi^{-1}} & P(S) \oplus F_1(S) \oplus F_2(S) \\ \downarrow{\scriptstyle \pi_{1,2}} & & & & & & \downarrow{\scriptstyle \pi_{1,2}} \\ P(S) \oplus F_1(S) & & \xrightarrow{\qquad\qquad id \qquad\qquad} & & & & P(S) \oplus F_1(S) \end{array}$$

commutes.

Proof. The proof is given in Section 3.2. □

Remark 3.6. (a) To understand the proofs of Theorems 3.4 and 3.5, it is helpful to understand the precise relationship between the free summands of $\mathbf{H}_G^*(Y)$ and $\mathbf{H}_G^*(B)$ in Theorem 3.5. The cell attachment in Theorem 3.5 is reflected in the usual long exact sequence in cohomology:

$$\ldots \longrightarrow \Sigma^{-V} T \longrightarrow \mathbf{H}_G^*(Y) \longrightarrow \mathbf{H}_G^*(B) \xrightarrow{\partial} \Sigma^{1-V} T \longrightarrow \ldots . \tag{3.2}$$

The case in which $\partial = 0$ gives $\mathbf{H}_G^*(Y) = \mathbf{H}_G^*(B) \oplus \Sigma^{-V} T$; the free summands of $\mathbf{H}_G^*(B)$ provide all but one of the free summands of $\mathbf{H}_G^*(Y)$. Our interest is in what happens if $\partial \neq 0$. In that case, it turns out that some, but not all, of the free summands of $\mathbf{H}_G^*(B)$ are preserved in $\mathbf{H}_G^*(Y)$. Finitely many of the free summands of $\mathbf{H}_G^*(B)$, say $\Sigma^{-\omega_1} T, \ldots, \Sigma^{-\omega_n} T$, get replaced by different summands, say $\Sigma^{-\omega_1'} T, \ldots, \Sigma^{-\omega_n'} T$, and further, a new summand $\Sigma^{-V'} T$, in a different dimension from $\Sigma^{-V} T$, gets added to form $\mathbf{H}_G^*(Y)$. The dimensions of the original summands satisfy $|\omega_1| < \ldots < |\omega_n| < |V|$ and $|V^G| < |\omega_1^G| < \ldots < |\omega_n^G|$. The dimensions of the replacement summands satisfy $|\omega_i'| = |\omega_i|$ for $1 \leq i \leq n$, $|(\omega_1')^G| = |V^G|$, and $|(\omega_i')^G| = |\omega_{i-1}^G|$ for $2 \leq i \leq n$. The

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dimension V' satisfies $|(V')^G| = |\omega_n^G|$ and $|V'| = |V|$. Let H' denote the module obtained from $\mathbf{H}_G^*(B)$ by replacing the summands $\Sigma^{-\omega_i}T$ with the summands $\Sigma^{-\omega_i'}T$. Then, the cohomology of Y is $\mathbf{H}_G^*(Y) \cong H' \oplus \Sigma^{-V'}T$. Note that $\mathbf{H}_G^*(Y)$ does not have generators in the same dimensions as the cells used to build Y. Briefly, the value of $\mathbf{H}_G^*(Y)$, comes from a shifting of the generators in $\mathbf{H}_G^*(B)$ and $\Sigma^{-V}T$. This shifting is pictured in Table 3.1.

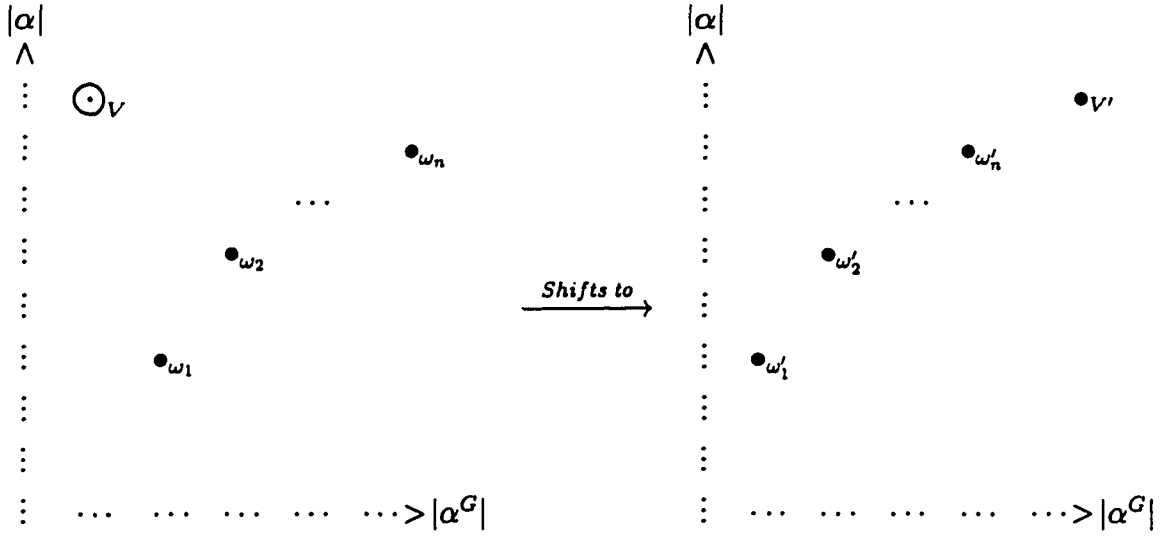


Table 3.1: Shifting of Generators

In the plot on the left, the symbol $\odot$ is used to denote the generator for $\Sigma^{-V}T$, thereby distinguishing it from the generators for $\bigoplus_{i=1}^n \Sigma^{-\omega_i}T$ which are denoted as usual. The plot on the right shows where these generators end up after the shifting induced by a nontrivial boundary map. Note that all of the shifting occurs in fixed dimensions greater than or equal to $|V^G|$.

(b) As in Theorem 2.6 of [9], the free summand F in our Theorem 3.4 has one summand $\Sigma^{-W}T$ for each representation cell DW that is involved in neither the domain nor range of a nontrivial boundary map.

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(c) Although Theorem 3.4 can tell us the coordinates of the generators for the cohomology of a generalized G-cell complex built with even-dimensional cells, it does not specify the exact dimensions for the generators. For example, let DV be a representaion cell whose attachment induces a nontrivial boundary map out of a single summand $\Sigma^{-\omega}T$ of the cohomology of an existing space. As stated in Corollary C.7, the exactness of the sequence

$$\longrightarrow \Sigma^{-V}T \longrightarrow \Sigma^{-V'}T \oplus \Sigma^{-\omega'}T \longrightarrow \Sigma^{-\omega}T \xrightarrow{\partial} \Sigma^{1-V}T \longrightarrow$$

is equivalent to the condition $d_\gamma = \pm 1$, where $\gamma = \omega' - \omega - V + V'$. Hence, given V and ω, the sum $V' + \omega'$ can be specified. However, neither V' nor ω' can be determined. Only the coordinates of V' and ω' are forced. This issue of uncertainty in the dimensions of the generators of a free module is explored further in Chapter 4.

3.1 The Proof of Theorem 3.4

Our proof proceeds by induction up the filtration

$$X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_s \longrightarrow \ldots$$

for X. Ultimately, the limit $\lim_s \mathbf{H}^*_G(X_s)$ must be computed. This computation is best understood via a comparison with the analogous computation in the nonequivariant case. Suppose that Z is a nonequivariant cell complex with even-dimensional cells and only finitely many cells in each dimension. Say Z has the filtration

$$Z_0 \longrightarrow Z_1 \longrightarrow \ldots \longrightarrow Z_s \longrightarrow \ldots .$$

For each $s \geq 0$, let $\mathcal{M}_s$ be the indexed set $\{m : D^m \text{ is a cell of } Z_s\}$. Since the cells are all even dimensional, the boundary maps associated with the cell attachments are

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all zero. Hence, for each $s \geq 0$, a simple argument gives that

$$H^*(Z_s) \cong \bigoplus_{m \in \mathcal{M}_s} \Sigma^{-m} H^*(*).$$

Moreover, the map in cohomology $H^*(Z_{s+1}) \longrightarrow H^*(Z_s)$ induced by the inclusion $Z_s \subseteq Z_{s+1}$ is just the obvious projection onto a summand. In this case, it is easy to use the standard Milnor short exact sequence

$$0 \longrightarrow \lim_s{}^1(\Sigma^{-1} H^*(Z_s)) \longrightarrow H^*(Z) \longrightarrow \lim_s H^*(Z_s) \longrightarrow 0$$

and argue that $\lim_s{}^1 (\Sigma^{-1} H^*(Z_s)) = 0$ and

$$H^*(Z) \cong \lim_s H^*(Z_s) \cong \bigoplus_{m \in \mathcal{M}} \Sigma^{-m} H^*(*).$$

Here, $\mathcal{M} = \bigcup_{s \geq 0} \mathcal{M}_s = \{m : D^m \text{ is a cell of } Z\}$.

In our equivariant case, X has two kinds of cells - orbit cells and representation cells. Hence, the cohomology of X has two kinds of summands - standard projective summands and free summands. What happens on the standard projective summands is exactly what happens in the nonequivariant case. However, the free summands are harder to handle. The difficulty arises from the dimension-shifting of the free generators caused by nonzero boundary maps in the cell attachment long exact sequence. As a result of this shifting, the map in cohomology

$$\mathbf{H}^*_G(X_{s+1}) \longrightarrow \mathbf{H}^*_G(X_s)$$

induced by the inclusion $X_s \subseteq X_{s+1}$ need not be a simple projection onto a summand.

It turns out that the standard Milnor short exact sequence can still be used as it was in the nonequivariant case. However, we have to work harder to argue that the $\lim^1$ term is zero and to compute $\mathbf{H}^*_G(X) \cong \lim_s \mathbf{H}^*_G(X_s)$. The key to our argument is the fact that, as $s \to \infty$, the minimum dimension of the representation cells added

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after X_s goes to infinity. Hence, the difference between $\mathbf{H}^*_G(X_s)$ and $\mathbf{H}^*_G(X_{s+1})$ caused by dimension-shifting gets restricted to higher and higher dimensions. Specifically, for each integer $s \geq 0$, we define

$$m_s = min\{|V^G| : \ DV \text{ is a cell of } X \text{ added after } X_s\}$$

The minimum of the empty set is taken to be infinity. The free generators of $\mathbf{H}^*_G(X_s)$ in fixed dimensions less than m_s are preserved in $\mathbf{H}^*_G(X_{s+k})$, for all $k \geq 0$, and ultimately in $\mathbf{H}^*_G(X)$. The free generators of $\mathbf{H}^*_G(X_s)$ in fixed dimensions at least m_s need not be preserved in $\mathbf{H}^*_G(X)$. Clearly

$$0 \leq m_0 \leq m_1 \leq ... \leq m_s \leq ... \leq \infty,$$

and our finiteness assumptions guarantee that $\lim_{s\to\infty} m_s = \infty$. Hence, in our case, $\lim_s \mathbf{H}^*_G(X_s)$ turns out to be computable.

To compute the $\lim^1$ and lim terms in our Milnor short exact sequence, we use Proposition D.1 which describes the limits of certain sequences of projective modules. Inductively, each module $\mathbf{H}^*_G(X_s)$ is shown to be projective of the form

$$\mathbf{H}^*_G(X_s) \cong \left(\bigoplus_{m\in\mathcal{M}_s} \Sigma^{-m}T_G\right) \oplus \left(\bigoplus_{\omega\in\overline{\mathcal{W}}_s} \Sigma^{-\omega}T\right).$$

Here, $\mathcal{M}_s$ is the indexed set $\{m : G \times D^m \text{ is a cell of } X_s\}$, and $\overline{\mathcal{W}}_s$ is an indexed set of even representation-like elements of $RO(G)$ such that $\overline{\mathcal{W}}_s$ is in one-to-one correspondence with the set of representation cells of X_s. From $\overline{\mathcal{W}}_s$ define the subset

$$\mathcal{W}_s = \{\omega \in \overline{\mathcal{W}}_s : |\omega^G| < m_s\}.$$

Let Q_s be the summand of $\mathbf{H}^*_G(X_s)$ defined by

$$Q_s = \left(\bigoplus_{m\in\mathcal{M}_s} \Sigma^{-m}T_G\right) \oplus \left(\bigoplus_{\omega\in\mathcal{W}_s} \Sigma^{-\omega}T\right).$$

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Our aim is to apply Proposition D.1 for the sequence of T-modules given by

$$\overline{Q}_s = \mathbf{H}^*_G(X_s).$$

Hence, we need to show that, for each $s \geq 0$, $\mathcal{W}_s \subseteq \mathcal{W}_{s+1}$, and, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, the diagram

$$\begin{array}{ccc} \mathbf{H}^*_G(X_{s+1};S) & \longrightarrow & \mathbf{H}^*_G(X_s;S) \\ \downarrow & & \downarrow \\ Q_{s+1}(S) & \longrightarrow & Q_s(S) \end{array} \tag{3.3}$$

commutes. Here, the top horizontal map is the map in cohomology induced by the inclusion $X_s \subseteq X_{s+1}$, and the remaining maps are the obvious projections onto a summand. Proposition D.1 can then be applied for the indexed sets $\mathcal{M} = \bigcup_{s\geq 0} \mathcal{M}_s$ and $\mathcal{W} = \bigcup_{s\geq 0} \mathcal{W}_s$, the sequence $\{m_s\}_{s\geq 0}$, the indexed sets

$$\mathcal{W}'_s = \{\omega \in \overline{\mathcal{W}}_s : |\omega^G| \geq m_s\},$$

and the sequence of T-modules given by $\overline{Q}_s = \mathbf{H}^*_G(X_s)$.

By our finiteness assumptions, X_0 is a disjoint union of orbits. Hence, $\mathbf{H}^*_G(X_0)$ is a direct sum, in which each summand has the form T_G or T. Clearly, the conclusions of Theorem 3.4 hold for X_0. Of course, $\mathcal{M}_0$ consists of one zero for each orbit G/e in X_0, and $\overline{\mathcal{W}}_0$ consists of one zero for each orbit G/G in X_0. Moreover,

$$\mathcal{W}_0 = \begin{cases} \emptyset & \text{if } m_0 = 0, \\ \overline{\mathcal{W}}_0 & \text{if } m_0 > 0, \end{cases} \quad \text{and} \quad Q_0 = \begin{cases} 0 & \text{if } m_0 = 0, \\ \mathbf{H}^*_G(X_0) & \text{if } m_0 > 0. \end{cases}$$

Suppose, for some $s \geq 0$, that the conclusions of Theorem 3.4 hold for X_s, and that

$$\begin{aligned} &\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}_s, \\ &\mathcal{W}_0 \subseteq \overline{\mathcal{W}}_0, \quad \mathcal{W}_1 \subseteq \overline{\mathcal{W}}_1, \quad \ldots \quad , \mathcal{W}_s \subseteq \overline{\mathcal{W}}_s, \\ &\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \ldots \subseteq \mathcal{W}_s, \text{ and} \\ &Q_0 \leq Q_1 \leq \ldots \leq Q_s \leq \mathbf{H}^*_G(X_s;S) \end{aligned}$$

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have been defined. Also suppose that, for $S = A$, R, and $\langle \mathbb{Z} \rangle$, the diagram

$$\begin{array}{ccccccc} \mathbf{H}^*_G(X_s;S) & \longrightarrow & \mathbf{H}^*_G(X_{s-1};S) & \longrightarrow & \cdots & \longrightarrow & \mathbf{H}^*_G(X_0;S) \\ \downarrow & & \downarrow & & & & \downarrow \\ Q_s(S) & \longrightarrow & Q_{s-1}(S) & \longrightarrow & \cdots & \longrightarrow & Q_0(S) \end{array}$$

commutes, where the maps in the top row are the induced maps in cohomology and the remaining maps are the obvious projections onto a summand.

Let C denote X_{s+1}/X_s. Hence, C is the wedge sum of spaces of the form $G_+ \wedge S^{2k}$ or S^V coming from the attached cells of the form $G \times D^{2k}$ or DV, respectively. This attachment is reflected in the cofibre sequence

$$(X_s)_+ \longrightarrow (X_{s+1})_+ \longrightarrow C.$$

For any coefficients S, this cofibre sequence gives rise to the expected long exact sequence in cohomology

$$\longrightarrow \tilde{\mathbf{H}}^\alpha_G(C;S) \longrightarrow \mathbf{H}^\alpha_G(X_{s+1};S) \longrightarrow \mathbf{H}^\alpha_G(X_s;S) \xrightarrow{\partial^\alpha_s} \tilde{\mathbf{H}}^{\alpha+1}_G(C;S) \longrightarrow$$
$$\mathbf{H}^{\alpha+1}_G(X_{s+1};S) \longrightarrow \mathbf{H}^{\alpha+1}_G(X_s;S) \longrightarrow. \quad (3.4)$$

For the case in which $S = A$, the boundary map $\partial : \mathbf{H}^*_G(X_s) \longrightarrow \Sigma\tilde{\mathbf{H}}^*_G(C)$ is a graded module map between projective T-modules. The summands in the domain of ∂ have the form $\Sigma^{-2j}T_G$ or $\Sigma^{-\omega}T$, where ω is an even representation-like element of $RO(G)$. The summands in the range have the form $\Sigma^{1-2k}T_G$ or $\Sigma^{1-V}T$, where V is an even-dimensional actual representation. Thus, the boundary map ∂ is made up of maps of the forms

$$\Sigma^{-2j}T_G \longrightarrow \Sigma^{1-2k}T_G,$$
$$\Sigma^{-2j}T_G \longrightarrow \Sigma^{1-V}T,$$
$$\Sigma^{-\omega}T \longrightarrow \Sigma^{1-2k}T_G, \text{ and}$$
$$\Sigma^{-\omega}T \longrightarrow \Sigma^{1-V}T.$$

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In the proof of Theorem 2.6 in [9], Lewis shows that only the last of these can be nonzero. Moreover, this can only occur if

$$|V| > |\omega| \text{ and } |V^G| < |\omega^G|.$$

We form X_{s+1} from X_s in two stages. First, form X'_{s+1} by attaching all of the cells of X_{s+1} on which the boundary map $\partial : \mathbf{H}^*_G(X_s) \longrightarrow \Sigma\tilde{\mathbf{H}}^*_G(C)$ vanishes. Let C' be the cofibre of the inclusion $X_s \subseteq X'_{s+1}$. By our observations about ∂, all of the orbit cells in C are included in C'. Since the boundary map

$$\partial' : \mathbf{H}^*_G(X_s) \longrightarrow \Sigma\tilde{\mathbf{H}}^*_G(C')$$

is zero, the long exact sequence in cohomology arising from the cofibre sequence

$$X_s \longrightarrow X'_{s+1} \longrightarrow C'$$

simply amounts to a short exact sequence

$$0 \longrightarrow \tilde{\mathbf{H}}^*_G(C') \longrightarrow \mathbf{H}^*_G(X'_{s+1}) \longrightarrow \mathbf{H}^*_G(X_s) \longrightarrow 0 \tag{3.5}$$

in *T-mod*. Moreover, the canonical map $A \longrightarrow R$ induces a diagram

$$\begin{array}{ccc} \mathbf{H}^*_G(X_s; A) & \xrightarrow{\partial'=0} & \Sigma\tilde{\mathbf{H}}^*_G(C') \\ \downarrow{\scriptstyle q} & & \downarrow \\ \mathbf{H}^*_G(X_s; R) & \xrightarrow{\partial'_R} & \Sigma\tilde{\mathbf{H}}^*_G(C'; R) \end{array}$$

in which the vertical maps are the canonical change of coefficients maps. It follows that $\partial'_R = 0$, and hence the canonical short exact sequence of coefficients

$$0 \longrightarrow \langle\mathbb{Z}\rangle \longrightarrow A \longrightarrow R \longrightarrow 0$$

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induces a diagram in cohomology

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{\mathbf{H}}^*_G(C';\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(X'_{s+1};\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(X_s;\langle\mathbb{Z}\rangle) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c'_1} & & \downarrow{\scriptstyle q'_1} & & \downarrow{\scriptstyle q_1} & & \\
0 & \longrightarrow & \tilde{\mathbf{H}}^*_G(C') & \longrightarrow & \mathbf{H}^*_G(X'_{s+1}) & \longrightarrow & \mathbf{H}^*_G(X_s) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c'_2} & & \downarrow{\scriptstyle q'_2} & & \downarrow{\scriptstyle q_2} & & \\
0 & \longrightarrow & \tilde{\mathbf{H}}^*_G(C';R) & \longrightarrow & \mathbf{H}^*_G(X'_{s+1};R) & \longrightarrow & \mathbf{H}^*_G(X_s;R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.6}$$

with exact rows and columns. Note that the long exact sequence which should have appeared in the middle column of this diagram, in fact, reduces to a short exact sequence. This can be proven, for example, by applying the 5-Lemma to the bottom two rows to show that q'_2 is an epimorphism. Clearly, the left column is the short exact sequence of change of coefficients maps associated with the projective module $\tilde{\mathbf{H}}^*_G(C')$. By the induction hypothesis, the right column is isomorphic to the short exact sequence of change of coefficients maps associated with the projective module $\overline{Q}_s = \mathbf{H}^*_G(X_s)$. Hence, Lemma C.9 tells us that the middle column splits as a sum of the left and right columns. Moreover, under the splitting isomorphisms, q'_i is identified with $c'_i \oplus q_i$, for $i = 1, 2$.

Recall that C' is a wedge sum of spaces of the form $G_+ \wedge S^{2k}$ or S^V. Our finiteness assumptions imply that, for any coefficients S, the cohomology $\tilde{\mathbf{H}}^*_G(C';S)$ is a direct sum with summands of the form $\Sigma^{-2k}T_G(S)$ or $\Sigma^{-V}T(S)$. Moreover, for any S, each free summand of $\tilde{\mathbf{H}}^*_G(C';S)$ has the form $\Sigma^{-V}T(S)$ for an actual representation V. It follows that, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, $\mathbf{H}^*_G(X'_{s+1};S)$ has the form asserted in Theorem 3.4, and the map $\mathbf{H}^*_G(X'_{s+1}) \longrightarrow \mathbf{H}^*_G(X'_{s+1};R)$ is the asserted change of coefficients

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map.

Having constructed X'_{s+1} as an intermediate step toward X_{s+1}, we also take an intermediate step toward specifying $\mathcal{M}_{s+1}$ and $\overline{\mathcal{W}}_{s+1}$. In fact, since C' contains all of the orbit cells of C, we are ready to specify $\mathcal{M}_{s+1}$. However, we use $\overline{\mathcal{W}}_{s+1,0}$ to denote an intermediate step between $\overline{\mathcal{W}}_s$ and $\overline{\mathcal{W}}_{s+1}$. Define the indexed sets

$$\mathcal{M}_{s+1} = \mathcal{M}_s \cup \{m : G \times D^m \text{ is a cell of } C'\}$$

and

$$\overline{\mathcal{W}}_{s+1,0} = \overline{\mathcal{W}}_s \cup \{V : DV \text{ is a cell of } C'\}$$

Clearly, $\mathcal{M}_s \subseteq \mathcal{M}_{s+1}$ and $\overline{\mathcal{W}}_s \subseteq \overline{\mathcal{W}}_{s+1,0}$. The splitting of sequence (3.5) gives that

$$\mathbf{H}^*_G(X'_{s+1}) \cong \left(\bigoplus_{m \in \mathcal{M}_{s+1}} \Sigma^{-m} T_G \right) \oplus \left(\bigoplus_{\omega \in \overline{\mathcal{W}}_{s+1,0}} \Sigma^{-\omega} T \right). \tag{3.7}$$

Also observe that, for $S = A$, R, and $\langle \mathbb{Z} \rangle$, the diagram

$$\begin{array}{ccc} \mathbf{H}^*_G(X'_{s+1}; S) & \longrightarrow & \mathbf{H}^*_G(X_s; S) \\ \downarrow & & \downarrow \\ Q_s(S) & \xrightarrow{id} & Q_s(S) \end{array} \tag{3.8}$$

commutes, where the vertical maps are the obvious projections.

To form X_{s+1} from X'_{s+1}, all of the cells of C not in C' are attached. Since our finiteness assumptions guarantee that X_{s+1}/X'_{s+1} contains only countably many cells, X_{s+1} can be formed from X'_{s+1} by a sequence of single cell attachments. This process is described by a filtration

$$X'_{s+1} = X'_{s+1,0} \longrightarrow X'_{s+1,1} \longrightarrow \cdots \longrightarrow X'_{s+1,t} \longrightarrow \cdots, \tag{3.9}$$

where, for each $t \geq 0$, $X'_{s+1,t+1}$ is formed from $X'_{s+1,t}$ by attaching a single new representation cell. Of course, this sequence may terminate after finitely many steps. However, in general, X_{s+1} is the colimit.

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The conclusions of Theorem 3.4 are shown to hold for each space $X'_{s+1,t}$ by induction on t. Our aim is to then apply Proposition D.1 to compute $\lim_t \mathbf{H}^*_G(X'_{s+1,t})$. For each integer $t \geq 0$, define

$$m_{s+1,t} = min\{|V^G| : DV \text{ is a cell of } X_{s+1} \text{ added after } X'_{s+1,t}\}$$

Clearly $m_s \leq m_{s+1,0} \leq m_{s+1,1} \leq \ldots \leq m_{s+1,t} \leq \ldots \leq \infty$. By our finiteness assumptions, $\lim_{t\to\infty} m_{s+1,t} = \infty$.

Inductively, we show that

$$\mathbf{H}^*_G(X'_{s+1,t}) \cong \left(\bigoplus_{m\in\mathcal{M}_{s+1}} \Sigma^{-m}T_G\right) \oplus \left(\bigoplus_{\omega\in\overline{\mathcal{W}}_{s+1,t}} \Sigma^{-\omega}T\right),$$

where $\overline{\mathcal{W}}_{s+1,t}$ is an indexed set of even representation-like elements of $RO(G)$ such that $\overline{\mathcal{W}}_{s+1,t}$ is in one-to-one correspondence with the set of representation cells of $X'_{s+1,t}$. For each $t \geq 0$, let

$$\begin{aligned}
&\mathcal{M}_{s+1,t} = \mathcal{M}_{s+1} \\
&\mathcal{W}_{s+1,t} = \{\omega \in \overline{\mathcal{W}}_{s+1,t} : |\omega^G| < m_{s+1,t}\}, \quad \text{and} \\
&Q_{s+1,t} = \left(\oplus_{m\in\mathcal{M}_{s+1}} \Sigma^{-m}T_G\right) \oplus \left(\oplus_{\omega\in\mathcal{W}_{s+1,t}} \Sigma^{-\omega}T\right).
\end{aligned}$$

Inductively, we show that, for each $t \geq 0$, $\mathcal{W}_{s+1,t} \subseteq \mathcal{W}_{s+1,t+1}$ and, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, the obvious diagram

$$\begin{array}{ccc}
\mathbf{H}^*_G(X'_{s+1,t+1}; S) & \longrightarrow & \mathbf{H}^*_G(X'_{s+1,t}; S) \\
\downarrow & & \downarrow \\
Q_{s+1,t+1}(S) & \longrightarrow & Q_{s+1,t}(S)
\end{array} \tag{3.10}$$

commutes. Ultimately, we apply Proposition D.1 for the indexed sets

$$\mathcal{M}_{s+1} = \bigcup_{t\geq 0} \mathcal{M}_{s+1,t} \quad \text{and} \quad \overline{\mathcal{W}}_{s+1} = \bigcup_{t\geq 0} \mathcal{W}_{s+1,t} \ ,$$

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the sequence $\{m_{s+1,t}\}_{t\geq 0}$, the indexed sets

$$\mathcal{W}'_{s+1,t} = \{\omega \in \overline{\mathcal{W}}_{s+1,t} : |\omega^G| \geq m_{s+1,t}\},$$

and the sequence of T-modules given by $\overline{Q}_{s+1,t} = \mathbf{H}^*_G(X'_{s+1,t})$.

The case $t = 0$ has already been handled, since $X'_{s+1,0} = X'_{s+1}$. This was the reason for the choice of notation $\overline{\mathcal{W}}_{s+1,0}$ in equation (3.7). Theorem 3.5 is used to argue the inductive step. Specifically, suppose that $X'_{s+1,t+1}/X'_{s+1,t} = S^V$. Apply Theorem 3.5 for $B = X'_{s+1,t}$, $Y = X'_{s+1,t+1}$, $P = \bigoplus_{m\in\mathcal{M}_{s+1}} \Sigma^{-m}T_G$,

$$F_1 = \bigoplus_{\substack{\omega\in\overline{\mathcal{W}}_{s+1,t} \\ \text{such that } |\omega^G|<|V^G|}} \Sigma^{-\omega}T,$$

$$F_2 = \bigoplus_{\substack{\omega\in\overline{\mathcal{W}}_{s+1,t} \\ \text{such that } |\omega^G|\geq|V^G|}} \Sigma^{-\omega}T,$$

and $F = F_1 \oplus F_2$. Define $\overline{\mathcal{W}}_{s+1,t+1}$ as the indexing set in the free module

$$F' = \bigoplus_{\omega\in\overline{\mathcal{W}}_{s+1,t+1}} \Sigma^{-\omega}T = F_1 \oplus F'_2$$

given by the conclusions of Theorem 3.5. For $S = A$, R, and $\langle\mathbb{Z}\rangle$, the commutativity of the diagram

$$\begin{array}{ccccccc} P(S)\oplus F_1(S)\oplus F'_2(S) & \xrightarrow[\cong]{\Psi'} & \mathbf{H}^*_G(Y;S) & \xrightarrow{i^*} & \mathbf{H}^*_G(B;S) & \xrightarrow[\cong]{\Psi^{-1}} & P(S)\oplus F_1(S)\oplus F_2(S) \\ \downarrow{\scriptstyle \pi_{1,2}} & & & & & & \downarrow{\scriptstyle \pi_{1,2}} \\ P(S)\oplus F_1(S) & & & \xrightarrow{\quad id \quad} & & & P(S)\oplus F_1(S) \end{array}$$

allows us to select $\overline{\mathcal{W}}_{s+1,t+1}$ so that $\mathcal{W}_{s+1,t} \subseteq \overline{\mathcal{W}}_{s+1,t+1}$. Since $m_{s+1,t} \leq m_{s+1,t+1}$, it follows that

$$\mathcal{W}_{s+1,t} \subseteq \mathcal{W}_{s+1,t+1} \subseteq \overline{\mathcal{W}}_{s+1,t+1}.$$

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Moreover, diagram (3.10) commutes. Of course, the conclusions of Theorem 3.5 give that the conclusions of Theorem 3.4 hold for $X'_{s+1,t+1}$.

The space X_{s+1} is the colimit of filtration (3.9) and each inclusion

$$X'_{s+1,t} \subseteq X'_{s+1,t+1}$$

is a cofibration. Hence, for any coefficients S, the standard Milnor short exact sequence takes the form

$$0 \longrightarrow \lim_t{}^1(\Sigma^{-1}\mathbf{H}^*_G(X'_{s+1,t}; S)) \longrightarrow \mathbf{H}^*_G(X_{s+1}; S) \longrightarrow \lim_t \mathbf{H}^*_G(X'_{s+1,t}; S) \longrightarrow 0. \quad (3.11)$$

Moreover, the short exact sequence

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \longrightarrow 0$$

induces a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim_t{}^1\, \Sigma^{-1}\mathbf{H}^*_G(X'_{s+1,t}; \langle \mathbb{Z} \rangle) & \longrightarrow & \mathbf{H}^*_G(X_{s+1}; \langle \mathbb{Z} \rangle) & \longrightarrow & \lim_t \mathbf{H}^*_G(X'_{s+1,t}; \langle \mathbb{Z} \rangle) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim_t{}^1\, \Sigma^{-1}\mathbf{H}^*_G(X'_{s+1,t}) & \longrightarrow & \mathbf{H}^*_G(X_{s+1}) & \longrightarrow & \lim_t \mathbf{H}^*_G(X'_{s+1,t}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim_t{}^1\, \Sigma^{-1}\mathbf{H}^*_G(X'_{s+1,t}; R) & \longrightarrow & \mathbf{H}^*_G(X_{s+1}; R) & \longrightarrow & \lim_t \mathbf{H}^*_G(X'_{s+1,t}; R) & \longrightarrow & 0
\end{array}$$

with short exact rows. The indexed set $\mathcal{M}_{s+1}$ has already been defined. Define $\overline{\mathcal{W}}_{s+1} = \bigcup_{t \geq 0} \mathcal{W}_{s+1,t}$ and hence $\mathcal{W}_{s+1} = \{\omega \in \overline{\mathcal{W}}_{s+1} : |\omega^G| < m_{s+1}\}$. For each $t \geq 0$, let $\overline{Q}_{s+1,t} = \mathbf{H}^*_G(X'_{s+1,t})$, and let

$$\overline{Q}_{s+1} = \left(\bigoplus_{m \in \mathcal{M}_{s+1}} \Sigma^{-m} T_G \right) \oplus \left(\bigoplus_{\omega \in \overline{\mathcal{W}}_{s+1}} \Sigma^{-\omega} T \right).$$

Proposition D.1 gives that the $\lim^1$ terms in sequence (3.11) are zero, and, for $S = A$, R, and $\langle \mathbb{Z} \rangle$,

$$\mathbf{H}^*_G(X_{s+1}; S) \cong \lim_t \mathbf{H}^*_G(X'_{s+1,t}; S) \cong \overline{Q}_{s+1}(S).$$

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Moreover, the sequence

$$0 \longrightarrow \mathbf{H}_G^*(X_{s+1}; \langle \mathbb{Z} \rangle) \longrightarrow \mathbf{H}_G^*(X_{s+1}) \longrightarrow \mathbf{H}_G^*(X_{s+1}; R) \longrightarrow 0$$

is isomorphic to the short exact sequence

$$0 \longrightarrow \overline{Q}_{s+1}(\langle \mathbb{Z} \rangle) \longrightarrow \overline{Q}_{s+1} \longrightarrow \overline{Q}_{s+1}(R) \longrightarrow 0$$

of change of coefficients maps.

Observe that Q_{s+1} is a summand of $\overline{Q}_{s+1}$ since $\mathcal{W}_{s+1} \subseteq \overline{\mathcal{W}}_{s+1}$. Since $m_s \leq m_{s+1,0}$, it follows that $\mathcal{W}_s \subseteq \mathcal{W}_{s+1,0} \subseteq \mathcal{W}_{s+1}$. Hence, Q_s is a summand of $Q_{s+1,0}$. In fact, for $S = A$, R, and $\langle \mathbb{Z} \rangle$ and for each $t \geq 0$, the diagram

$$\begin{array}{ccc} \mathbf{H}_G^*(X'_{s+1,t+1}; S) & \longrightarrow & \mathbf{H}_G^*(X'_{s+1,t}; S) \\ \downarrow & & \downarrow \\ Q_{s+1,t+1}(S) & \longrightarrow & Q_{s+1,t}(S) \\ \downarrow & & \downarrow \\ Q_{s+1,0}(S) & \xrightarrow{id} & Q_{s+1,0}(S) \\ \downarrow & & \downarrow \\ Q_s(S) & \xrightarrow{id} & Q_s(S) \end{array} \tag{3.12}$$

commutes. Hence, so does

$$\begin{array}{ccc} \mathbf{H}_G^*(X_{s+1}; S) & \longrightarrow & \mathbf{H}_G^*(X'_{s+1}; S) \\ \downarrow & & \downarrow \\ Q_{s+1}(S) & \longrightarrow & Q_s(S). \end{array} \tag{3.13}$$

Combining diagrams (3.13) and (3.8) gives that diagram (3.3) commutes.

It has now been established, by induction on s, that the conclusions Theorem 3.4 hold for each space X_s in our filtration

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \ldots \longrightarrow X_s \xrightarrow{i_s} \ldots$$

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for X. The conclusions of Theorem 3.4 are seen to hold for X by using arguments similar to those given for the passage from X_s to X_{s+1}. Here, for any coefficients S, the relevant Milnor short exact sequence is

$$0 \longrightarrow \lim_s{}^1(\Sigma^{-1}\mathbf{H}^*_G(X_s;S)) \longrightarrow \mathbf{H}^*_G(X;S) \longrightarrow \lim_s \mathbf{H}^*_G(X_s;S) \longrightarrow 0.$$

Proposition D.1 gives that the $\lim^1$ terms vanish, and, for $S = A$, R, and $\langle\mathbb{Z}\rangle$,

$$\mathbf{H}^*_G(X;S) \cong \lim_s \mathbf{H}^*_G(X_s;S) \cong Q(S),$$

where

$$Q = \left(\bigoplus_{m\in\mathcal{M}} \Sigma^{-m}T_G\right) \oplus \left(\bigoplus_{\omega\in\mathcal{W}} \Sigma^{-\omega}T\right).$$

Moreover, the sequence

$$0 \longrightarrow \mathbf{H}^*_G(X;\langle\mathbb{Z}\rangle) \longrightarrow \mathbf{H}^*_G(X) \longrightarrow \mathbf{H}^*_G(X;R) \longrightarrow 0$$

is isomorphic to the short exact sequence

$$0 \longrightarrow Q(\langle\mathbb{Z}\rangle) \longrightarrow Q \longrightarrow Q(R) \longrightarrow 0$$

of change of coefficients maps. Hence, the conclusions of Theorem 3.4 hold for X.

3.2 The Proof of Theorem 3.5

Let Y be a G-space formed by attaching an even dimensional representation cell DV to B. For $S = A$, R, and $\langle\mathbb{Z}\rangle$, let $N(S) = \Sigma^{-V}T(S)$. The cell attachment is reflected in a long exact sequence

$$\longrightarrow N \longrightarrow \mathbf{H}^*_G(Y) \xrightarrow{i^*} \mathbf{H}^*_G(B) \xrightarrow{\partial} \Sigma N \longrightarrow .$$

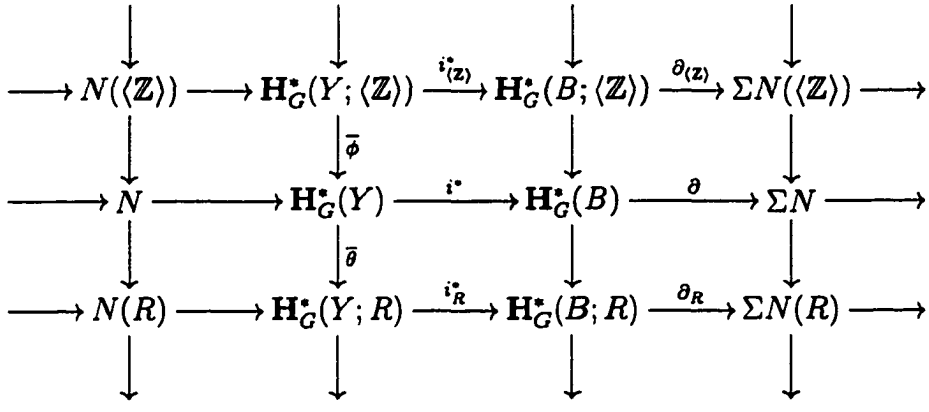
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Moreover, the short exact sequence of coefficients

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \dashrightarrow 0$$

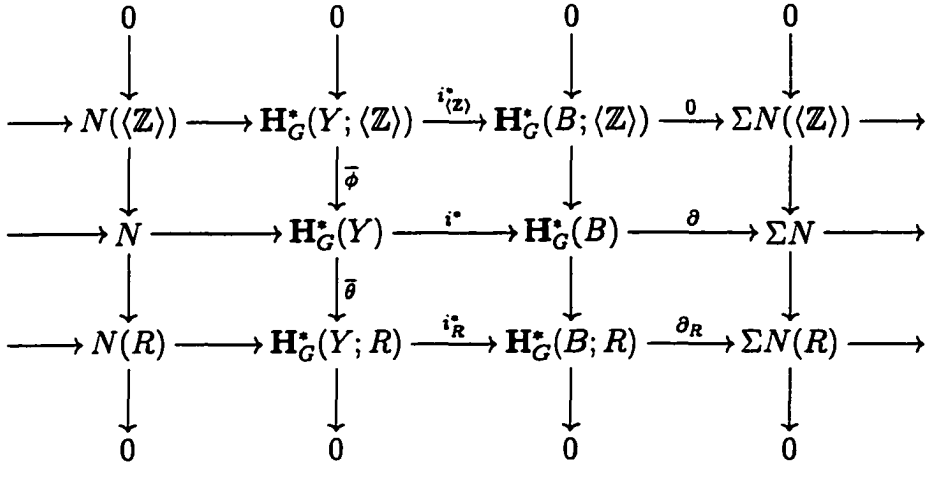
yields a commuting diagram

(3.14)



with long exact rows and columns. Note that $\partial_{\langle \mathbb{Z} \rangle} = 0$, since the generators of $\mathbf{H}^*_G(B; \langle \mathbb{Z} \rangle)$ are in even dimensions and $\Sigma N(\langle \mathbb{Z} \rangle)$ is zero in even dimensions. Hence, $\mathbf{H}^*_G(Y; \langle \mathbb{Z} \rangle) \cong N(\langle \mathbb{Z} \rangle) \oplus \mathbf{H}^*_G(B; \langle \mathbb{Z} \rangle)$ is a free $T(\langle \mathbb{Z} \rangle)$-module. Since $\mathbf{H}^*_G(Y; R)$ is an R-module in every dimension and $\Sigma \mathbf{H}^*_G(Y; \langle \mathbb{Z} \rangle)(G/G)$ is a free abelian group in every dimension, Lemma A.8 tells us that the boundary map $\mathbf{H}^*_G(Y; R) \longrightarrow \Sigma \mathbf{H}^*_G(Y; \langle \mathbb{Z} \rangle)$ associated with the second column of diagram (3.14) is zero. It follows that diagram (3.14) reduces to the diagram

(3.15)



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with long exact rows and short exact columns.

Recall that, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, $\mathbf{H}^*_G(B;S) \cong P(S) \oplus F(S)$, where P is a standard projective T-module and F is a free T-module. Note that $\partial_S|_{P(S)} = 0$. However, it is possible that ∂ and ∂_R are nonzero on F and $F(R)$, respectively. The goal of the proof is to split $\mathbf{H}^*_G(B)$ into two summands. One summand is a finitely generated free T-module such that ∂ is nonzero on each generator. The complementary summand sits inside the kernel of ∂. This splitting of $\mathbf{H}^*_G(B)$ allows us to analogously split diagram (3.15) as a sum of two diagrams. We then analyze the diagram corresponding to the nonzero part of ∂ to complete the proof.

We start by giving, in Subsection 3.2.1, an outline of the major steps. There, the proof is broken up into a sequence of propositions and other theorems. A couple of the propositions have particularly short proofs, which are given directly after their statements. However, the theorems and most of the propositions have more involved proofs, which are delayed to later sections and subsections.

3.2.1 Overview of the Proof of Theorem 3.5

For $S = A$, R, and $\langle\mathbb{Z}\rangle$, let $\Psi_S : P(S) \oplus F(S) \longrightarrow \mathbf{H}^*_G(B;S)$ be the isomorphism given in the hypotheses of the theorem. Our first goal is to rewrite $\partial : \mathbf{H}^*_G(B) \longrightarrow \Sigma N$ in a very specific form that isolates the components of $\mathbf{H}^*_G(B)$ on which it is nontrivial. The following proposition is applied for $H = P \oplus F$ and the map

$$H \overset{\Psi}{\cong} \mathbf{H}^*_G(B) \xrightarrow{\partial} \Sigma N.$$

Proposition 3.7. *Let $P = \bigoplus_{m\in\mathcal{M}} \Sigma^{-m} T_G$ be a standard projective module with generators in even dimensions. Let $N = \Sigma^{-V} T$ and $F = \bigoplus_{\omega\in\mathcal{W}} \Sigma^{-\omega} T$ be free modules with generators in even representation-like dimensions. Let $H = P \oplus F$, and let $\partial : H \longrightarrow \Sigma N$ be a map. Let $\mathcal{W}_0 = \{\omega \in \mathcal{W} : \partial|_{\Sigma^{-\omega}T} = 0\}$ and $F_0 = \bigoplus_{\omega\in\mathcal{W}_0} \Sigma^{-\omega} T$.*

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Then, there exist finitely many elements $\omega_1, \ldots, \omega_n$ *of* $\mathcal{W} - \mathcal{W}_0$ *and an isomorphism* $\tilde{f} : H \longrightarrow H$ *such that*

(i) $|\omega_j| < |\omega_{j+1}|$ *and* $|\omega_j^G| < |\omega_{j+1}^G|$ *for each* $1 \leq j \leq n-1$;

(ii) every $\omega \in \mathcal{W} - \mathcal{W}_0$ *must satisfy* $|\omega| \geq |\omega_j|$ *and* $|\omega^G| \leq |\omega_j^G|$ *for some* $1 \leq j \leq n$; *and*

(iii) if $J = \bigoplus_{j=1}^{n} \Sigma^{-\omega_j} T$, $J' = \bigoplus_{\omega \in \mathcal{W} - \mathcal{W}_0 - \{\omega_j\}} \Sigma^{-\omega} T$, $\bar{J} = J \oplus J'$, *and* $\tilde{\partial} = \partial|_J$, *then* $\tilde{f}$ *decomposes as a sum*

$$id \oplus id \oplus f : P \oplus F_0 \oplus \bar{J} \longrightarrow P \oplus F_0 \oplus \bar{J}$$

and the composite $\partial \circ \tilde{f}$ *is just*

$$P \oplus F_0 \oplus J' \oplus J \xrightarrow{\pi_4} J \xrightarrow{\tilde{\partial}} \Sigma N$$

Proof. The proof is given in Subsection 3.2.2. □

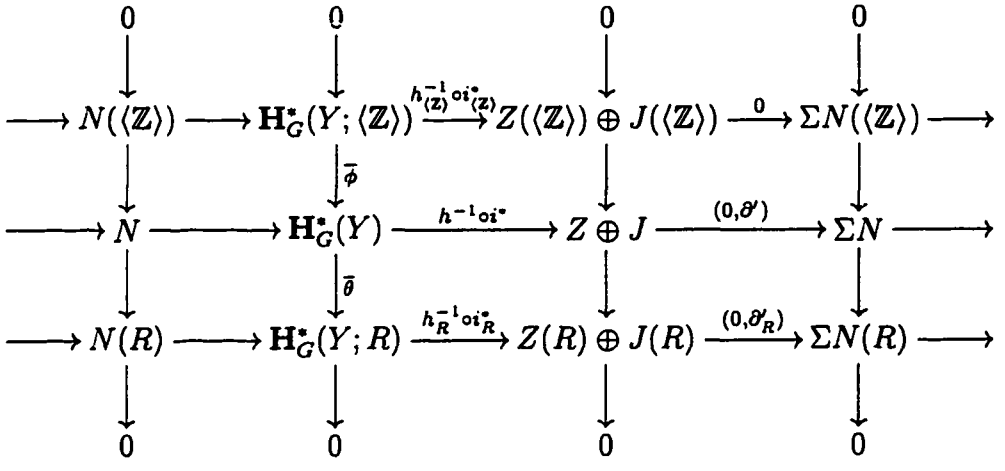
Let $\mathcal{W}_0$, $\{\omega_1, \ldots, \omega_n\}$, F_0, J, J', and $\tilde{f}$ be the data obtained by applying Proposition 3.7 to $H = P \oplus F$ and $\partial \circ \Psi : H \longrightarrow \Sigma N$. Let $Z = P \oplus F_0 \oplus J'$. Since $\tilde{f}$ is an isomorphism, so are $\tilde{f}(\langle \mathbb{Z} \rangle)$ and $\tilde{f}(R)$. Hence, the isomorphisms $h_S = \Psi_S \circ \tilde{f}(S)$, for $S = A$, R, and $\langle \mathbb{Z} \rangle$, fit into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle \mathbb{Z} \rangle) \oplus J(\langle \mathbb{Z} \rangle) & \longrightarrow & Z \oplus J & \longrightarrow & Z(R) \oplus J(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_{\langle \mathbb{Z} \rangle} \cong} & & \downarrow{\scriptstyle h = \Psi \circ \tilde{f} \cong} & & \downarrow{\scriptstyle h_R \cong} & & \\ 0 & \longrightarrow & \mathbf{H}_G^*(B; \langle \mathbb{Z} \rangle) & \longrightarrow & \mathbf{H}_G^*(B) & \longrightarrow & \mathbf{H}_G^*(B; R) & \longrightarrow & 0 \end{array} \tag{3.16}$$

with short exact rows. Here, the top row is the canonical short exact sequence of change of coefficients maps and the bottom row is the third column of diagram (3.15). For $S = A$ and R, let $\partial_S' = (\partial_S \circ h_S)|_{J(S)}$. Proposition 3.7 also gives that, under the

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isomorphisms in diagram (3.16), diagram (3.15) can be identified with the diagram



with exact rows and columns.

For $S = A$, R, and $\langle\mathbb{Z}\rangle$, the module $\mathbf{H}^*_G(B;S)$ has been split as a sum $Z(S)\oplus J(S)$. Our aim is to analogously split $\mathbf{H}^*_G(Y;S)$ as a sum of $Z(S)$ and another module. Since the rows of diagram (3.14) are exact and $\partial_S|_{Z(S)} = 0$, it follows that, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, the composite

$$q_S : \mathbf{H}^*_G(Y;S) \xrightarrow{i^*_S} \mathbf{H}^*_G(B;S) \xrightarrow{h_S^{-1}} Z(S)\oplus J(S) \xrightarrow{\pi_1} Z(S) \tag{3.17}$$

is an epimorphism.

Definition 3.8. For $S = A$, R, and $\langle\mathbb{Z}\rangle$, let M_S denote the kernel of the map

$$q_S : \mathbf{H}^*_G(Y;S) \longrightarrow Z(S)$$

in equation (3.17). Let $u_S : M_S \longrightarrow \mathbf{H}^*_G(Y;S)$ be the inclusion of the kernel of q_s. For convenience, M_A, u_A, and q_A are denoted M, u, and q, respectively. Note that the short exact sequence

$$0 \longrightarrow \mathbf{H}^*_G(Y;\langle\mathbb{Z}\rangle) \xrightarrow{\bar{\phi}} \mathbf{H}^*_G(Y) \xrightarrow{\bar{\theta}} \mathbf{H}^*_G(Y;R) \longrightarrow 0$$

restricts to give a short exact sequence

$$0 \longrightarrow M_{\langle\mathbf{Z}\rangle} \xrightarrow{\phi} M \xrightarrow{\theta} M_R \longrightarrow 0.$$

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Since Z is a projective T-module, the map $q : \mathbf{H}^*_G(Y) \longrightarrow Z$ splits.

Definition 3.9. Let $r : Z \longrightarrow \mathbf{H}^*_G(Y)$ be a splitting of the map $q : \mathbf{H}^*_G(Y) \longrightarrow Z$ in equation (3.17). Note that Lemma 3.3 gives us unique maps $r_{\langle \mathbb{Z} \rangle}$ and r_R that fit into the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle \mathbb{Z} \rangle) & \longrightarrow & Z & \longrightarrow & Z(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r_{\langle \mathbb{Z} \rangle}} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_R} & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(Y;\langle \mathbb{Z} \rangle) & \xrightarrow{\bar{\phi}} & \mathbf{H}^*_G(Y) & \xrightarrow{\bar{\theta}} & \mathbf{H}^*_G(Y;R) & \longrightarrow & 0 \end{array}$$

with exact rows. Here, the top row is the canonical short exact sequence of change of coefficients maps.

The next three propositions show how r can be used to obtain splittings of the module $\mathbf{H}^*_G(Y)$, the short exact sequence

$$0 \longrightarrow \mathbf{H}^*_G(Y;\langle \mathbb{Z} \rangle) \longrightarrow \mathbf{H}^*_G(Y) \longrightarrow \mathbf{H}^*_G(Y;R) \longrightarrow 0,$$

and the map $i^* : \mathbf{H}^*_G(Y) \longrightarrow \mathbf{H}^*_G(B)$.

Proposition 3.10. *The maps r_S in Definition 3.9 give a splitting of the diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_{\langle \mathbb{Z} \rangle} & \xrightarrow{u_{\langle \mathbb{Z} \rangle}} & \mathbf{H}^*_G(Y;\langle \mathbb{Z} \rangle) & \xrightarrow{q_{\langle \mathbb{Z} \rangle}} & Z(\langle \mathbb{Z} \rangle) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \bar{\phi}} & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{u} & \mathbf{H}^*_G(Y) & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \bar{\theta}} & & \downarrow & & \\ 0 & \longrightarrow & M_R & \xrightarrow{u_R} & \mathbf{H}^*_G(Y;R) & \xrightarrow{q_R} & Z(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

in which the rows and columns and columns are exact. Here, the right vertical maps are the canonical change of coefficients maps.

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Proof. This is an immediate consequence of Lemma C.9. □

Proposition 3.11. **(a)** *For $S = A$, R, and $\langle\mathbb{Z}\rangle$, there is a diagram*

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N(S) & \xrightarrow{f_S} & M_S & \xrightarrow{g_S} & J(S) & \xrightarrow{\partial'_S} & \Sigma N(S) & \longrightarrow \\
 & \| & & \downarrow{\scriptstyle u_S} & & \downarrow & & \| & \\
\longrightarrow & N(S) & \longrightarrow & \mathbf{H}^*_G(Y;S) & \xrightarrow{i^*_S} & \mathbf{H}^*_G(B;S) & \xrightarrow{\partial_S} & \Sigma N(S) & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle \pi_1 \circ h_S^{-1}} & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & Z(S) & = & Z(S) & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0, &
\end{array} \tag{3.18}$$

with exact rows and columns, in which the maps

$$f_S : N(S) \longrightarrow M_S \quad \text{and} \quad g_S : M_S \longrightarrow J(S)$$

are the obvious induced maps between kernels.

(b) *The T-modules M_S of Definition 3.8 fit into a diagram*

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N(\langle\mathbb{Z}\rangle) & \xrightarrow{f_{\langle\mathbb{Z}\rangle}} & M_{\langle\mathbb{Z}\rangle} & \xrightarrow{g_{\langle\mathbb{Z}\rangle}} & J(\langle\mathbb{Z}\rangle) & \xrightarrow{0} & \Sigma N(\langle\mathbb{Z}\rangle) & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow & \\
\longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J & \xrightarrow{\partial'} & \Sigma N & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \downarrow & \\
\longrightarrow & N(R) & \xrightarrow{f_R} & M_R & \xrightarrow{g_R} & J(R) & \xrightarrow{\partial'_R} & \Sigma N(R) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array} \tag{3.19}$$

with exact rows and columns. Here, the unlabeled vertical maps are the obvious change of coefficients maps.

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Proof. The proof is given in Subsection 3.2.3. □

Recall that $Z = P \oplus F_0 \oplus J'$ and the map $\Psi_S : Z(S) \oplus J(S) \longrightarrow \mathbf{H}^*_G(B; S)$ is an isomorphism for $S = A$, R, and $\langle \mathbb{Z} \rangle$.

Proposition 3.12. *For $S = A$, R, and $\langle \mathbb{Z} \rangle$, let*

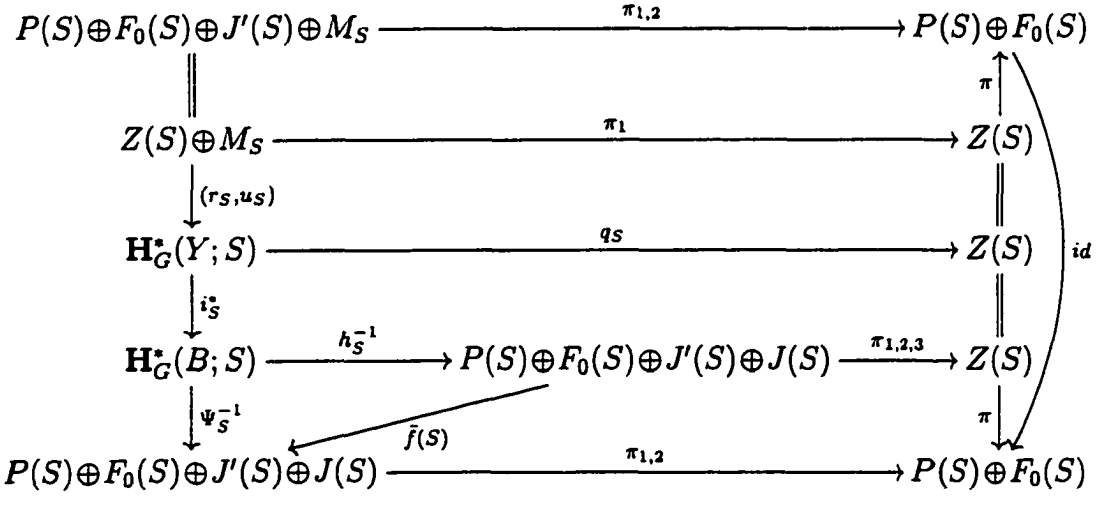
$$\tilde{\Psi}'_S = (r_S, u_S) : Z(S) \oplus M_S \longrightarrow \mathbf{H}^*_G(Y; S).$$

Then, the diagram

$$\begin{array}{ccc}
\mathbf{H}^*_G(Y;S) & \xrightarrow{\quad i^*_S \quad} & \mathbf{H}^*_G(B;S) \\
{\scriptstyle \tilde{\Psi}'_S}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Psi_S^{-1}} \\
P(S) \oplus F_0(S) \oplus J'(S) \oplus M_S & & P(S) \oplus F_0(S) \oplus J'(S) \oplus J(S) \\
{\scriptstyle \pi_{1,2}}\downarrow & & {\scriptstyle \pi_{1,2}}\downarrow \\
P(S) \oplus F_0(S) & \xrightarrow{\quad id \quad} & P(S) \oplus F_0(S)
\end{array}$$

commutes.

Proof. Recall that $\bar{J} = J \oplus J'$, the map $\tilde{f} : P \oplus F_0 \oplus \bar{J} \longrightarrow P \oplus F_0 \oplus \bar{J}$ has the form $id \oplus id \oplus f$, and $h = \Psi \circ \tilde{f}$. Let $\pi : Z \longrightarrow P \oplus F_0$ denote the projection onto the first two summands of $Z = P \oplus F_0 \oplus J'$. For $S = A$, R, and $\langle \mathbb{Z} \rangle$, the diagram asserted in the proposition follows from the diagram



□

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Diagram (3.14) has now been replaced by diagram (3.19), which isolates the support of ∂ and is much easier to study. In particular, the second row of diagram (3.19) gives a long exact sequence

$$\longrightarrow N \xrightarrow{f} M \xrightarrow{g} J \xrightarrow{\partial'} \Sigma N \longrightarrow . \tag{3.20}$$

that is central in our computation of M.

Theorem 3.13. *The T-module M of Definition 3.8 is free. Moreover,*

(a) *There exist elements $V', \omega_1', \ldots, \omega_n'$ of $RO(G)$ satisfying $|V'| = |V|$; $|\omega_i'| = |\omega_i|$, for $1 \leq i \leq n$; $|(\omega_1')^G| = |V^G|$; $|(\omega_i')^G| = |\omega_{i-1}^G|$, for $2 \leq i \leq n$; and $|(V')^G| = |\omega_n^G|$, such that*

$$M \cong \Sigma^{-V'}T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'}T. \tag{3.21}$$

(b) *If we let $M' = \Sigma^{-V'}T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'}T$, then there are isomorphisms*

$$M'(\langle \mathbb{Z} \rangle) \cong M_{\langle \mathbb{Z} \rangle} \quad \text{and} \quad M'(R) \cong M_R$$

making the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'(\langle \mathbb{Z} \rangle) & \longrightarrow & M' & \longrightarrow & M'(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_{\langle \mathbb{Z} \rangle}}\cong & & \downarrow{\scriptstyle \iota}\cong & & \downarrow{\scriptstyle \iota_R}\cong & & \\ 0 & \longrightarrow & M_{\langle \mathbb{Z} \rangle} & \xrightarrow{\phi} & M & \xrightarrow{\theta} & M_R & \longrightarrow & 0 \end{array}$$

commute. Here, the maps on the top row are the canonical change of coefficients maps.

Before giving the proof of Theorem 3.13, we show that Theorem 3.13 and the earlier propositions can by used to prove Theorem 3.5.

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The proof of Theorem 3.5. For $S = A$, R, and $\langle\mathbb{Z}\rangle$, let

$$\tilde{\Psi}'_S = (r_S, u_S) : Z(S) \oplus M_S \longrightarrow \mathbf{H}^*_G(Y; S)$$

By Proposition 3.10, the maps $\tilde{\Psi}'_S$ are isomorphisms, and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) \oplus M_{\langle\mathbb{Z}\rangle} & \longrightarrow & Z \oplus M & \longrightarrow & Z(R) \oplus M_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{\Psi}'_{\langle\mathbb{Z}\rangle} \cong} & & \downarrow{\scriptstyle \tilde{\Psi}' \cong} & & \downarrow{\scriptstyle \tilde{\Psi}'_R \cong} & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(Y; \langle\mathbb{Z}\rangle) & \xrightarrow{\bar{\phi}} & \mathbf{H}^*_G(Y) & \xrightarrow{\bar{\theta}} & \mathbf{H}^*_G(Y; R) & \longrightarrow & 0 \end{array} \tag{3.22}$$

commutes. Recall that $Z = P \oplus F_0 \oplus J'$ and $F = F_0 \oplus J' \oplus J$. Theorem 3.13 gives a free T-module M' with $Rank(M') = 1 + Rank(J)$. Moreover, it gives, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, an isomorphism $l_S : M'(S) \longrightarrow M_S$ such that the digram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) \oplus M'(\langle\mathbb{Z}\rangle) & \longrightarrow & Z \oplus M' & \longrightarrow & Z(R) \oplus M'(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id \oplus l_{\langle\mathbb{Z}\rangle}} & & \downarrow{\scriptstyle id \oplus l} & & \downarrow{\scriptstyle id \oplus l_R} & & \\ 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) \oplus M_{\langle\mathbb{Z}\rangle} & \longrightarrow & Z \oplus M & \longrightarrow & Z(R) \oplus M_R & \longrightarrow & 0 \end{array} \tag{3.23}$$

commutes. Here the top row is the canonical short exact sequence of change of coefficients maps, and the bottom row is the sum of the canonical short exact sequence of change of coefficient maps associated with Z and the sequence

$$0 \longrightarrow M_{\langle\mathbb{Z}\rangle} \xrightarrow{\phi} M \xrightarrow{\theta} M_R \longrightarrow 0.$$

Let $F' = F_0 \oplus J' \oplus M'$. Hence, $Rank(F') = 1 + Rank(F)$. For $S = A$, R, and $\langle\mathbb{Z}\rangle$, let $\Psi'_S = \tilde{\Psi}'_S \circ (id \oplus l_S)$. Since $P \oplus F' = Z \oplus M'$, diagram (3.22) and diagram (3.23) together give the canonical diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(\langle\mathbb{Z}\rangle) \oplus F'(\langle\mathbb{Z}\rangle) & \longrightarrow & P \oplus F' & \longrightarrow & P(R) \oplus F'(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi'_{\langle\mathbb{Z}\rangle} \cong} & & \downarrow{\scriptstyle \Psi' \cong} & & \downarrow{\scriptstyle \Psi'_R \cong} & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(Y; \langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(Y) & \longrightarrow & \mathbf{H}^*_G(Y; R) & \longrightarrow & 0 \end{array}$$

asserted in Theorem 3.5.

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Let F_1 be the sum of all of the summands $\Sigma^{-\omega}T$ of F such that $|\omega^G| < |V^G|$, and write $F = F_1 \oplus F_2$. Observe that F_1 is a summand of F_0. Proposition 3.12 gives that, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, the top half of the diagram

$$\begin{array}{ccccccc} Z(S) \oplus M_S & \xrightarrow[\cong]{\tilde{\Psi}'_S} & \mathbf{H}^*_G(Y;S) & \xrightarrow{i^*_S} & \mathbf{H}^*_G(B;S) & \xrightarrow[\cong]{\Psi_S^{-1}} & P(S) \oplus F(S) \\ \downarrow & & & & & & \downarrow \\ P(S) \oplus F_0(S) & & \xrightarrow{\quad id \quad} & & & & P(S) \oplus F_0(S) \\ \downarrow & & & & & & \downarrow \\ P(S) \oplus F_1(S) & & \xrightarrow{\quad id \quad} & & & & P(S) \oplus F_1(S) \end{array} \tag{3.24}$$

commutes. Here, the vertical maps are the obvious projections onto a summand. Of course, the bottom half of diagram (3.24) commutes. It is clear that the diagram

$$\begin{array}{ccccc} & & \overset{\Psi'_S}{\frown} & & \\ Z(S) \oplus M'(S) & \xrightarrow{id \oplus i_S} & Z(S) \oplus M_S & \xrightarrow{\tilde{\Psi}'_S} & \mathbf{H}^*_G(Y;S) \\ \downarrow & & \downarrow & & \\ Z(S) & = & Z(S) & & \\ \downarrow & & \downarrow & & \\ P(S) \oplus F_1(S) & = & P(S) \oplus F_1(S) & & \end{array} \tag{3.25}$$

commutes, where the vertical maps are the obvious projections. Combining diagrams (3.24) and (3.25) gives, for $S = A$, R, and $\langle\mathbb{Z}\rangle$, the diagram

$$\begin{array}{ccccccc} P(S) \oplus F'(S) & \xrightarrow[\cong]{\Psi'_S} & \mathbf{H}^*_G(Y;S) & \xrightarrow{i^*_S} & \mathbf{H}^*_G(B;S) & \xrightarrow[\cong]{\Psi_S^{-1}} & P(S) \oplus F(S) \\ \downarrow & & & & & & \downarrow \\ P(S) \oplus F_1(S) & & \xrightarrow{\quad id \quad} & & & & P(S) \oplus F_1(S) \end{array}$$

asserted in Theorem 3.5. □

The proof of Theorem 3.13 is accomplished in three steps.

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Step 1. If M satisfies equation (3.21) in Theorem 3.13(a), then, in the dimension ($\omega_1', \ldots, \omega_n'$, or V') of each generator, M consists only of the A associated to that generator. This separation of the generators forces the structure of the maps in the short exact sequence

$$0 \longrightarrow M_{\langle \mathbf{Z} \rangle} \xrightarrow{\phi} M \xrightarrow{\theta} M_R \longrightarrow 0$$

in Theorem 3.13(b). Hence, our first step in the proof of Theorem 3.13 is to reduce our work by showing that Theorem 3.13(a) implies that Theorem 3.13(b) holds.

Proposition 3.14. *Suppose that there exist elements $V', \omega_1', \ldots, \omega_n'$ of $RO(G)$ satisfying $|V'| = |V|$; $|\omega_i'| = |\omega_i|$, for $1 \leq i \leq n$; $|(\omega_1')^G| = |V^G|$; $|(\omega_i')^G| = |\omega_{i-1}^G|$, for $2 \leq i \leq n$; and $|(V')^G| = |\omega_n^G|$, such that*

$$M \cong \Sigma^{-V'} T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'} T.$$

If we let $M' = \Sigma^{-V'} T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'} T$, then there are isomorphisms $M_{\langle \mathbf{Z} \rangle} \cong M'(\langle \mathbb{Z} \rangle)$ and $M_R \cong M'(R)$ making the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'(\langle \mathbb{Z} \rangle) & \longrightarrow & M' & \longrightarrow & M'(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_{\langle \mathbf{Z} \rangle}}\cong & & \downarrow{\scriptstyle \iota}\cong & & \downarrow{\scriptstyle \iota_R}\cong & & \\ 0 & \longrightarrow & M_{\langle \mathbf{Z} \rangle} & \xrightarrow{\phi} & M & \xrightarrow{\theta} & M_R & \longrightarrow & 0 \end{array}$$

commute. Here, the maps on the top row are the canonical change of coefficients maps.

Proof. The proof is given in Subsection 3.2.4 □

Step 2. The first step has reduced our work to determining the value of M. The main tool for doing this is diagram (3.19) and, in particular, sequence (3.20). Specifically, our computation of M relies on an algebraic result.

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Theorem 3.15. *Let $\omega_1, \ldots, \omega_n$, and $\tilde{V}$ be even elements of $RO(G)$ such that*

$$|\tilde{V}^G| < |\omega_1^G| < |\omega_2^G| < \ldots < |\omega_n^G| \quad \text{and} \quad |\omega_1| < |\omega_2| < \ldots < |\omega_n| < |\tilde{V}|.$$

Let $N = \Sigma^{-\tilde{V}}T$ and $J = \bigoplus_{i=1}^{n} \Sigma^{-\omega_i}T$. Let M be a T-module for which there is a long exact sequence

$$\longrightarrow N \longrightarrow M \longrightarrow J \xrightarrow{\partial} \Sigma N \longrightarrow$$

in which $\partial : J \longrightarrow \Sigma N$ is a module map that is nontrivial on each summand of J. If there exists an element ω_1' in $RO(G)$ such that $(|(\omega_1')^G|, |\omega_1'|) = (|\tilde{V}^G|, |\omega_1|)$ and $M^{\omega_1'} = A$, then there exist elements $\tilde{V}', \omega_2', \ldots, \omega_n'$ such that $|\tilde{V}'| = |\tilde{V}|$; $|\omega_i'| = |\omega_i|$, for $1 \leq i \leq n$; $|(\omega_1')^G| = |\tilde{V}^G|$; $|(\omega_i')^G| = |\omega_{i-1}^G|$, for $2 \leq i \leq n$; $|(\tilde{V}')^G| = |\omega_n^G|$, and

$$M \cong \Sigma^{-\tilde{V}'}T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'}T$$

Proof. The proof is given in Section 3.3. □

Remark 3.16. Obviously, it is necessary for M to take on the value A in the the dimension ω_1' specified in Theorem 3.15. However, this value is not forced by the long exact sequence. Theorem 3.35 shows that, if the hypothesis $M^{\omega_1'} = A$ is removed from Theorem 3.15, then M need not be free.

Step 3. The second step has further reduced our work to computing the value of M in a key dimension. However, the constraints on M given by diagram (3.19) are not sufficient to determine this value. Our third step is to construct a diagram similar to diagram (3.19), but involving L-coefficients. The additional constraints imposed by this diagram enable us to determine the value of M in the required dimension. The key to both the proof and utility of the following proposition is the relationship between the functors $\mathbf{H}_G^*(?; L)$ and $\mathbf{H}_G^*(?; R)$ given in Proposition B.13.

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Proposition 3.17. *Let λ be an irreducible complex representation. Let*

$$h_L : Z(L) \oplus J(L) \longrightarrow \mathbf{H}^*_G(B; L)$$

be the composite

$$Z(L) \oplus J(L) \cong \Sigma^{2-\lambda} Z(R) \oplus \Sigma^{2-\lambda} J(R) \xrightarrow{\Sigma^{2-\lambda} h_R} \Sigma^{2-\lambda} \mathbf{H}^*_G(B; R) \cong \mathbf{H}^*_G(B; L),$$

where the unlabeled isomorphisms are given by Proposition B.13. Let

$$q_L : \mathbf{H}^*_G(Y; L) \longrightarrow Z(L)$$

be the composite

$$\mathbf{H}^*_G(Y; L) \xrightarrow{i^*_L} \mathbf{H}^*_G(B; L) \xrightarrow{h_L^{-1}} Z(L) \oplus J(L) \xrightarrow{\pi_1} Z(L),$$

and let $M_L = \operatorname{Ker} q_L$. Then, there is a commuting diagram

$$\begin{array}{ccccccccc}
\longrightarrow & N(L) & \xrightarrow{f_L} & M_L & \xrightarrow{g_L} & J(L) & \xrightarrow{\partial'_L} & \Sigma N(L) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J & \xrightarrow{\partial'} & \Sigma N & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N(\langle \mathbb{Z} \rangle) & \xrightarrow{f_{\langle \mathbb{Z} \rangle}} & M_{\langle \mathbb{Z} \rangle} & \xrightarrow{f_{\langle \mathbb{Z} \rangle}} & J(\langle \mathbb{Z} \rangle) & \xrightarrow{0} & \Sigma N(\langle \mathbb{Z} \rangle) & \longrightarrow
\end{array} \tag{3.26}$$

with long exact rows. The maps in the first, third, and fourth columns are the obvious change of coefficients maps. In even dimensions, each column is exact at the middle row. Moreover, the top row of this diagram is isomorphic to the suspension by $2 - \lambda$ of the sequence

$$\longrightarrow N(R) \xrightarrow{f_R} M_R \xrightarrow{g_R} J(R) \xrightarrow{\partial'_R} \Sigma N(R) \longrightarrow$$

in the bottom row of diagram (3.19).

Proof. The proof is given in Subsection 3.2.5. □

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Diagrams (3.19) and (3.26) together allow us to determine M^α for all α such that $(|\alpha^G|, |\alpha|) = (|V^G|, |\omega_1|)$. The second row of diagram (3.19) and the second column of diagram (3.26) give two short exact sequences that contain M^α. The key to computing M^α is that Lemma A.3 tells us that there is only one common solution to these two extension problems with solution M^α. Hence the second hypothesis of Theorem 3.15 is seen to hold for our module M.

Proposition 3.18. *There is some ω_1' in $RO(G)$ such that*

$$(|(\omega_1')^G|, |\omega_1'|) = (|V^G|, |\omega_1|) \quad \textit{and} \quad M^{\omega_1'} = A.$$

Proof. Let α be in $RO(G)$ with $(|\alpha^G|, |\alpha|) = (|V^G|, |\omega_1|)$, and let λ be a nontrivial complex irreducible representation. By Proposition 3.17, the top row of diagram (3.26) may be identified with the suspension by $2 - \lambda$ of the bottom row of diagram (3.19). In dimension α, this sequence reduces to the sequence

$$0 \longrightarrow M_L^\alpha \longrightarrow R \xrightarrow{\partial_R^{\alpha+2-\lambda}} \langle \mathbb{Z}/p \rangle \longrightarrow M_L^{\alpha+1} \longrightarrow 0.$$

in which $\partial_R^{\alpha+2-\lambda} \neq 0$. Lemma A.5 tell us that $M_L^\alpha = L$. In dimension α, diagram (3.26) reduces to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & L & \xrightarrow{g_L} & R & \xrightarrow{\partial_L'} & \langle \mathbb{Z}/p \rangle \\
 & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \langle \mathbb{Z} \rangle & \xrightarrow{f} & M^\alpha & \xrightarrow{g} & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \zeta} & & \downarrow & & \\
 & & \langle \mathbb{Z} \rangle & \xrightarrow{\cong} & \langle \mathbb{Z} \rangle & \longrightarrow & 0 & &
\end{array}$$

Since g_L is a monomorphism, so is η. The lower left-hand square shows that $\zeta \neq 0$. Hence, the image of ζ is isomorphic to $\langle \mathbb{Z} \rangle$. Let $\zeta' : M^\alpha \longrightarrow \langle \mathbb{Z} \rangle$ be the induced epimorphism onto the image of ζ. It follows that we have a short exact sequence

$$0 \longrightarrow L \xrightarrow{\eta} M^\alpha \xrightarrow{\zeta'} \langle \mathbb{Z} \rangle \longrightarrow 0. \tag{3.27}$$

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Also observe that the middle row of diagram (3.26) gives us the short exact sequence

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow M^{\alpha} \longrightarrow R \longrightarrow 0. \tag{3.28}$$

Lemma A.3 lists all of the possible solutions to the extension problems given by sequences (3.27) and (3.28). In fact, the only solution common to both sequences is $M^{\alpha} = A[d]$ for some integer d prime to p. Corollary C.2 therefore gives that there is some ω_1' such that $(|(\omega_1')^G|, |\omega_1'|) = (|\alpha^G|, |\alpha|)$ and $M^{\omega_1'} = A$. □

3.2.2 The Proof of Proposition 3.7

Since V is finite dimensional and the elements of $\mathcal{W}$ are in representation-like dimensions, a plot of $\mathcal{W} - \mathcal{W}_0$ must be contained within the triangle whose vertices are $(|V^G| + 2, |V| - 2)$, $(|V^G| + 2, |V^G| + 2)$, and $(|V| - 2, |V| - 2)$. Hence, it is possible to pick elements $\omega_1, \ldots, \omega_n$ of $\mathcal{W} - \mathcal{W}_0$ satisfying the assertions in Proposition 3.7. First, pick ω_1 such that $|\omega_1^G|$ is the maximum of

$$\{|\omega^G| : \omega \in \mathcal{W} - \mathcal{W}_0 \text{ and } \forall \omega' \in \mathcal{W} - \mathcal{W}_0, |\omega| \leq |\omega'|\}.$$

Then, pick ω_2 such that $|\omega_2^G|$ is the maximum of

$$\{|\omega^G| : \omega \in \mathcal{W} - \mathcal{W}_0;\ |\omega^G| > |\omega_1^G|;\ \text{and } \forall \omega' \in \mathcal{W} - \mathcal{W}_0 \text{ s.t. } |(\omega')^G| > |\omega_1^G|,\ |\omega| \leq |\omega'|\},$$

and so forth. The triangular shape of the plot of $\mathcal{W} - \mathcal{W}_0$ guarantees that this process is well defined and terminates after finitely many steps. Of course, n is taken to be the number of these steps. To construct the desired isomorphism $\tilde{f} : H \longrightarrow H$, we use of the following lemma n times.

Lemma 3.19. *Let $N = \Sigma^{-V}T$ and $\tilde{J} = \bigoplus_{\beta \in I} \Sigma^{-\beta}T$ be free modules with generators in even dimensions, and let $\partial : \tilde{J} \longrightarrow \Sigma N$ be a map which is nonzero on each generator*

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of $\tilde{J}$. Assume that there is an element α of I such that $|\beta^G| \leq |\alpha^G|$ and $|\beta| \geq |\alpha|$ for each $\beta \in I$. Let $\tilde{\partial} = \partial|_{\Sigma^{-\alpha}T}$. Then, there is an isomorphism $f : \tilde{J} \longrightarrow \tilde{J}$ such that

$$\partial \circ f = (0, \tilde{\partial}) : \left(\bigoplus_{\beta \in I - \{\alpha\}} \Sigma^{-\beta} T \right) \oplus \Sigma^{-\alpha} T \longrightarrow \Sigma N$$

Proof. In order to write down a formula for the isomorphism $f : \tilde{J} \longrightarrow \tilde{J}$, we need to partition I. Define

$$\begin{aligned} I_1 &= \{\beta \in I : |\beta^G| < |\alpha^G| \text{ and } |\beta| = |\alpha|\}, \\ I_2 &= \{\beta \in I - \alpha : |\beta^G| = |\alpha^G| \text{ and } |\beta| = |\alpha|\}, \\ I_3 &= \{\beta \in I : |\beta^G| = |\alpha^G| \text{ and } |\beta| > |\alpha|\}, \text{ and} \\ I_4 &= \{\beta \in I : |\beta^G| < |\alpha^G| \text{ and } |\beta| > |\alpha|\}. \end{aligned}$$

Observe that I is the disjoint union

$$I_1 \cup I_2 \cup I_3 \cup I_4 \cup \{\alpha\}.$$

Let $j_\alpha : \Sigma^{-\alpha}T \longrightarrow \tilde{J}$ denote the inclusion of $\Sigma^{-\alpha}T$ as a summand of $\tilde{J}$. For each $\beta \in I - \{\alpha\}$, we define maps $f_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$ and $g_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$. such that $\partial \circ f_\beta = 0$. Ultimately, the map

$$f : \tilde{J} = \left(\bigoplus_{\beta \in I - \{\alpha\}} \Sigma^{-\beta} T \right) \oplus \Sigma^{-\alpha} T \xrightarrow{((f_\beta), j_\alpha)} \tilde{J}$$

is the isomorphism that we seek, and $g = ((g_\beta), j_\alpha) : \tilde{J} \longrightarrow \tilde{J}$ is the inverse for f. The definitions of f_β and g_β depend on which subset of I_j contains β. For each $\beta \in I - \{\alpha\}$, let $\{\mu'_\beta, \tau'_\beta, \iota'_\beta\}$ be the standard basis for $(\Sigma^{-\beta}T)^\beta = A$.

For each $\beta \in I_1$, define a_β by the equation $\partial^\beta(\mu'_\beta) = a_\beta \partial^\beta(\xi_{\beta-\alpha})$, where $\xi_{\beta-\alpha}$ is the standard generator in $(\Sigma^{-\alpha}T)^\beta = R$. Observe that $\mu'_\beta - a_\beta \xi_{\beta-\alpha}$, τ'_β, ι'_β, $\iota_{\beta-\alpha}$, and $p\xi_{\beta-\alpha}$ are all sent to zero by ∂. Define $f_\beta : \Sigma^{-\beta}T \longrightarrow \Sigma^{-\beta}T \oplus \Sigma^{-\alpha}T \subseteq \tilde{J}$ by the map

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$A \longrightarrow A \oplus R$ sending μ'_β to $\mu'_\beta - a_\beta \xi_{\beta_\beta - \alpha}$, and observe that $\tau'_\beta \mapsto \tau'_\beta - a_\beta p \xi_{\beta-\alpha}$ and $\iota'_\beta \mapsto \iota'_\beta - a_\beta \iota_{\beta-\alpha}$. Similarly define $g_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$ by $\mu'_\beta \mapsto \mu'_\beta + a_\beta \xi_{\beta-\alpha}$.

For each $\beta \in I_2$, define a_β by the equation $\partial^\beta(\mu'_\beta) = a_\beta \partial^\beta(\mu_{\beta-\alpha})$, where $\mu_{\beta-\alpha}$ is the standard generator in $(\Sigma^{-\alpha}T)^\beta = A[d_{\beta-\alpha}]$. Observe that $\mu'_\beta - a_\beta \mu_{\beta-\alpha}$, τ'_β, $p\mu_{\beta-\alpha}$, and $\tau_{\beta-\alpha}$ are all sent to zero by ∂. Define $f_\beta : \Sigma^{-\beta}T \longrightarrow \Sigma^{-\beta}T \oplus \Sigma^\alpha T \subseteq \tilde{J}$ by the map $A \longrightarrow A \oplus A[d_{\beta-\alpha}]$ sending μ'_β to $\mu'_\beta - a_\beta \mu_{\beta-\alpha}$, and observe that $\tau'_\beta \mapsto \tau'_\beta - a_\beta d_{\beta-\alpha}\tau_{\beta-\alpha}$ and $\iota'_\beta \mapsto \iota'_\beta - a_\beta d_{\beta-\alpha}\iota_{\beta-\alpha}$. Similarly define $g_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$ by $\mu'_\beta \mapsto \mu'_\beta + a_\beta \mu_{\beta-\alpha}$.

For each $\beta \in I_3$, define a_β by the equation $\partial^\beta(\mu'_\beta) = a_\beta \partial^\beta(\epsilon_{\beta-\alpha})$, where $\epsilon_{\beta-\alpha}$ is a generator for $(\Sigma^{-\alpha}T)^\beta = \langle \mathbb{Z} \rangle$. Observe that $\mu'_\beta - a_\beta \epsilon_{\beta-\alpha}$, τ'_β, and $p\epsilon_{\beta-\alpha}$ are all sent to zero by ∂. Define $f_\beta : \Sigma^{-\beta}T \longrightarrow \Sigma^{-\beta}T \oplus \Sigma^\alpha T \subseteq \tilde{J}$ by the map $A \longrightarrow A \oplus \langle \mathbb{Z} \rangle$ sending μ'_β to $\mu'_\beta - a_\beta \epsilon_{\beta-\alpha}$, and observe that $\tau'_\beta \mapsto \tau'_\beta$ and $\iota'_\beta \mapsto \iota'_\beta$. Similarly define $g_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$ by $\mu'_\beta \mapsto \mu'_\beta + a_\beta \epsilon_{\beta-\alpha}$.

For each $\beta \in I_4$, define a_β by the equation $\partial^\beta(\mu'_\beta) = a_\beta \partial^\beta(\zeta_{\beta-\alpha})$, where $\zeta_{\beta-\alpha}$ is a generator for $(\Sigma^{-\alpha}T)^\beta = \langle \mathbb{Z}/p \rangle$. Observe that $\mu'_\beta - a_\beta \zeta_{\beta-\alpha}$ and τ'_β are sent to zero by ∂. Define $f_\beta : \Sigma^{-\beta}T \longrightarrow \Sigma^{-\beta}T \oplus \Sigma^\alpha T \subseteq \tilde{J}$ by the map $A \longrightarrow A \oplus \langle \mathbb{Z}/p \rangle$ sending μ'_β to $\mu'_\beta - a_\beta \zeta_{\beta-\alpha}$, and observe that $\tau'_\beta \mapsto \tau'_\beta$ and $\iota'_\beta \mapsto \iota'_\beta$. Similarly define $g_\beta : \Sigma^{-\beta}T \longrightarrow \tilde{J}$ by $\mu'_\beta \mapsto \mu'_\beta + a_\beta \zeta_{\beta-\alpha}$.

It is straightforward to check that $g = f^{-1}$, and hence f is an isomorphism. Also, since $\partial \circ f_\beta = 0$ for each $\beta \in I - \{\alpha\}$, it follows that $\partial \circ f$ has the asserted form. □

Recall that $\mathcal{W}_0 = \{\omega \in \mathcal{W} : \partial|_{\Sigma^{-\omega}T} = 0\}$. In order to use Lemma 3.19 to prove Proposition 3.7, we partition $\mathcal{W} - \mathcal{W}_0$. Set $\omega_0 = V$. For each $1 \leq j \leq n$, let

$$\mathcal{W}_j = \{\omega \in \mathcal{W} - \mathcal{W}_0 : |\omega^G_{j-1}| < |\omega^G| \leq |\omega^G_j|\}.$$

Note that $\mathcal{W}$ is the disjoint union $\mathcal{W}_0 \cup \cdots \cup \mathcal{W}_n$. For each $1 \leq j \leq n$, apply Lemma 3.19 for $N = \Sigma^{-V}T$, $\tilde{J}_j = \bigoplus_{\beta \in \mathcal{W}_j} \Sigma^{-\beta}T$, $\partial|_{\tilde{J}_j}$, and $\alpha = \omega_j$. Hence, Lemma 3.19 gives us an isomorphism $f_j : \tilde{J}_j \longrightarrow \tilde{J}_j$. Note that $\tilde{J} = \bigoplus_{j=1}^n \tilde{J}_j$, and define $f : \tilde{J} \longrightarrow \tilde{J}$ by

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$f = \oplus f_j$. Note that $H = P \oplus F_0 \oplus \tilde{J}$, and define $\tilde{f} : H \longrightarrow H$ by $\tilde{f} = id_P \oplus id_{F_0} \oplus f$. For each $1 \leq j \leq n$, let $\tilde{J}'_j = \bigoplus_{\beta \in \mathcal{W}_j - \omega_j} \Sigma^{-\beta} T$. Let $J = \bigoplus_{j=1}^n \Sigma^{-\omega_j} T$ and $J' = \bigoplus_{j=1}^n \tilde{J}'_j$. It is clear that $\partial \circ \tilde{f} = (0, 0, 0, \tilde{\partial}) : P \oplus F_0 \oplus J' \oplus J \longrightarrow \Sigma N$, where $\tilde{\partial} = \partial|_J$.

3.2.3 The Proof of Proposition 3.11

Recall that diagram (3.16) gives us an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) \oplus J(\langle\mathbb{Z}\rangle) & \longrightarrow & Z \oplus J & \longrightarrow & Z(R) \oplus J(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_{\langle\mathbb{Z}\rangle}}\cong & & \downarrow{\scriptstyle h}\cong & & \downarrow{\scriptstyle h_R}\cong & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(B; \langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(B) & \longrightarrow & \mathbf{H}^*_G(B; R) & \longrightarrow & 0. \end{array}$$

For $S = A$, R, and $\langle\mathbb{Z}\rangle$, we have epimorphisms

$$N(S) \longrightarrow 0,$$

$$q'_S : \mathbf{H}^*_G(B; S) \xrightarrow{h_S^{-1}} Z(S) \oplus J(S) \xrightarrow{\pi_1} Z(S),$$

and

$$q_S : \mathbf{H}^*_G(Y; S) \xrightarrow{i^*_S} \mathbf{H}^*_G(B; S) \xrightarrow{q'_S} Z(S).$$

These epimorphisms provide a surjection from diagram (3.15), which is

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \longrightarrow & N(\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(Y; \langle\mathbb{Z}\rangle) & \xrightarrow{i^*_{\langle\mathbb{Z}\rangle}} & \mathbf{H}^*_G(B; \langle\mathbb{Z}\rangle) & \xrightarrow{0} & \Sigma N(\langle\mathbb{Z}\rangle) & \longrightarrow \\ & \downarrow & & \downarrow{\scriptstyle \overline{\phi}} & & \downarrow & & \downarrow & \\ \longrightarrow & N & \longrightarrow & \mathbf{H}^*_G(Y) & \xrightarrow{i^*} & \mathbf{H}^*_G(B) & \xrightarrow{\partial} & \Sigma N & \longrightarrow \\ & \downarrow & & \downarrow{\scriptstyle \overline{\theta}} & & \downarrow & & \downarrow & \\ \longrightarrow & N(R) & \longrightarrow & \mathbf{H}^*_G(Y; R) & \xrightarrow{i^*_R} & \mathbf{H}^*_G(B; R) & \xrightarrow{\partial_R} & \Sigma N(R) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & & 0, & \end{array}$$

onto the diagram

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$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) & = & Z(\langle\mathbb{Z}\rangle) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z & = & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z(R) & = & Z(R) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array} \tag{3.29}$$

whose rows and columns are clearly exact. Since M_S is the kernel of the map q_S, the kernel of this map from diagram (3.15) to diagram (3.29) is the diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
N(\langle\mathbb{Z}\rangle) & \xrightarrow{f_{\langle\mathbb{Z}\rangle}} & M_{\langle\mathbb{Z}\rangle} & \xrightarrow{g_{\langle\mathbb{Z}\rangle}} & J(\langle\mathbb{Z}\rangle) & \xrightarrow{0} & \Sigma N(\langle\mathbb{Z}\rangle) & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow & & \downarrow & & \\
N & \xrightarrow{f} & M & \xrightarrow{g} & J & \xrightarrow{\partial'} & \Sigma N & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow & & \\
N(R) & \xrightarrow{f_R} & M_R & \xrightarrow{g_R} & J(R) & \xrightarrow{\partial'_R} & \Sigma N(R) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0. & &
\end{array} \tag{3.30}$$

whose rows and columns must be exact. This last diagram is precisely the one whose existence is asserted by part (b).

For part (a), note that we have a short exact sequence

$$0 \longrightarrow (3.30) \longrightarrow (3.15) \longrightarrow (3.29) \longrightarrow 0$$

of diagrams. The three diagrams of part (a) are just slices of this 3-dimensional diagram.

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3.2.4 The Proof of Proposition 3.14

Let $M' = \Sigma^{-V'}T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega_i'}T$, and suppose that $l : M' \longrightarrow M$ is an isomorphism. Since the top row of diagram (3.19) has $\partial_{\langle \mathbb{Z} \rangle} = 0$, it follows that

$$M_{\langle \mathbb{Z} \rangle} \cong N(\langle \mathbb{Z} \rangle) \oplus J(\langle \mathbb{Z} \rangle)$$

is a free $T(\langle \mathbb{Z} \rangle)$-module with generators in dimensions V, $\omega_1, \ldots, \omega_n$. However, Lemma 4.7 tells us that this list of dimensions can be replaced by any other list that preserves fixed dimensions. Hence, $M_{\langle \mathbb{Z} \rangle}$ is also seen to be a free $T(\langle \mathbb{Z} \rangle)$-module with generators in dimensions $\omega_1', \ldots, \omega_n', V'$.

We must show that there is a particular isomorphism $l_{\langle \mathbb{Z} \rangle} : M'(\langle \mathbb{Z} \rangle) \longrightarrow M_{\langle \mathbb{Z} \rangle}$ such that the square

$$\begin{array}{ccc} M'(\langle \mathbb{Z} \rangle) & \xrightarrow{c} & M' \\ {\scriptstyle l_{\langle \mathbb{Z} \rangle}}\downarrow{\scriptstyle \cong} & & {\scriptstyle l}\downarrow{\scriptstyle \cong} \\ M_{\langle \mathbb{Z} \rangle} & \xrightarrow{\phi} & M \end{array}$$

commutes, where c is the canonical change of coefficients map. In that case, we can choose $l_R : M'(R) \longrightarrow M_R$ to be the induced map between cokernels in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'(\langle \mathbb{Z} \rangle) & \longrightarrow & M' & \longrightarrow & M'(R) & \longrightarrow & 0 \\ & & {\scriptstyle l_{\langle \mathbb{Z} \rangle}}\downarrow{\scriptstyle \cong} & & {\scriptstyle l}\downarrow{\scriptstyle \cong} & & {\scriptstyle l_R}\downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & M_{\langle \mathbb{Z} \rangle} & \xrightarrow{\phi} & M & \xrightarrow{\theta} & M_R & \longrightarrow & 0 \end{array}$$

with short exact rows. Of course, l_R must then be an isomorphism too.

Lemma 3.20. *There is an isomorphism $l_{\langle \mathbb{Z} \rangle} : M'(\langle \mathbb{Z} \rangle) \longrightarrow M_{\langle \mathbb{Z} \rangle}$ such that the diagram*

$$\begin{array}{ccc} & & M'(\langle \mathbb{Z} \rangle) \\ & {\scriptstyle l_{\langle \mathbb{Z} \rangle}}\swarrow & \downarrow{\scriptstyle c} \\ M_{\langle \mathbb{Z} \rangle} & \xrightarrow{l^{-1} \circ \phi} & M' \end{array}$$

commutes, where c is the canonical change of coefficients map.

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Proof. Let $l'_{\langle \mathbb{Z} \rangle} : M'(\langle \mathbb{Z} \rangle) \longrightarrow M_{\langle \mathbb{Z} \rangle}$ be an isomorphism, and let $\phi' = l^{-1} \circ \phi \circ l'_{\langle \mathbb{Z} \rangle}$. Diagram (3.19) then gives a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & N(\langle \mathbb{Z} \rangle) & \xrightarrow{f'_{\langle \mathbb{Z} \rangle}} & M'(\langle \mathbb{Z} \rangle) & \xrightarrow{g'_{\langle \mathbb{Z} \rangle}} & J(\langle \mathbb{Z} \rangle) & \xrightarrow{0} & \Sigma N(\langle \mathbb{Z} \rangle) & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \phi'} & & \downarrow & & \downarrow & \\
\longrightarrow & N & \xrightarrow{f'} & M' & \xrightarrow{g'} & J & \xrightarrow{\partial'} & \Sigma N & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \theta \circ l} & & \downarrow & & \downarrow & \\
\longrightarrow & N(R) & \xrightarrow{f_R} & M_R & \xrightarrow{g_R} & J(R) & \xrightarrow{\partial'_R} & \Sigma N(R) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array} \tag{3.31}$$

with exact rows and columns. By using this diagram in the dimensions of the generators for $M'(\langle \mathbb{Z} \rangle)$, the structure of the map ϕ' can easily be determined.

In dimension ω'_1, the first two rows of diagram (3.31) are

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & \langle \mathbb{Z} \rangle & \xrightarrow{\cong} & \langle \mathbb{Z} \rangle & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \phi'} & & \downarrow & & \downarrow & \\
0 \longrightarrow & \langle \mathbb{Z} \rangle & \longrightarrow & A & \longrightarrow & R & \longrightarrow & 0 & \longrightarrow .
\end{array}$$

It follows that ϕ' is a standard inclusion $\langle \mathbb{Z} \rangle \longrightarrow A$ that sends 1 to $\pm\kappa$.

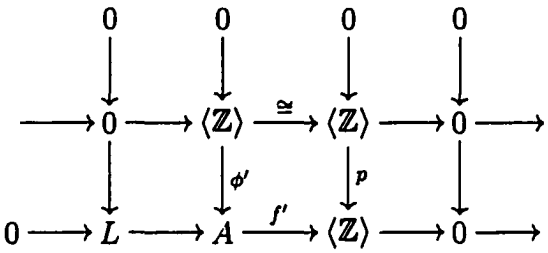
In dimension ω'_i, for $2 \leq i \leq n$, the first two rows of diagram (3.31) are

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & 0 & \longrightarrow & \langle \mathbb{Z} \rangle & \xrightarrow{\cong} & \langle \mathbb{Z} \rangle & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle pi_2} & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & A & \xrightarrow{f'} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{\partial} & \langle \mathbb{Z}/p \rangle & \longrightarrow 0.
\end{array}$$

Proposition A.4 gives that the map f' sends μ to $(\pm\xi, \pm 1)$, and hence ϕ' is a standard inclusion as before.

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In dimension V', the first two rows of diagram (3.31) are

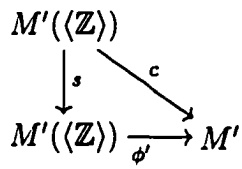


Proposition A.4 gives that the map f' is a standard map that sends μ to ± 1, and hence ϕ' must again be a standard inclusion.

It is now clear that there is some automorphism

$$s : M'(\langle \mathbb{Z} \rangle) \longrightarrow M'(\langle \mathbb{Z} \rangle)$$

such that the diagram



commutes. At most, s flips the signs of the generators. Let $l_{\langle \mathbf{Z} \rangle} = l'_{\langle \mathbf{Z} \rangle} \circ s$. □

3.2.5 The Proof of Proposition 3.17

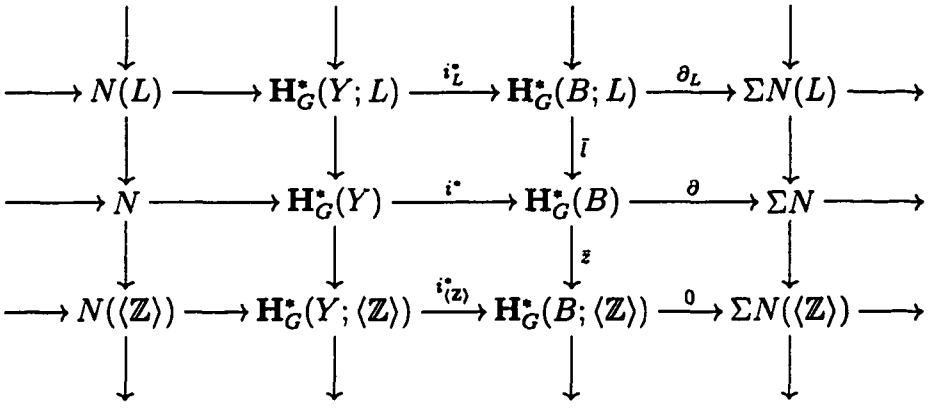
The short exact sequence of coefficients

$$0 \longrightarrow L \longrightarrow A \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow 0$$

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yields a commuting diagram

(3.32)



with long exact rows and columns. Proposition B.13 provides an isomorphism

$$\mathbf{H}^*_G(?;L) \cong \Sigma^{2-\lambda}\mathbf{H}^*_G(?;R).$$

This gives $T_G(L) \cong \Sigma^{2-\lambda}T_G(R)$ and $T(L) \cong \Sigma^{2-\lambda}T(R)$. Moreover, we can identify the top row of diagram (3.32) with the suspension by $2-\lambda$ of the long exact sequence

$$\longrightarrow N(R) \longrightarrow \mathbf{H}^*_G(Y;R) \longrightarrow \mathbf{H}^*_G(B;R) \xrightarrow{\partial_R} \Sigma N(R) \longrightarrow$$

in cohomology with R-coefficients. Note that the diagram

$$\begin{array}{ccccccc}
\mathbf{H}^*_G(Y;L) & \xrightarrow{i^*_L} & \mathbf{H}^*_G(B;L) & \xrightarrow{h_L^{-1}} & Z(L)\oplus J(L) & \xrightarrow{\pi_1} & Z(L) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\Sigma^{2-\lambda}\mathbf{H}^*_G(Y;R) & \xrightarrow{\Sigma^{2-\lambda}i^*_R} & \Sigma^{2-\lambda}\mathbf{H}^*_G(B;R) & \xrightarrow{\Sigma^{2-\lambda}h_R^{-1}} & \Sigma^{2-\lambda}Z(R)\oplus\Sigma^{2-\lambda}J(R) & \xrightarrow{\pi_1} & \Sigma^{2-\lambda}Z(R)
\end{array}$$

commutes, where the vertical isomorphisms are given by Proposition B.13. The top composite is the map q_L, and the bottom composite is $\Sigma^{2-\lambda}q_R$. Hence, $M_L \cong \Sigma^{2-\lambda}M_R$.

Our aim is to split the diagram in Proposition 3.17 off of diagram (3.32). In order to do this, we need to understand the vertical maps in diagram (3.32). Clearly, the maps in the left and right columns of diagram (3.32) are the obvious change of

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coefficients maps. The isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle)\oplus J(\langle\mathbb{Z}\rangle) & \longrightarrow & Z\oplus J & \longrightarrow & Z(R)\oplus J(R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_{\langle\mathbb{Z}\rangle}}\cong & & \downarrow{\scriptstyle h}\cong & & \downarrow{\scriptstyle h_R}\cong & & \\ 0 & \longrightarrow & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(B) & \longrightarrow & \mathbf{H}^*_G(B;R) & \longrightarrow & 0. \end{array}$$

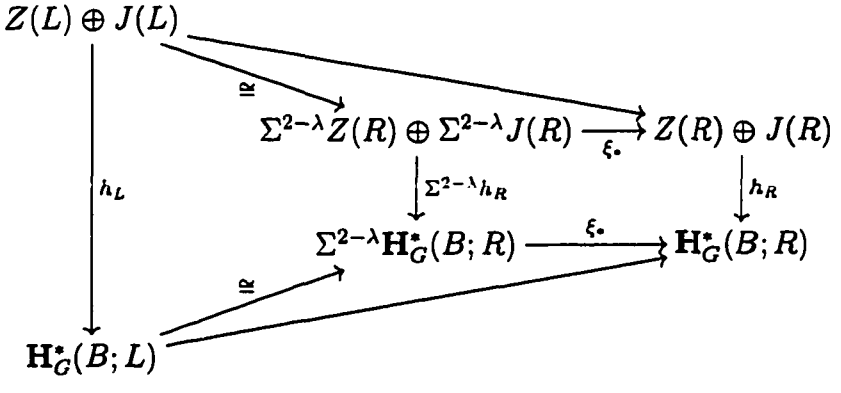
is used to understand the maps in the third column of diagram (3.32).

Lemma 3.21. *The diagrams*

$$\begin{array}{ccc} Z(L)\oplus J(L) & \longrightarrow & Z(R)\oplus J(R) \\ \cong\downarrow{\scriptstyle h_L} & & \cong\downarrow{\scriptstyle h_R} \\ \mathbf{H}^*_G(B;L) & \xrightarrow{j} & \mathbf{H}^*_G(B;R) \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} Z(L)\oplus J(L) & \longrightarrow & Z\oplus J \\ \cong\downarrow{\scriptstyle h_L} & & \cong\downarrow{\scriptstyle h} \\ \mathbf{H}^*_G(B;L) & \xrightarrow{\bar{\imath}} & \mathbf{H}^*_G(B) \end{array}$$

commute, where the horizontal maps are the canonical change of coefficients maps.

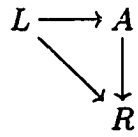
Proof. In the diagram



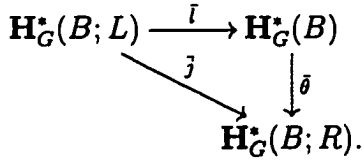
the horizontal maps are given by muliplication by $\xi \in T^{\lambda-2}(R)(G/G)$, and the unlabeled isomorphisms are given by Proposition B.13. The right-hand square commutes since h_R is a $T(R)$-module map. The left-hand trapezoid commutes by definition of h_L. The triangles commute by Proposition B.13. Hence, the outer trapezoid commutes and gives the first diagram assserted in the lemma.

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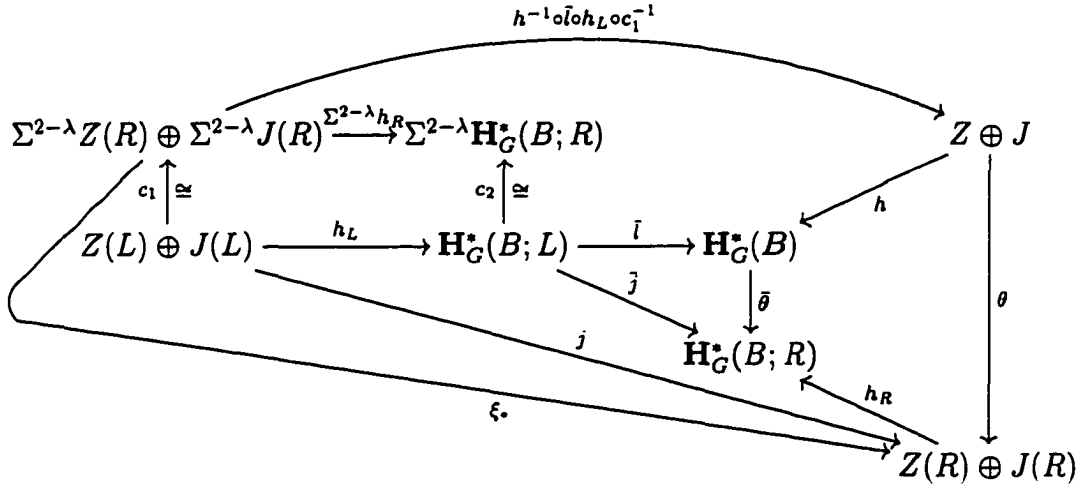
The diagram



of canonical maps yields the diagram in cohomology



This diagram fits into the commuting diagram



Here, θ and j are the canonical change of coefficients maps, and c_1 and c_2 are the isomorphisms given by Proposition B.13. Let $l = h^{-1}\circ\bar{l}\circ h_L\circ c_1^{-1}$.

Our aim is to show that $l\circ c_1$ is the canonical change of coefficients map

$$Z(L)\oplus J(L)\longrightarrow Z\oplus J.$$

Hence, it suffices to show that l maps each summand $\Sigma^{2-\lambda-m}T_G(R)$ isomorphically onto the summand $\Sigma^{-m}T_G$, and l maps each summand $\Sigma^{2-\lambda-\omega}T(R)$ to the summand $\Sigma^{-\omega}T$ via the T-module map determined by the isomorphism

$$(\Sigma^{2-\lambda-\omega}T(R))^{\lambda-2+\omega} = R\xrightarrow{\cong} R = (\Sigma^{-\omega}T)^{\lambda-2+\omega}.$$

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We justify this by using the properties of the maps ξ_* and θ. Since ξ_* is multiplication by $\xi \in T^{\lambda-2}(R)(G/G)$, it follows that ξ_* maps each summand $\Sigma^{2-\lambda-m}T_G(R)$ isomorphically onto the summand $\Sigma^{-m}T_G(R)$. Also, ξ_* maps each summand $\Sigma^{2-\lambda-\omega}T(R)$ to the summand $\Sigma^{-\omega}T(R)$ via the $T(R)$-module map determined by the isomorphism

$$(\Sigma^{2-\lambda-\omega}T(R))^{\lambda-2+\omega} = R \xrightarrow{\cong} R = (\Sigma^{-\omega}T(R))^{\lambda-2+\omega}.$$

Note that θ maps each summand $\Sigma^{-m}T_G$ isomorphically onto $\Sigma^{-m}T_G(R)$, and θ maps each summand $\Sigma^{-\omega}T$ onto $\Sigma^{-\omega}T(R)$ via the canonical epimorphism.

The map l out of a summand $\Sigma^{2-\lambda-m}T_G(R)$ is determined by its value at the G/e-level. By assumption, the map θ is an isomorphism at the G/e-level. It follows that l must map the summand $\Sigma^{2-\lambda-m}T_G(R)$ isomorphically onto the summand $\Sigma^{-m}T_G$ as asserted.

To determine the map l on a summand $\Sigma^{2-\lambda-\omega}T(R)$, it suffices to determine its value on $R = (\Sigma^{2-\lambda-\omega}T(R))^{\lambda-2+\omega}$. Let $\xi_{\lambda-2+\omega}$ denote a standard generator for this copy of R. The map ξ_* sends this R isomorphically onto $R = (\Sigma^{-\omega}T(R))^{\lambda-2+\omega}$, and θ sends $R = (\Sigma^{-\omega}T)^{\lambda-2+\omega}$ isomorphically onto $R = (\Sigma^{-\omega}T(R))^{\lambda-2+\omega}$. Therefore $(l_G^{\lambda-2+\omega})(\xi_{\lambda-2+\omega}) = \xi' + \psi$. Here, ξ' denotes a standard generator for $R = (\Sigma^{-\omega}T)^{\lambda-2+\omega}$, ψ is in $(Z \oplus J)^{\lambda-2+\omega}$, and $(\theta_G^{\lambda-2+\omega})(\psi) = 0$. Since $0 = p\epsilon_\lambda \xi_{\lambda-2+\omega}$, it follows that

$$0 = l_G^{\lambda-2+\omega}(0) = l_G^{\lambda-2+\omega}(p\epsilon_\lambda \xi_{\lambda-2+\omega}) = l_G^{\lambda-2+\omega}(p\epsilon_\lambda \xi_{\lambda-2+\omega}) = p\epsilon_\lambda \psi.$$

Since $(\theta_G^{\lambda-2+\omega})(\psi) = 0$, the element ψ must be a sum of generators of the form $p\epsilon$, κ, or $\epsilon^{-1}\kappa$ from summands of $Z \oplus J$ of the form $\Sigma^{-\omega'}T$. However, $p\epsilon_\lambda$ times any of these elements is not zero. Hence, it must be that $\psi = 0$, and therefore $l^{\lambda-2+\omega}$ maps $R = (\Sigma^{2-\lambda-\omega}T(R))^{\lambda-2+\omega}$ isomorphically onto $R = (\Sigma^{-\omega}T)^{\lambda-2+\omega}$ as asserted. □

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Lemma 3.22. *The diagram*

$$\begin{array}{ccc} Z \oplus J & \longrightarrow & Z(\langle\mathbb{Z}\rangle) \oplus J(\langle\mathbb{Z}\rangle) \\ \cong \downarrow h & & \cong \downarrow h_{\langle\mathbb{Z}\rangle} \\ \mathbf{H}^*_G(B) & \xrightarrow{z} & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) \end{array}$$

commutes, where the horizontal maps are the canonical change of coefficients maps.

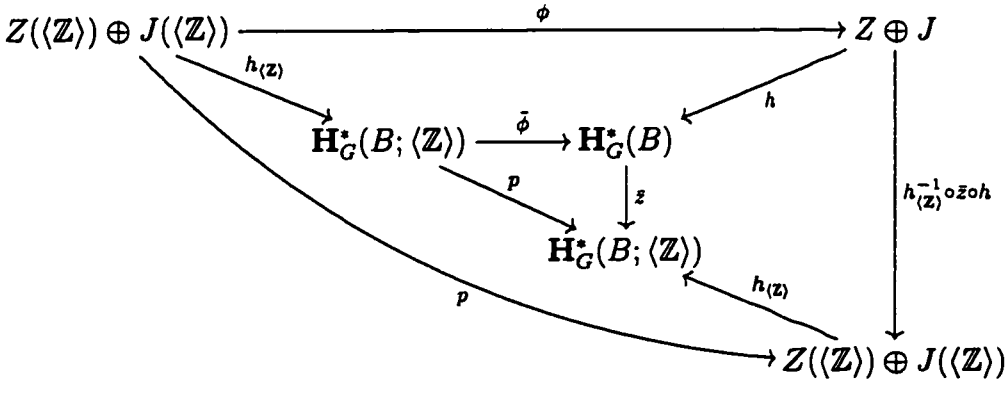
Proof. The canonical diagram of coefficients

$$\begin{array}{ccc} \langle\mathbb{Z}\rangle & \longrightarrow & A \\ & \searrow^{p} & \downarrow \\ & & \langle\mathbb{Z}\rangle \end{array}$$

yields a diagram in cohomology

$$\begin{array}{ccc} \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) & \xrightarrow{\bar{\phi}} & \mathbf{H}^*_G(B) \\ & \searrow^{p} & \downarrow z \\ & & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle), \end{array}$$

where the diagonal map is multiplication by p. This diagram fits into the commuting diagram



Let $z = h^{-1}_{\langle\mathbb{Z}\rangle} \circ \bar{z} \circ h$. It is straightforward to check that the map z must therefore send each summand $\Sigma^{-\omega}T$ to the summand $\Sigma^{-\omega}T(\langle\mathbb{Z}\rangle)$ via the canonical map. Since $\mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle)(G/e) = 0$, each summand $\Sigma^{-m}T_G$ of $Z \oplus J$ must be killed by z. □

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Let $\partial'_L = (\partial_L \circ h_L)|_{J(L)}$. The diagram

$$\begin{array}{ccccc} Z(L)\oplus J(L) & \xrightarrow{h_L} & \mathbf{H}^*_G(B;L) & \xrightarrow{\partial_L} & \Sigma N(L) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \Sigma^{2-\lambda}Z(R)\oplus\Sigma^{2-\lambda}J(R) & \xrightarrow{\Sigma^{2-\lambda}h_R} & \Sigma^{2-\lambda}\mathbf{H}^*_G(B;R) & \xrightarrow{\Sigma^{2-\lambda}\partial_R} & \Sigma^{3-\lambda}N(R) \end{array}$$

commutes, where the vertical isomorphisms are given by Proposition B.13. Since the composite in the bottom row has the form $(0, \Sigma^{2-\lambda}\partial'_R)$, the composite in the top row has the form $(0, \partial'_L)$.

Lemmas 3.21 and 3.22 together give us an isomorphism of sequences

$$\begin{array}{ccccc} Z(L)\oplus J(L) & \longrightarrow & Z\oplus J & \longrightarrow & Z(\langle\mathbb{Z}\rangle)\oplus J(\langle\mathbb{Z}\rangle) \\ {\scriptstyle h_L}\downarrow\cong & & {\scriptstyle h}\downarrow\cong & & {\scriptstyle h_{\langle\mathbb{Z}\rangle}}\downarrow\cong \\ \mathbf{H}^*_G(B;L) & \xrightarrow{\bar{l}} & \mathbf{H}^*_G(B) & \xrightarrow{\bar{z}} & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) \end{array}$$

such that the rows are exact at the middle column. For $S = L$, A, and $\langle\mathbb{Z}\rangle$, we have epimorphisms

$$N(S) \longrightarrow 0,$$

$$q'_S : \mathbf{H}^*_G(B;S) \xrightarrow{h_S^{-1}} Z(S)\oplus J(S) \xrightarrow{\pi_1} Z(S),$$

and

$$q_S : \mathbf{H}^*_G(Y;S) \xrightarrow{i^*_S} \mathbf{H}^*_G(B;S) \xrightarrow{q'_S} Z(S).$$

These epimorphisms provide a surjection from the diagram

$$\begin{array}{ccccccccc} \longrightarrow & N(L) & \longrightarrow & \mathbf{H}^*_G(Y;L) & \xrightarrow{i^*_L} & \mathbf{H}^*_G(B;L) & \xrightarrow{\partial_L} & \Sigma N(L) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & N & \longrightarrow & \mathbf{H}^*_G(Y) & \xrightarrow{i^*} & \mathbf{H}^*_G(B) & \xrightarrow{\partial} & \Sigma N & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & N(\langle\mathbb{Z}\rangle) & \longrightarrow & \mathbf{H}^*_G(Y;\langle\mathbb{Z}\rangle) & \xrightarrow{i^*_{\langle\mathbb{Z}\rangle}} & \mathbf{H}^*_G(B;\langle\mathbb{Z}\rangle) & \xrightarrow{0} & \Sigma N(\langle\mathbb{Z}\rangle) & \longrightarrow \end{array} \tag{3.33}$$

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onto the diagram

$$\begin{array}{ccccccccc}
\longrightarrow & 0 & \longrightarrow & Z(L) & = & Z(L) & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & Z & = & Z & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & Z(\langle\mathbb{Z}\rangle) & = & Z(\langle\mathbb{Z}\rangle) & \longrightarrow & 0 & \longrightarrow
\end{array} \tag{3.34}$$

whose rows are clearly exact. Moreover the columns of diagrams (3.33) and (3.34) are certainly exact at the middle row. Since M_S is the kernel of the map q_S, the kernel of this map from diagram (3.33) to diagram (3.34) is the diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & N(L) & \xrightarrow{f_L} & M_L & \xrightarrow{g_L} & J(L) & \xrightarrow{\partial'_L} & \Sigma N(L) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J & \xrightarrow{\partial'} & \Sigma N & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & N(\langle\mathbb{Z}\rangle) & \xrightarrow{f_{\langle\mathbb{Z}\rangle}} & M_{\langle\mathbb{Z}\rangle} & \xrightarrow{f_{\langle\mathbb{Z}\rangle}} & J(\langle\mathbb{Z}\rangle) & \xrightarrow{0} & \Sigma N(\langle\mathbb{Z}\rangle) & \longrightarrow
\end{array}$$

asserted in Proposition 3.17. Let α be an even element of $RO(G)$. By Proposition B.15, the map

$$\eta^\alpha : Z(L) \longrightarrow Z$$

is a monomorphism. It follows by a routine diagram chase that the sequence

$$M_L^\alpha \longrightarrow M^\alpha \longrightarrow M_{\langle\mathbb{Z}\rangle}^\alpha$$

is exact at M^α.

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3.3 The Proof of Theorem 3.15

Our computation of M comes primarily from the long exact sequence

$$\longrightarrow N \xrightarrow{f} M \xrightarrow{g} J \xrightarrow{\partial} \Sigma N \longrightarrow \tag{3.35}$$

in which $N = \Sigma^{-\tilde{V}}T$ and $J = \bigoplus_{i=1}^{n} \Sigma^{-\omega_i}T$. A plot of the generators for N and J is given in Table 3.2. Of course, the assumption that $M^{\omega_1'} = A$ is also used to show that M has the desired form. In fact, there are just a few key dimensions in which it is necessary to compute M. From the values in those dimensions and the constraints on M given by sequence (3.35), the value of M is forced.

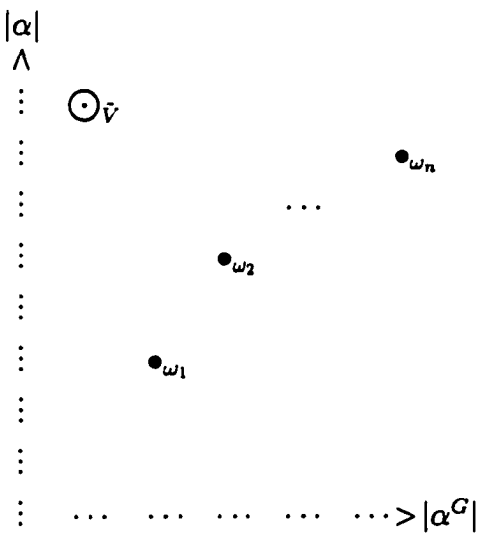


Table 3.2: Plot of Sequence (3.35)

To describe the key dimensions, it is convenient to have some notation for specifying coordinates relative to the points plotted in Table 3.2. Let $l = |\tilde{V}^G|$ and $t = |\tilde{V}|$. For each $1 \leq i \leq n$, let $l_i = |\omega_i^G|$ and $b_i = |\omega_i|$. To avoid having to give descriptions like: α is in $RO(G)$ and $(|\alpha^G|, |\alpha|) = (|\tilde{V}^G|, |\omega_1|)$, we simply say that α is in $[l, b_1]$. Hence, $[l, b_1]$ denotes an equivalence class of elements of $RO(G)$ that all plot to coordinates (l, b_1). We refer to such an equivalence class as a location. Note that

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ω_1' is in location $[l, b_1]$. Table 3.3 gives a plot of some important locations associated with the points plotted in Table 3.2. Column vectors are used in Table 3.3 strictly for aesthetic reasons.

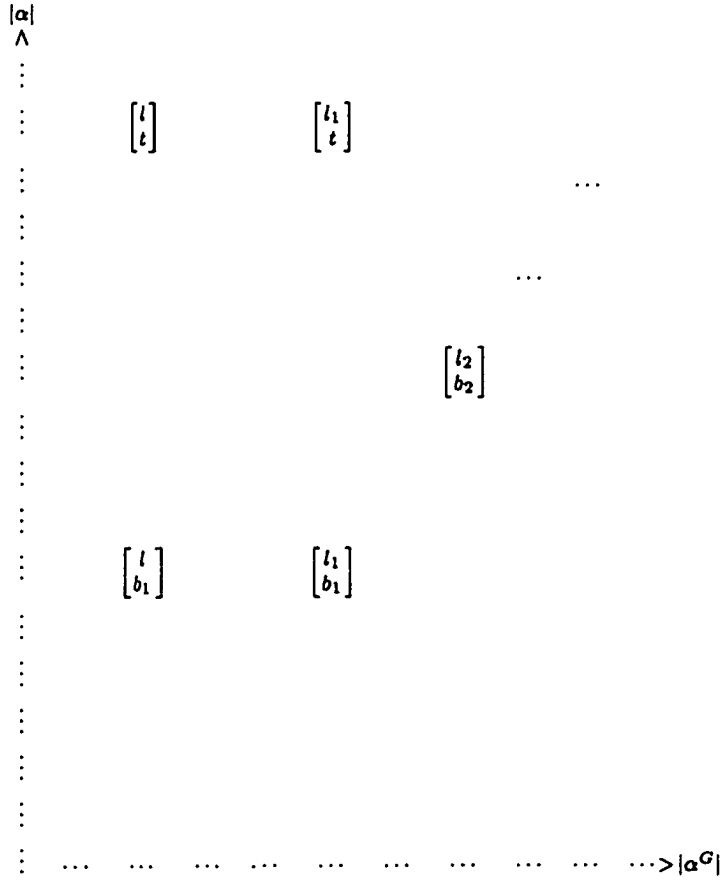


Table 3.3: Location Notation

The key locations are $[l, b_1]$, $[l, t]$, and $[l_1, t]$. Our aim is to construct a map

$$\theta : \Sigma^{-\tilde{V}_1'} T \oplus \Sigma^{-\omega_1'} T \longrightarrow M,$$

where $\tilde{V}_1'$ is in $[l_1, t]$. The dimensions ω_1' and $\tilde{V}_1'$ are the reason for our particular interest in the locations $[l, b_1]$ and $[l_1, t]$. However, all three locations turn out to be connected. Our proof of Theorem 3.15 is by induction on n. For $n = 1$, θ is an isomorphism. For $n > 1$, θ is used to construct a long exact sequence for which

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the inductive hypothesis can be applied. Before starting our induction argument, we compute the value of M^α for all α in the key locations.

Lemma C.5 specifies the dimensions α in which $\partial^\alpha : J^\alpha \longrightarrow N^{\alpha+1}$ is nonzero. In particular, for α in any of $[l, b_1]$, $[l, t]$, or $[l_1, t]$, ∂^α and $\partial^{\alpha-1}$ are both zero. In these cases, sequence (3.35) reduces to the short exact sequence

$$0 \longrightarrow N^\alpha \xrightarrow{f} M^\alpha \xrightarrow{g} J^\alpha \longrightarrow 0. \tag{3.36}$$

In dimension ω_1', sequence (3.36) has the form

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \longrightarrow 0.$$

Clearly sequence (3.36) does not split in that dimension. We claim that, in fact, sequence (3.36) does not split for α in any of $[l, b_1]$, $[l, t]$, or $[l_1, t]$. This is certainly necessary for M to have the asserted form. Moreover, this fact together with the solutions to the extension problems listed in Lemma A.3 enables us to ultimately compute M^α for α in any of $[l, b_1]$, $[l, t]$, or $[l_1, t]$.

First observe that, for a fixed location, a short exact sequence from one dimension in that location splits if and only if all short exact sequences in that location split.

Lemma 3.23. *Let α and β be elements of $RO(G)$ such that $(|\alpha^G|, |\alpha|) = (|\beta^G|, |\beta|)$ and $\partial^\alpha = \partial^{\alpha-1} = 0$. Then, the short exact sequence*

$$0 \longrightarrow N^\alpha \xrightarrow{f} M^\alpha \xrightarrow{g} J^\alpha \longrightarrow 0$$

splits if and only if the short exact sequence

$$0 \longrightarrow N^\beta \xrightarrow{f} M^\beta \xrightarrow{g} J^\beta \longrightarrow 0$$

splits.

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Proof. Suppose that the short exact sequence in dimension β splits. Module maps given by multiplication by $\mu_{\alpha-\beta}$ in $T^{\alpha-\beta}(G/G)$ yield the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^{\alpha} & \longrightarrow & M^{\alpha} & \longrightarrow & J^{\alpha} & \longrightarrow & 0 \\ & & \mu_N \uparrow \cong & & \mu_M \uparrow \cong & & \mu_J \uparrow \cong & & \\ 0 & \longrightarrow & T^{\alpha-\beta} \square N^{\beta} & \longrightarrow & T^{\alpha-\beta} \square M^{\beta} & \longrightarrow & T^{\alpha-\beta} \square J^{\beta} & \longrightarrow & 0 \end{array}$$

with exact rows. Proposition C.1 tells us that, since $\alpha-\beta \in RO_0(G)$, the cup product maps μ_N, μ_M, and μ_J are isomorphisms. Since $T^{\alpha-\beta}\square$? preserves split short exact sequences, the sequence in dimension α must also split. The converse is clearly true by a symmetric argument. □

Since the sequence

$$0 \longrightarrow N^{\omega_1'} \xrightarrow{f} M^{\omega_1'} \xrightarrow{g} J^{\omega_1'} \longrightarrow 0$$

does not split, and ω_1' is in $[l, b_1]$, it follows from Lemma 3.23 that sequence (3.36) does not split for any α in $[l, b_1]$. To show that there are no splittings in locations $[l, t]$ and $[l_1, t]$, we use a lemma that relates those locations to location $[l, b_1]$.

Lemma 3.24. **(a)** *If sequence (3.36) splits for any γ in $[l_1, t]$, then sequence (3.36) splits for all α in $[l, t]$.*

(b) *If sequence (3.36) splits for any α in $[l, t]$, then sequence (3.36) splits for all β in $[l, b_1]$.*

Proof. The proof is given in Subsection 3.3.3. □

Observe that Lemma 3.24 implies that sequence (3.36) does not split for α in any of $[l, b_1]$, $[l, t]$, or $[l_1, t]$. Lemma A.3 lists all of the possible solutions to the extension

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problems

$$0 \longrightarrow \langle \mathbb{Z} \rangle \xrightarrow{f} M^\alpha \xrightarrow{g} R \longrightarrow 0,$$
$$0 \longrightarrow A[d_{\alpha - \tilde{V}}] \xrightarrow{f} M^\alpha \xrightarrow{g} \langle \mathbb{Z}/p \rangle^n \longrightarrow 0, \text{ and}$$
$$0 \longrightarrow L \xrightarrow{f} M^\alpha \xrightarrow{g} \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} \longrightarrow 0,$$

arising from α in $[l, b_1]$, $[l, t]$, and $[l_1, t]$, respectively. In fact, since these sequences do not split, Lemma A.3 gives very strong restrictions on the values of M^α.

Lemma 3.25. *Let α be in $RO(G)$.*

(a) *If α is in $[l, b_1]$, then $M^\alpha = A[d]$ for some integer d prime to p.*

(b) *If α is in $[l, t]$, then $M^\alpha = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1}$.*

(c) *If α is in $[l_1, t]$ and $n = 1$, then $M^\alpha = A[d]$ for some integer d prime to p.*

(d) *If α is in $[l_1, t]$ and $n > 1$, then M^α is either $R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2}$ or $A[d] \oplus \langle \mathbb{Z}/p \rangle^{n-1}$ for some integer d prime to p.*

Parts (c) and (d) of Lemma 3.25 show that it is necessary to consider the case in which $n = 1$ seperately from the case in which $n > 1$. Hence, our proof now proceeds by induction on n.

3.3.1 The Base Case

Since $n = 1$, we simplify notation by letting $r = l_1 = l_n$ and $b = b_1 = b_n$. The important locations are now plotted in Table 3.4. In this case, the switch described by Table 3.5 is shown to occur. That is, we claim that there is some $\tilde{V}'$ in $[r, t]$ such that $M = \Sigma^{-\tilde{V}'} T \oplus \Sigma^{-\omega_1'} T$. In particular, we need $M^{\tilde{V}'} = A$. This is precisely what Corollary C.2 accomplishes for us. It tells us that Lemma 3.25 implies that there is

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some dimension $\tilde{V}'$ in $[r,t] = [l_1,t]$ in which M takes the value A. So let $\tilde{V}'$ be such a dimension.

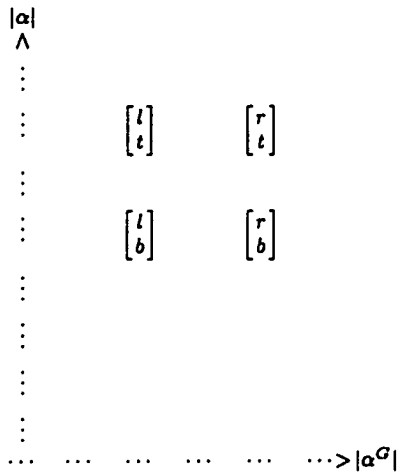


Table 3.4: Notation for the Base Case

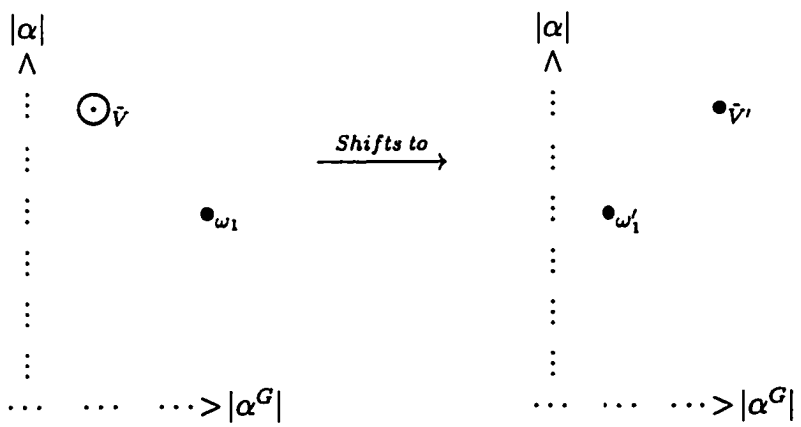


Table 3.5: Basic Shift of Generators

Define $N' = \Sigma^{-\tilde{V}'}T$ and $J' = \Sigma^{\omega_1'}T$. Note that

$$(N')^{\tilde{V}'} = M^{\tilde{V}'} = A \quad \text{and} \quad (J')^{\omega_1'} = M^{\omega_1'} = A.$$

Define T-module maps $\theta_1 : N' \longrightarrow M$ and $\theta_2 : J' \longrightarrow M$ by defining $\theta_1^{\tilde{V}'} : A \longrightarrow A$ and

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$\theta_2^{\omega_1'} : A \longrightarrow A$ to be the identity maps. We claim that

$$\theta = (\theta_1, \theta_2) : N' \oplus J' \longrightarrow M$$

is an isomorphism. A critical step in proving this is to observe that $\theta^{\tilde{V}}$ is an isomorphism. Of course, $(N' \oplus J')^{\tilde{V}} = R \oplus \langle \mathbb{Z} \rangle$, and Lemma 3.25 gives that $M^{\tilde{V}} = R \oplus \langle \mathbb{Z} \rangle$.

Lemma 3.26. *The map $\theta^{\tilde{V}} : R \oplus \langle \mathbb{Z} \rangle \longrightarrow R \oplus \langle \mathbb{Z} \rangle$ can be written as $\theta^{\tilde{V}} = \phi_1 \oplus \phi_2$, where $\phi_1 : R \longrightarrow R$ and $\phi_2 : \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle$ are isomorphisms.*

Proof. The proof is given in Subsection 3.3.4. □

Our plan is to use the 5-Lemma to show that θ is an isomorphism. Hence, we define maps $f' : N \longrightarrow N' \oplus J'$ and $g' : N' \oplus J' \longrightarrow J$ such that the diagram

$$\begin{array}{ccccccccc} \longrightarrow & N & \xrightarrow{f'} & N' \oplus J' & \xrightarrow{g'} & J & \xrightarrow{\partial} & \Sigma N & \longrightarrow \\ & \| id & & \downarrow \theta & & \| id & & \| id & \\ \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J & \xrightarrow{\partial} & \Sigma N & \longrightarrow \end{array} \tag{3.37}$$

commutes and has exact rows. Define $g' = g \circ \theta$. Since $N = \Sigma^{-\tilde{V}} T$, it suffices to specify f' in dimension $\tilde{V}$. That is, we need to define $(f')^{\tilde{V}}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{(f')^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{(g')^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \\ & & \| id & & \downarrow \theta^{\tilde{V}} & & \| id & & \\ 0 & \longrightarrow & A & \xrightarrow{f^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{g^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \end{array} \tag{3.38}$$

commutes and has exact rows. By Lemma 3.26, $\theta^{\tilde{V}}$ is an isomorphism, so define $(f')^{\tilde{V}} = (\theta^{\tilde{V}})^{-1} \circ f^{\tilde{V}}$. Proposition A.4(c) tells us that, since the top row of diagram (3.38) is a short exact sequence, $(f')^{\tilde{V}}(\mu) = (\pm\xi, \pm\epsilon)$. It follows that

$$f' = (f_1', f_2') : N \longrightarrow N' \oplus J',$$

where $f_1' : N \longrightarrow N'$ and $f_2' : N \longrightarrow J'$ are standard shift maps. Also, observe from the definition of g' that $g' = (g_1', g_2')$, where $g_1' : N' \longrightarrow J$ and $g_2' : J' \longrightarrow J$ are standard

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shift maps. Since the top row of diagram (3.38) is a short exact sequence and g_1' and g_2' are standard shift maps, Theorem C.6 tells us that the top row of diagram (3.37) is a long exact sequence. The 5-Lemma thus gives that $\theta : N' \oplus J' \longrightarrow M$ is an isomorphism. Hence, the base case of our induction is complete.

3.3.2 The Inductive Step

Let $n \geq 2$, and suppose that Theorem 3.15 holds if J has $n-1$ generators. Define $J_1 = \Sigma^{-\omega_1} T$ and $J_2 = \oplus_{i=2}^{n} \Sigma^{-\omega_i} T$, so $J = J_1 \oplus J_2$. The boundary map $\partial : J_1 \oplus J_2 \longrightarrow \Sigma N$ is thus written as $\partial = (\partial_1, \partial_2)$ where $\partial_i = \partial|_{J_i}$ is nonzero for $i = 1, 2$. Recall that there is an element ω_1' in $[l, b_1]$ such that $M^{\omega_1'} = A$. Define $J_1' = \Sigma^{-\omega_1'} T$.

For α in location $[l_1, t]$, Lemma 3.25 lists two possible values for M^α. The following lemma establishes the correct value and gives a property of the map $g^\alpha : M^\alpha \longrightarrow J^\alpha$.

Lemma 3.27. *For all α in $[l_1, t]$, $M^\alpha = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2}$. Moreover, the map $g^\alpha : M^\alpha = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2} \longrightarrow J^\alpha = \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1}$ is such that there is no map s making the diagram*

$$\begin{array}{ccc} & & {J_1}^\alpha \\ & \nexists s \nearrow & \downarrow i_1 \\ M^\alpha & \xrightarrow{g^\alpha} & {J_1}^\alpha \oplus {J_2}^\alpha \end{array}$$

commute.

Proof. The proof is given in Subsection 3.3.5. □

Our aim is to use the inductive hypothesis by constructing a new long exact sequence

$$\longrightarrow N' \longrightarrow M' \longrightarrow J_2 \xrightarrow{\partial_2'} \Sigma N' \longrightarrow, \tag{3.39}$$

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in which $J_2 = \bigoplus_{i=2}^{n} \Sigma^{-\omega_i} T$ has $n-1$ generators, M' is a quotient of M, and N' is a free T-module generated in a dimension from location $[l_1, t]$. As an intermediate step on the way to constructing sequence (3.39), we first make the reduction pictured in Table 3.6. That is, sequence (3.35), which gives the plot pictured on the left side, is traded in for a new long exact sequence, which gives the plot pictured on the right side.

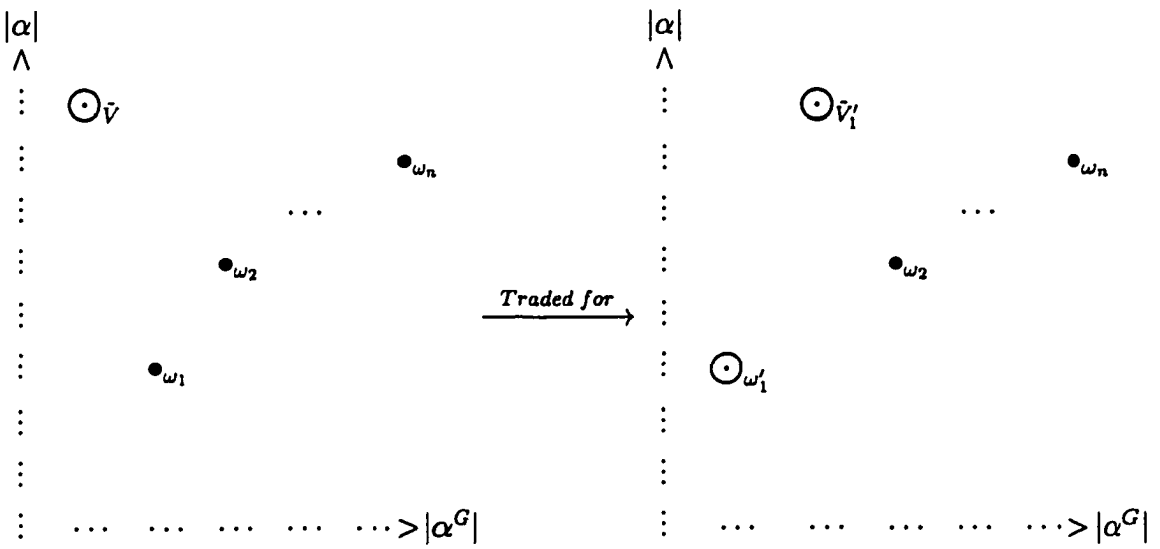


Table 3.6: The Inductive Step

To be precise, we claim that the graded module M from sequence (3.35) also fits into a long exact sequence

$$\longrightarrow N' \oplus J_1' \xrightarrow{\theta} M \longrightarrow J_2 \xrightarrow{(\partial_2', 0)} \Sigma N' \oplus \Sigma J_1' \longrightarrow$$

in which $N' = \Sigma^{\tilde{V}_1'} T$, for some $\tilde{V}_1'$ in $[l_1, t]$, and $\partial_2' : J_2 \longrightarrow \Sigma N'$ is nonzero on each summand of J_2. Recall that $J_1' = \Sigma^{-\omega_1'} T$. The dimension $\tilde{V}_1'$ and the map

$$\theta : N' \oplus J_1' \longrightarrow M$$

are given as part of the following proposition.

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Proposition 3.28. *The dimension $\tilde{V}_1'$ of the generator of the free T-module*

$$N' = \Sigma^{-\tilde{V}_1'} T$$

can be chosen in $[l_1, t]$ so that there are maps

$$\begin{array}{lll} i_1 : N \longrightarrow N' & r_1 : N' \longrightarrow J_1 & \theta_1 : N' \longrightarrow M \\ i_2 : N \longrightarrow J_1' & r_2 : J_1' \longrightarrow J_1 & \theta_2 : J_1' \longrightarrow M \end{array}$$

which make the diagram

$$\begin{array}{ccccccccc} \longrightarrow & N & \xrightarrow{i=(i_1,i_2)} & N' \oplus {J_1}' & \xrightarrow{r=(r_1,r_2)} & J_1 & \xrightarrow{\partial_1} & \Sigma N & \longrightarrow \\ & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta=(\theta_1,\theta_2)} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle id} & \\ \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J_1 \oplus J_2 & \xrightarrow{(\partial_1,\partial_2)} & \Sigma N & \longrightarrow . \end{array} \tag{3.40}$$

commute and make the top row exact. Here, the bottom row is sequence (3.35).

Proposition 3.28 is proven by a sequence of four lemmas.

Lemma 3.29. *There is a map $\theta_2 : J_1' \longrightarrow M$ and a standard shift map $r_2 : J_1' \longrightarrow J_1$ making the square*

$$\begin{array}{ccc} J_1' & \xrightarrow{r_2} & J_1 \\ {\scriptstyle \theta_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ M & \xrightarrow{g} & J_1 \oplus J_2. \end{array}$$

commute.

Proof. Proposition 2.5 tells us that is suffices to define θ_2 and r_2 in dimension ω_1'. Observe that $J_2^{\omega_1'} = 0$ and hence $i_1^{\omega_1'} = id$. Define $\theta_2^{\omega_1'} = id : A \longrightarrow A$, and define $r_2^{\omega_1'} = g^{\omega_1'} : A \longrightarrow R$. This gives the commuting diagram

$$\begin{array}{ccccccccc} & & & & A & \xrightarrow{r_2^{\omega_1'}} & J_1^{\omega_1'} = R & & \\ & & & & {\scriptstyle \theta_2^{\omega_1'}}\downarrow & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & N^{\omega_1'} = \langle \mathbb{Z} \rangle & \xrightarrow{f^{\omega_1'}} & M^{\omega_1'} = A & \xrightarrow{g^{\omega_1'}} & J^{\omega_1'} = R & \longrightarrow & 0 \end{array} \tag{3.41}$$

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in which the bottom row is sequence (3.35) in dimension ω_1'. Since the square in diagram (3.41) commutes, it follows that the square asserted in the lemma commutes. Also observe that, since $r_2^{\omega_1'} = g^{\omega_1'}$ is an epimorphism, r_2 is a horizontal shift map by definition. □

Lemma 3.30. *The dimension $\tilde{V}_1'$ of the generator of $N' = \Sigma^{-\tilde{V}_1'}T$ can be chosen in $[l_1, t]$ so that there are*

(i) a map $\theta_1 : N' \longrightarrow M$ such that $\theta_1^{\tilde{V}_1'}$ is an isomorphism at G/e, and

(ii) a standard shift map $r_1 : N' \longrightarrow J_1$

which make the diagram

$$\begin{array}{ccc} N' & \xrightarrow{r_1} & J_1 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ M & \xrightarrow{g} & J_1 \oplus J_2. \end{array}$$

commute.

Proof. The proof is given in Subsection 3.3.6. □

Observe that the portion

$$\begin{array}{ccccccccc} \longrightarrow & N & & N' \oplus J_1' & \xrightarrow{r=(r_1,r_2)} & J_1 & \xrightarrow{\partial_1} & \Sigma N & \longrightarrow \\ & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta=(\theta_1,\theta_2)} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle id} & \\ \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J_1 \oplus J_2 & \xrightarrow{(\partial_1,\partial_2)} & \Sigma N & \longrightarrow \end{array} \tag{3.42}$$

of diagram (3.40) is complete. To finish proving Proposition 3.28, we need to define the map $i : N \longrightarrow N' \oplus J_1'$ which completes diagram (3.40) and to show that the resulting top row of diagram (3.40) is exact. In order to define i, we first need some technical properties of diagram (3.42) in dimension $\tilde{V}$. Recall from Lemma 3.25 that $M^{\tilde{V}} = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1}$.

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Lemma 3.31. *In the diagram*

$$\begin{array}{ccccccccc} & & & & R\oplus\langle\mathbb{Z}\rangle & \xrightarrow{r^{\tilde{V}}} & \langle\mathbb{Z}/p\rangle & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\theta^{\tilde{V}}} & & \downarrow{\scriptstyle i_1} & & \\ 0 & \longrightarrow & A & \xrightarrow{f^{\tilde{V}}} & R\oplus\langle\mathbb{Z}\rangle\oplus\langle\mathbb{Z}/p\rangle^{n-1} & \xrightarrow{g^{\tilde{V}}} & \langle\mathbb{Z}/p\rangle\oplus\langle\mathbb{Z}/p\rangle^{n-1} & \longrightarrow & 0 \end{array} \tag{3.43}$$

obtained by evaluating diagram (3.42) at $\tilde{V}$,

(a) $\theta_1^{\tilde{V}}$ *is an isomorphism at G/e;*

(b) $\operatorname{Coker}\theta_2^{\tilde{V}} = R\oplus\langle\mathbb{Z}/p\rangle^{n-1}$; *and*

(c) $\operatorname{Ker} r^{\tilde{V}} = A$, $r_2(1) = \pm r_1(\xi)$, *and there is a map $\tilde{i} : A \longrightarrow R\oplus\langle\mathbb{Z}\rangle$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\tilde{i}} & R\oplus\langle\mathbb{Z}\rangle & \xrightarrow{r^{\tilde{V}}} & \langle\mathbb{Z}/p\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle\theta^{\tilde{V}}} & & \downarrow{\scriptstyle i_1} & & \\ 0 & \longrightarrow & A & \xrightarrow{f^{\tilde{V}}} & R\oplus\langle\mathbb{Z}\rangle\oplus\langle\mathbb{Z}/p\rangle^{n-1} & \xrightarrow{g^{\tilde{V}}} & \langle\mathbb{Z}/p\rangle\oplus\langle\mathbb{Z}/p\rangle^{n-1} & \longrightarrow & 0 \end{array}$$

commutes and has exact rows.

Proof. The proof is given in Subsection 3.3.7. □

Lemma 3.32. *There are standard shift maps $i_1 : N \longrightarrow N'$ and $i_2 : N \longrightarrow J_1'$ making the square*

$$\begin{array}{ccc} N & \xrightarrow{i=(i_1,i_2)} & N'\oplus J_1' \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle\theta} \\ N & \xrightarrow{f} & M \end{array}$$

commute. Moreover, the sequence

$$\longrightarrow N \xrightarrow{i} N'\oplus J_1' \xrightarrow{r} J_1 \xrightarrow{\partial_1} \Sigma N \longrightarrow$$

is exact.

Proof. Let $i^{\tilde{V}}$ be the map $\tilde{i}$ given in Lemma 3.31(c). Hence, the diagram

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{r^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \theta^{\tilde{V}}} & & \downarrow{\scriptstyle i_1} & & \\ 0 & \longrightarrow & A & \xrightarrow{f^{\tilde{V}}} & M^{\tilde{V}} & \xrightarrow{g^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle^n & \longrightarrow & 0 \end{array}$$

commutes and has exact rows. Therefore, $i^{\tilde{V}}$ gives a map $i : N \longrightarrow N' \oplus J'$ that fits into the square in the statement of the lemma. Write $i^{\tilde{V}} = (i_1^{\tilde{V}}, i_2^{\tilde{V}})$, where $i_1^{\tilde{V}} : A \longrightarrow R$ and $i_2^{\tilde{V}} : A \longrightarrow \langle \mathbb{Z} \rangle$. By Lemma 3.31(c), $i_1^{\tilde{V}}$ and $i_2^{\tilde{V}}$ are both epimorphisms. Therefore, Lemma 2.8 tells us that the module maps i_1 and i_2 are standard shift maps. Since $\partial_1 : J_1 \longrightarrow \Sigma N$ is a nontrivial map, Theorem C.6 indicates that the sequence displayed in the lemma is a long exact sequence. □

Having completed the proof of Proposition 3.28, we want to exploit the diagram introduced there to produce a diagram yielding sequence (3.39)

$$\longrightarrow N' \longrightarrow M' \longrightarrow J_2 \xrightarrow{\partial_2'} \Sigma N' \longrightarrow .$$

This is done in two stages. In the first stage, we expand diagram (3.40) vertically. Let $\partial_2' = (\Sigma i_1) \circ \partial_2$ and $M' = \operatorname{Coker} \theta_2$.

Proposition 3.33. *The diagram* (3.44)

$$\begin{array}{ccccccccc} & & & & \downarrow & & \downarrow & & \\ & & & & \Sigma^{-1}J_2 & \xrightarrow{id} & \Sigma^{-1}J_2 & & \\ & & & & \downarrow{\scriptstyle (\Sigma^{-1}\partial_2', 0)} & & \downarrow{\scriptstyle 0} & & \\ \longrightarrow & N & \xrightarrow{i} & N' \oplus J_1' & \xrightarrow{r} & J_1 & \xrightarrow{\partial_1} & \Sigma N & \longrightarrow \\ & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle id} & \\ \longrightarrow & N & \xrightarrow{f} & M & \xrightarrow{g} & J_1 \oplus J_2 & \xrightarrow{(\partial_1, \partial_2)} & \Sigma N & \longrightarrow \\ & & & \downarrow{\scriptstyle \pi_2 g} & & \downarrow{\scriptstyle \pi_2} & & & \\ & & & J_2 & \xrightarrow{id} & J_2 & & & \\ & & & \downarrow{\scriptstyle (\partial_2', 0)} & & \downarrow{\scriptstyle 0} & & & \\ & & & \Sigma N' \oplus \Sigma J_1' & \xrightarrow{\Sigma r} & \Sigma J_1 & & & \\ & & & \downarrow & & \downarrow & & & \end{array}$$

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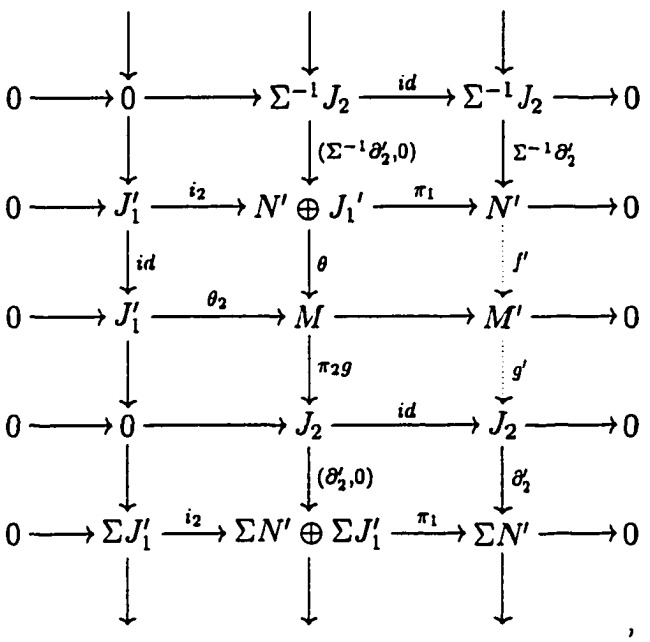
commutes and has exact rows and columns.

Proof. The proof is given in Subsection 3.3.8. □

In the second stage, we form sequence (3.39) as a quotient of the left column of diagram (3.44).

Proposition 3.34. *The diagram*

(3.45)



in which $f' : N' \longrightarrow M'$ and $g' : M' \longrightarrow J_2$ are the obvious induced maps between cokernels, has exact rows and columns.

Proof. The rows and the left column are clearly exact. The middle column is exact by Proposition 3.33. A routine diagram chase therefore gives that the right column is also exact. □

The right column of diagram (3.45) is the desired sequence (3.39). A plot of N' and J_2 is given in Table 3.7. In order to apply the inductive hypothesis and get that M' is free, we need to compute M' in location $[l_1, b_2]$.

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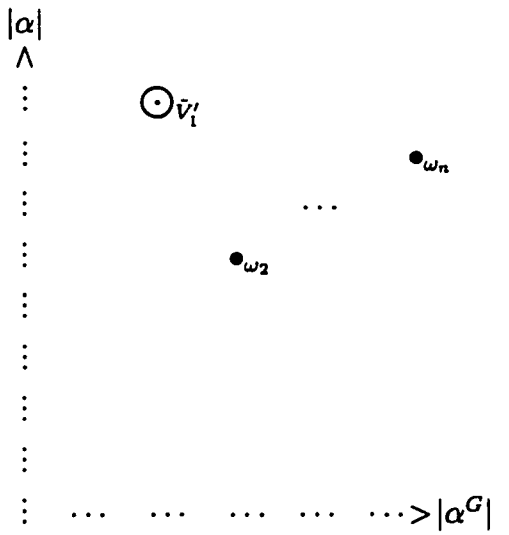


Table 3.7: Plot of Sequence (3.39)

For α in $[l_1, b_2]$, $(J'_1)^\alpha = 0$ and hence $(M')^\alpha = M^\alpha$. Thus, in dimension α, sequence (3.35) reduces to the exact sequence

$$0 \xrightarrow{f^\alpha} (M')^\alpha = M^\alpha \xrightarrow{g^\alpha} \langle \mathbb{Z} \rangle \oplus R \xrightarrow{(\partial_1^\alpha, \partial_2^\alpha)} \langle \mathbb{Z}/p \rangle \longrightarrow 0$$

in which both ∂_1^α and ∂_2^α are nonzero. Lemma A.5 specifies the kernel of the map $(\partial_1^\alpha, \partial_2^\alpha)$ and gives that $(M')^\alpha = A[d]$ for some d. Corollary C.2 therefore tells us that there must be some ω'_2 in $[l_1, b_2]$ such that $(M')^{\omega'_2} = A$.

By the inductive hypothesis, M' must be free and have the form

$$M' = \Sigma^{-\tilde{V}'} T \oplus \bigoplus_{i=2}^{n} \Sigma^{-\omega'_i} T$$

where $\tilde{V}', \omega'_2, \ldots, \omega'_n$ are elements of $RO(G)$ such that $|\tilde{V}'| = |\tilde{V}'_1| = |\tilde{V}|$; $|\omega'_i| = |\omega_i|$, for $2 \leq i \leq n$; $|(\omega'_i)^G| = |\omega_{i-1}^G|$, for $2 \leq i \leq n$; and $|(\tilde{V}')^G| = |\omega_n^G|$. Since M' is free, the middle row of diagram (3.45) splits and gives

$$M \cong J'_1 \oplus M' = \Sigma^{-\tilde{V}'} T \oplus \bigoplus_{i=1}^{n} \Sigma^{-\omega'_i} T$$

as asserted.

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3.3.3 The Proof of Lemma 3.24

(a): Let γ be in $[l_1, t]$ and α be in $[l, t]$. Let $d = d_{\alpha-\tilde{\gamma}}$. Suppose that the sequence

$$0 \longrightarrow N^\gamma = L \xrightarrow{f} M^\gamma \xrightarrow{g} J^\gamma = \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} \longrightarrow 0$$

splits. Since $? \square R$ preserves split short exact sequences, multiplication by $\xi_{\alpha-\gamma}$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[d] & \longrightarrow & M^\alpha & \longrightarrow & \langle \mathbb{Z}/p \rangle^n & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \xi_N} & & \uparrow{\scriptstyle \xi_M} & & \uparrow{\scriptstyle \xi_J}{\scriptstyle \cong} & & \\ 0 & \longrightarrow & N^\gamma \square R & \longrightarrow & M^\gamma \square R & \underset{s}{\longrightarrow} & J^\gamma \square R & \longrightarrow & 0 \end{array}$$

with exact rows. Let $s : J^\gamma \square R \longrightarrow M^\gamma \square R$ be a map splitting the bottom row. Corollary C.4 tells us that the cup product map $\xi_J : J^\gamma \square T^{\alpha-\gamma} \longrightarrow J^\alpha$ is an isomorphism. Therefore, $\xi_M \circ s \circ \xi_J^{-1}$ gives the desired splitting of the top row.

(b): Let α be in $[l, t]$ and β be in $[l, b_1]$. Let $d = d_{\alpha-\tilde{\gamma}}$. Suppose that the short exact sequence

$$0 \longrightarrow N^\alpha \longrightarrow M^\alpha \longrightarrow J^\alpha \longrightarrow 0$$

splits. Since $N^\alpha = A[d]$ and $J^\alpha = \langle \mathbb{Z}/p \rangle^n$, this sequence can be written in the form

$$0 \longrightarrow A[d] \xrightarrow{i_1} A[d] \oplus \langle \mathbb{Z}/p \rangle^n \xrightarrow{\pi_2} \langle \mathbb{Z}/p \rangle^n \longrightarrow 0.$$

Since $N^\beta = \langle \mathbb{Z} \rangle$ and $J^\beta = R$, the short exact sequence in dimension β has the form

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow M^\beta \longrightarrow R \longrightarrow 0. \tag{3.46}$$

Assume that sequence (3.46) does not split. Lemma A.3(b) lists all of the possible solutions to the extension problem given in sequence (3.46). Since sequence (3.46) does not split, it must have the form

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A[d'] \longrightarrow R \longrightarrow 0 \tag{3.47}$$

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for some integer d' prime to p. Applying $? \square \langle \mathbb{Z} \rangle$ to sequence (3.47) gives the exact sequence

$$\langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

Clearly, the first map must be multiplication by p. Therefore, the multiplication map $\epsilon_{\alpha-\beta} : ?^{\beta} \square \langle \mathbb{Z} \rangle \longrightarrow ?^{\alpha}$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[d] & \xrightarrow{i_1} & A[d] \oplus \langle \mathbb{Z}/p \rangle^n & \xrightarrow{\pi_2} & \langle \mathbb{Z}/p \rangle^n & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \epsilon_N} & & \uparrow{\scriptstyle \epsilon_M} & & \uparrow{\scriptstyle \epsilon_J} & & \\ 0 & \longrightarrow & \langle \mathbb{Z} \rangle & \xrightarrow{p} & \langle \mathbb{Z} \rangle & \longrightarrow & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \end{array}$$

with exact rows. Corollary C.4 specifies the cup product map $\epsilon_N : N^{\beta} \square T^{\alpha-\beta} \longrightarrow N^{\alpha}$ and gives that $(\epsilon_N)_G(1) = \pm\kappa$. Let $x = (\epsilon_M)_G(1)$. Since the left hand square commutes, it follows that $\kappa = \pm px$ in $A[d]$ regarded as a summand of M^{α}. Since $\kappa = p\mu - d\tau$, it follows that $p(\mu \mp x) = d\tau$, and hence $p \mid d$. This contradiction shows that sequence (3.46) must split.

3.3.4 The Proof of Lemma 3.26

Note that $|\tilde{V}^G - (\tilde{V}')^G| < 0$ and $|\tilde{V} - \tilde{V}'| = 0$. Multiplication by $\xi_{\tilde{V}-\tilde{V}'}$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{g} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \xi_N} & & \uparrow{\scriptstyle \xi_M} & & \cong \uparrow{\scriptstyle \xi_J} & & \\ & & L \square R & \longrightarrow & A \square R & \longrightarrow & \langle \mathbb{Z} \rangle \square R & \longrightarrow & 0 \\ & & \| \cong & & \| \cong & & \| \cong & & \\ & & L & \xrightarrow{h} & R & \longrightarrow & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \end{array}$$

with exact rows. Here, the top row is sequence (3.36) for $\alpha = \tilde{V}$, and the middle row is $? \square R$ applied to (3.36) for $\alpha = \tilde{V}'$. The bottom row simply gives the values of

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the box products in the middle row. Lemma A.6(b) tells us that the exactness of the bottom row forces $h(\tau) = \pm p\xi$. Corollary C.4 tells us that the cup product map ξ_N is an isomorphism at G/e. If we identify the last two rows, then the square

$$\begin{array}{ccc} A & \xrightarrow{f} & R \oplus \langle \mathbb{Z} \rangle \\ {\scriptstyle \xi_N}\uparrow & & \uparrow{\scriptstyle \xi_M} \\ L & \xrightarrow{h} & R \end{array}$$

gives that

$$\xi_M(p\xi) = \pm f(\xi_N(\tau)) = \pm f(\tau) = (\pm p\xi, 0).$$

Therefore, $\xi_M = \pm i_1$, where $i_1 : R \longrightarrow R \oplus \langle \mathbb{Z} \rangle$ is the inclusion into the first factor.

Note that $|\tilde{V}^G - (\omega_1')^G| = 0$ and $|\tilde{V} - \omega_1'| > 0$. Multiplication by $\epsilon_{\tilde{V}-\omega_1'}$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{g} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \epsilon_N} & & \uparrow{\scriptstyle \epsilon_M} & & \cong \uparrow{\scriptstyle \epsilon_J} & & \\ & & \langle \mathbb{Z} \rangle \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & A \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & R \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & 0 \\ & & \| \cong & & \| \cong & & \| \cong & & \\ & & \langle \mathbb{Z} \rangle & \xrightarrow{h} & \langle \mathbb{Z} \rangle & \longrightarrow & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \end{array}$$

with exact rows. Here, the top row is sequence (3.36) for $\alpha = \tilde{V}$, and the middle row is $? \,\square\, \langle \mathbb{Z} \rangle$ applied to sequence (3.36) for $\alpha = \omega_1'$. The bottom row simply gives the values of the box products in the middle row. The exactness of the bottom row clearly implies that h is multiplication by p. Proposition A.4(c) tells us that the exactness of the top row forces $f(\mu) = (\pm\xi, \pm\epsilon)$. Corollary C.4 gives the value of the cup product map ϵ_N. If we identify the bottom two rows, then the square

$$\begin{array}{ccc} A & \xrightarrow{f} & R \oplus \langle \mathbb{Z} \rangle \\ {\scriptstyle \epsilon_N}\uparrow & & \uparrow{\scriptstyle \epsilon_M} \\ \langle \mathbb{Z} \rangle & \xrightarrow{p} & \langle \mathbb{Z} \rangle \end{array}$$

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gives that

$$\epsilon_M(p) = f(\epsilon_N(1)) = \pm f(\kappa) = (0, \pm p).$$

Therefore, $\epsilon_M = \pm i_2$, where $i_2 : \langle \mathbb{Z} \rangle \longrightarrow R \oplus \langle \mathbb{Z} \rangle$ is the inclusion into the second factor.

Using diagram (2.2) separately for $\theta_1 : N' \longrightarrow M$ and $\theta_2 : J' \longrightarrow M$ gives the diagrams

$$\begin{array}{ccccccc} R & \xrightarrow{\cong} & A \,\square\, R & \xrightarrow[\cong]{\theta^{\tilde{V}'} \square id} & A \,\square\, R & \xrightarrow{\cong} & R \\ & \searrow^{i_1} & \downarrow{\scriptstyle \epsilon_{N'}} & & \downarrow{\scriptstyle \epsilon_M} & \swarrow_{i_1} & \\ & & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{\theta^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle, & & \end{array}$$

and

$$\begin{array}{ccccccc} \langle \mathbb{Z} \rangle & \xrightarrow{\cong} & A \,\square\, \langle \mathbb{Z} \rangle & \xrightarrow[\cong]{\theta^{\omega_1'} \square id} & A \,\square\, \langle \mathbb{Z} \rangle & \xrightarrow{\cong} & \langle \mathbb{Z} \rangle \\ & \searrow^{i_2} & \downarrow{\scriptstyle \epsilon_{J'}} & & \downarrow{\scriptstyle \epsilon_M} & \swarrow_{i_2} & \\ & & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{\theta^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle & & \end{array}$$

which show that $\theta^{\tilde{V}} \circ i_1 = \pm i_1$ and $\theta^{\tilde{V}} \circ i_2 = \pm i_2$. Putting these results together gives that $\theta^{\tilde{V}}$ has the desired properties.

3.3.5 The Proof of Lemma 3.27

Suppose that α is in $[l_1, t]$, and let β be in $[l_1, t-2]$. In dimension β, sequence (3.35) reduces to a short exact sequence,

$$0 \longrightarrow M^\beta \longrightarrow J^\beta \xrightarrow{\partial^\beta} \langle \mathbb{Z}/p \rangle \longrightarrow 0, \tag{3.48}$$

where J^β is either

$$\langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} \quad \text{or} \quad \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2} \oplus R,$$

depending on the value of $|\omega_n|$. If $n \geq 3$, then there is an element of order p in $J^\beta(G/G)$ that ∂^β_G sends to a generator, and hence sequence (3.48) splits. If $n = 2$,

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then ∂^β is either

$$(\partial_1^\beta, \partial_2^\beta) : \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \quad \text{or} \quad (\partial_1^\beta, \partial_2^\beta) : \langle \mathbb{Z} \rangle \oplus R \longrightarrow \langle \mathbb{Z}/p \rangle.$$

Since $\partial_1^\beta \neq 0$ and $\partial_2^\beta \neq 0$, Lemma A.5 lists the kernels of these maps and gives that M^β is either $\langle \mathbb{Z} \rangle$ or $A[d]$, respectively. In any case, applying $? \square \langle \mathbb{Z} \rangle$ to sequence (3.48) gives the short exact sequence,

$$0 \longrightarrow M^\beta \square \langle \mathbb{Z} \rangle \xrightarrow{\tilde{g}} \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} \xrightarrow{\tilde{\partial}} \langle \mathbb{Z}/p \rangle \longrightarrow 0, \tag{3.49}$$

in which $\tilde{\partial}$ is nonzero on each summand. If $n \geq 3$, then the exactness of sequence (3.49) follows from the split exactness of sequence (3.48). If $n = 2$, then it follows from Lemma A.6, which gives examples of right exact sequences that must, in fact, be exact. Since $\tilde{\partial}$ sends an element of order p to a generator, sequence (3.49) must split. Hence $M \square \langle \mathbb{Z} \rangle = \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2}$, and $\tilde{g}$ has a left inverse, say $\tilde{r}$. Therefore, multiplication by $\epsilon_{\alpha-\beta}$ gives the diagram

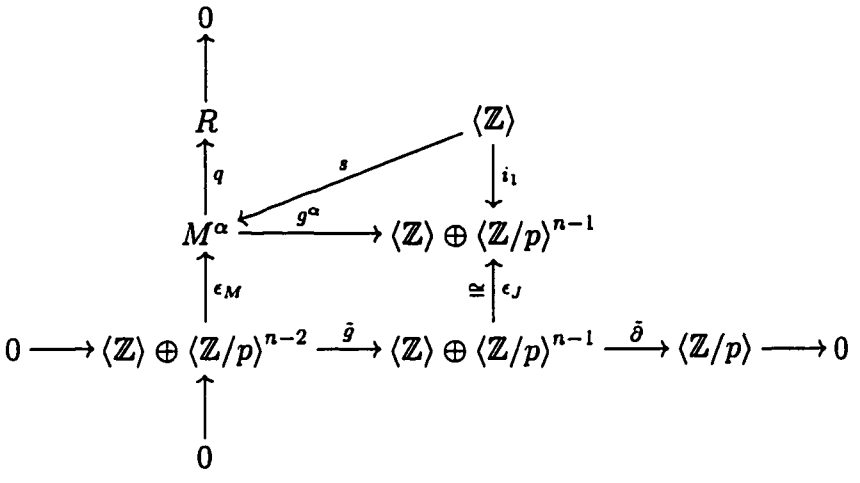
$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f^\alpha} & M^\alpha & \xrightarrow{g^\alpha} & \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle \epsilon_M} & & \cong \uparrow{\scriptstyle \epsilon_J} & & \\ 0 & \longrightarrow & \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2} & \xrightarrow[\tilde{r}]{\tilde{g}} & \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} & \xrightarrow{\tilde{\partial}} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \end{array} \tag{3.50}$$

with exact rows. This shows that $\tilde{r} \circ \epsilon_J^{-1} \circ g^\alpha$ is a left inverse for ϵ_M, and hence ϵ_M is a split injection that displays $\langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2}$ as a summand of M^α. Lemma A.3(f) therefore gives $M^\alpha = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-2}$. Moreover, it follows that $\operatorname{Coker} \epsilon_M = R$.

To establish the asserted property of the map g^α, suppose that there is a map $s : \langle \mathbb{Z} \rangle = J_1^\alpha \longrightarrow M^\alpha$ such that $g^\alpha \circ s = i_1$. Here $i_1 : J_1^\alpha \longrightarrow J_1^\alpha \oplus J_2^\alpha$ is inclusion into

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the first factor. Since $\mathrm{Coker}\,\epsilon_M = R$, there is a commuting diagram



in which the left column and bottom row are exact. Since $Mack(\langle\mathbb{Z}\rangle, R) = 0$, it follows that $q \circ s = 0$. Hence there is a map $s' : \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}\rangle \oplus \langle\mathbb{Z}/p\rangle^{n-2}$ such that $\epsilon_M \circ s' = s$. This gives $i_1 = g^\alpha \circ \epsilon_M \circ s' = \epsilon_J \circ \tilde{g} \circ s'$, and hence i_1 is factored through the kernel of $\tilde{\partial}$. However, $\tilde{\partial}$ is nontrivial on each summand. Hence $\tilde{\partial} \circ i_1 \neq 0$, a contradiction.

3.3.6 The Proof of Lemma 3.30

Let P be the pullback in the diagram

$$\begin{array}{ccc} P & \xrightarrow{h} & J_1 \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle i_1} \\ M & \xrightarrow{g} & J_1 \oplus J_2 \end{array} \tag{3.51}$$

in T-Mod. Let α be in $[l_1, t]$. Hence, $\partial^\alpha = \partial^{\alpha-1} = 0$. Sequence (3.35) reduces to the short exact sequence

$$0 \longrightarrow L \xrightarrow{f^\alpha} M^\alpha \xrightarrow{g^\alpha} \langle\mathbb{Z}\rangle \oplus \langle\mathbb{Z}/p\rangle^{n-1} \longrightarrow 0$$

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in dimension α. Since P^α is a pullback in $Mack$, there is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & P^\alpha & \xrightarrow{h^\alpha} & \langle\mathbb{Z}\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle k^\alpha} & & \downarrow{\scriptstyle i_1} & & \\ 0 & \longrightarrow & L & \xrightarrow{f^\alpha} & M^\alpha & \xrightarrow{g^\alpha} & \langle\mathbb{Z}\rangle \oplus \langle\mathbb{Z}/p\rangle^{n-1} & \longrightarrow & 0. \end{array} \tag{3.52}$$

with exact rows. Lemma A.3(a) indicates that either P^α is $A[d]$, for some integer d prime to p, or the top row of diagram (3.52) splits. If $s' : \langle\mathbb{Z}\rangle \longrightarrow P^\alpha$ were a splitting of the top row, then $s = k^\alpha \circ s'$ would contradict Lemma 3.27. Thus, $P^\alpha = A[d]$ for some integer d prime to p

Corollary C.2 tells us that there must be some $\tilde{V}_1'$ in $[l_1, t]$ such that $P^{\tilde{V}_1'} = A$. Moreover, taking the α in diagram (3.52) to be $\tilde{V}_1'$ gives the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & A & \xrightarrow{h^{\tilde{V}_1'}} & \langle\mathbb{Z}\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle k^{\tilde{V}_1'}} & & \downarrow{\scriptstyle i_1} & & \\ 0 & \longrightarrow & L & \xrightarrow{f^{\tilde{V}_1'}} & M^{\tilde{V}_1'} & \xrightarrow{g^{\tilde{V}_1'}} & \langle\mathbb{Z}\rangle \oplus \langle\mathbb{Z}/p\rangle^{n-1} & \longrightarrow & 0 \end{array} \tag{3.53}$$

with exact rows. Let $N' = \Sigma^{-\tilde{V}_1'}T$, $r_1^{\tilde{V}_1'} = h^{\tilde{V}_1'}$, and $\theta_1^{\tilde{V}_1'} = k^{\tilde{V}_1'}$. Since the right-hand square of diagram (3.53) commutes, the square asserted in the lemma must also commute. The top row of diagram (3.53) shows that $r_1^{\tilde{V}_1'} = h^{\tilde{V}_1'}$ is an epimorphism. Hence, Lemma 2.8 tells us that r_1 is a vertical shift map. The left-hand square of diagram (3.53) shows that $\theta_1^{\tilde{V}_1'} = k^{\tilde{V}_1'}$ must be an isomorphism at G/e.

3.3.7 The Proof of Lemma 3.31

(a): By Lemma 3.30, $\theta_1^{\tilde{V}_1'}$ is an isomorphism at G/e. By using diagram (2.2), $\theta_1^{\tilde{V}}$ can be written as the composite

$$(N')^{\tilde{V}} = R \xrightarrow{\cong} A \,\square\, R \xrightarrow{\theta_1^{\tilde{V}_1'} \square id} M^{\tilde{V}_1'} \,\square\, R \xrightarrow{\xi} M^{\tilde{V}}.$$

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The map $\theta_1^{\tilde{V}}$ is therefore seen to be an isomorphism at G/e by showing that the multiplication map ξ is an isomorphism at G/e. Since sequence (3.35) is a long exact sequence of T-module maps, multiplication by $\xi_{\tilde{V}-\tilde{V}_1'}$ in $T^{\tilde{V}-\tilde{V}_1'} = R$ gives the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^{\tilde{V}} & \longrightarrow & M^{\tilde{V}} & \longrightarrow & J^{\tilde{V}} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \xi^N} & & \uparrow{\scriptstyle \xi} & & \uparrow{\scriptstyle \xi^J} & & \\ & & N^{\tilde{V}_1'} \square R & \longrightarrow & M^{\tilde{V}_1'} \square R & \longrightarrow & J^{\tilde{V}_1'} \square R & \longrightarrow & 0 \end{array}$$

with exact rows. At G/e, this diagram reduces to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cong} & M^{\tilde{V}}(G/e) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \xi^N} & & \uparrow{\scriptstyle \xi_e} & & \uparrow & & \\ & & \mathbb{Z} \otimes \mathbb{Z} & \longrightarrow & M^{\tilde{V}_1'}(G/e) \otimes \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

with exact rows. Since $M^{\tilde{V}_1'}(G/e) = \mathbb{Z}$, the first map on the bottom row is clearly an isomorphism. Hence, the displayed map $\xi_e : (M^{\tilde{V}_1'} \square R)(G/e) \longrightarrow M^{\tilde{V}}(G/e)$ must also be an isomorphism. It follows that $\theta_1^{\tilde{V}}$ is an isomorphism at G/e.

(b): Diagram (2.2) can also be used to factor $\theta_2^{\tilde{V}} : (J_1')^{\tilde{V}} \longrightarrow M^{\tilde{V}}$ as follows:

$$\begin{array}{ccc} A \square \langle \mathbb{Z} \rangle & \xrightarrow{\theta_2^{\omega_1'} \square id} & A \square \langle \mathbb{Z} \rangle \\ \uparrow \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \epsilon_M} \\ \langle \mathbb{Z} \rangle & \xrightarrow{\theta_2^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1}. \end{array}$$

Since $\theta_2^{\omega_1'} = id$, it follows that $\operatorname{Coker} \theta_2^{\tilde{V}}$ is the cokernel of the multiplication map

$$\epsilon_M : A \square \langle \mathbb{Z} \rangle \longrightarrow R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1}.$$

Since multiplication by $\epsilon_{\tilde{V}-\omega_1'}$ gives a module map for each of N, M, and J, the

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following commuting diagram is obtained from sequence (3.35):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M^{\tilde{V}} & \longrightarrow & \langle \mathbb{Z}/p \rangle^{n-1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \epsilon_N} & & \uparrow{\scriptstyle \epsilon_M} & & \uparrow{\scriptstyle \epsilon_J} & & \\
 & & \langle \mathbb{Z} \rangle \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & A \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & R \,\square\, \langle \mathbb{Z} \rangle & \longrightarrow & 0 \\
 & & \| \cong & & \| \cong & & \| \cong & & \\
 & & \langle \mathbb{Z} \rangle & \longrightarrow & \langle \mathbb{Z} \rangle & \longrightarrow & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0.
\end{array}$$

Here, the top row is sequence (3.35) in dimension $\tilde{V}$, the middle row is $? \,\square \langle \mathbb{Z} \rangle$ applied to sequence (3.35) in dimension ω_1', and the bottom row identifies the box products in the middle row. Clearly, the first map in the bottom row must be multiplication by p. Identifying the last two rows therefore gives the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & M^{\tilde{V}} & \longrightarrow & \langle \mathbb{Z}/p \rangle^{n} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \epsilon_N} & & \uparrow{\scriptstyle \epsilon_M} & & \uparrow{\scriptstyle \epsilon_J} & & \\
0 & \longrightarrow & \langle \mathbb{Z} \rangle & \xrightarrow{p} & \langle \mathbb{Z} \rangle & \longrightarrow & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0
\end{array}$$

with exact rows. Corollary C.4 tells us that $\operatorname{Coker} \epsilon_N = R$. Clearly, ϵ_J is injective with $\operatorname{Coker} \epsilon_J = \langle \mathbb{Z}/p \rangle^{n-1}$. Finally, the Snake Lemma gives us the short exact sequence of cokernels

$$0 \longrightarrow R \longrightarrow \operatorname{Coker} \theta_2^{\tilde{V}} \longrightarrow \langle \mathbb{Z}/p \rangle^{n-1} \longrightarrow 0$$

which splits by Propostion A.2.

(c): Since r_1 and r_2 are standard shift maps, Proposition C.3 gives that both $r_1^{\tilde{V}}$ and $r_2^{\tilde{V}}$ are epimorphisms. Therefore, by Lemma A.5, $\operatorname{Ker} r^{\tilde{V}} = A[d]$ where d is defined by the equation $r_2^{\tilde{V}}(1) = d r_1^{\tilde{V}}(\xi)$. Let h be the induced map between the kernels of $r^{\tilde{V}}$ and $g^{\tilde{V}}$. Thus, diagram (3.43) can be expanded to the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A[d] & \xrightarrow{i} & R \oplus \langle \mathbb{Z} \rangle & \xrightarrow{r^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle (\theta_1^{\tilde{V}}, \theta_2^{\tilde{V}})} & & \downarrow{\scriptstyle i_1^{\tilde{V}}} & & \\
0 & \longrightarrow & A & \xrightarrow{f^{\tilde{V}}} & R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{n-1} & \xrightarrow{g^{\tilde{V}}} & \langle \mathbb{Z}/p \rangle^{n} & \longrightarrow & 0
\end{array} \tag{3.54}$$

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with short exact rows. By part (a), the map $\theta_1^{\tilde{V}}$, and hence the map $\theta^{\tilde{V}} = (\theta_1^{\tilde{V}}, \theta_2^{\tilde{V}})$, is an isomorphism at G/e. Since i and $f^{\tilde{V}}$ are clearly isomorphisms at G/e, so is h. Therefore, by Lemma A.7, h is a monomorphism with $\operatorname{Coker} h = \langle \mathbb{Z}/x \rangle$ for some $x \not\equiv 0 \bmod p$. Combining parts (a) and (b) gives that $\operatorname{Coker} \theta^{\tilde{V}} = \langle \mathbb{Z}/p \rangle^{n-1}$. Clearly $i_1^{\tilde{V}}$ is injective with $\operatorname{Coker} i_1 = \langle \mathbb{Z}/p \rangle^{n-1}$. The Snake Lemma therefore gives the short exact sequence

$$0 \longrightarrow \langle \mathbb{Z}/x \rangle \longrightarrow \langle \mathbb{Z}/p \rangle^{n-1} \longrightarrow \langle \mathbb{Z}/p \rangle^{n-1} \longrightarrow 0$$

of cokernels of the vertical maps in diagram (3.54). Hence, $x = \pm 1$, h is an isomorphism, and $d \equiv \pm 1$ modulo p. Therefore, $\operatorname{Ker} r^{\tilde{V}} = A$ and $r_2(1) = \pm r_1(\xi)$. It follows that a map $\tilde{i} : A \longrightarrow R \oplus \langle \mathbb{Z} \rangle$ can be chosen with the asserted properties.

3.3.8 The Proof of Proposition 3.33

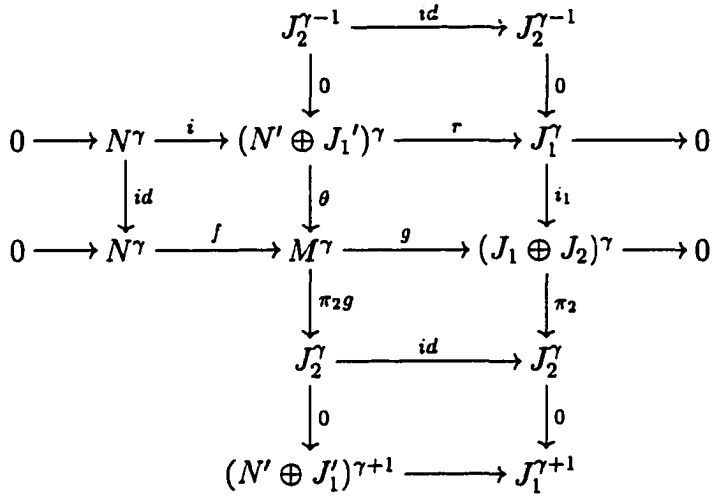
To verify that diagram (3.44) commutes, we need only check that $(\Sigma r) \circ (\partial_2', 0) = 0$. In fact, it suffices to check that this composite is zero in the dimensions $\omega_1, \ldots, \omega_n$ of the generators for J_2. Note that $\Sigma J_1'$ is zero in those dimensions. By definition, $r_1 \circ i_1 + r_2 \circ i_2 = r \circ i = 0$. For each $2 \leq j \leq n$, $r_2^{\omega_j+1} = 0$. Hence, $r_1^{\omega_j+1} \circ i_1^{\omega_j+1} = 0$. It follows that $(\Sigma r_1)^{\omega_j} \circ (\partial_2')^{\omega_j} = r_1^{\omega_j+1} \circ i_1^{\omega_j+1} \circ \partial_2^{\omega_j} = 0$. Hence diagram (3.44) commutes. Proposition 3.28 tells us that its rows are exact. The right column is exact by definition. It remains to prove that the left column is also exact.

Exactness is shown at each of $N' \oplus J_1'$, M, and J_2 for each dimension γ. Our proof is broken into cases based on whether or not the maps ∂_1^γ and ∂_2^γ are zero. Lemma C.5 can be used to determine the range of dimensions handled in each case. Note that $\partial_i^\gamma \neq 0$ implies $\partial_i^{\gamma-1} = 0$, for $i = 1, 2$. Also note that $\partial_2^\gamma = 0$ implies $(\partial_2')^\gamma = 0$.

Case 1: ∂^γ and $\partial^{\gamma-1}$ are both zero.

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Diagram (3.44) reduces to the diagram

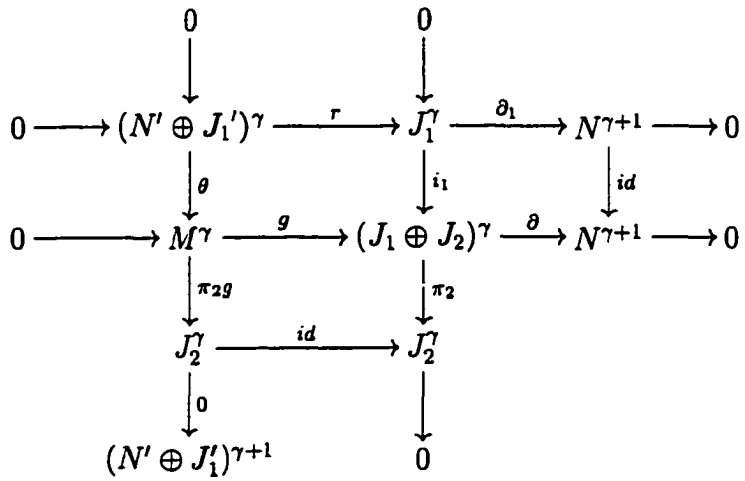


in which the right column is exact. Since the two rows are short exact sequences, the Snake Lemma gives that the middle column reduces to a short exact sequence in dimension γ.

Case 2: $\partial^\gamma \neq 0$ and hence $\partial^{\gamma-1} = 0$.

Note that $J_2^{\gamma-1} = 0$. Also note that $\partial_1^\gamma \neq 0$ implies that $i_1^{\gamma+1} = 0$ and hence $(\partial_2')^\gamma = 0$.

(2a) Suppose $\partial_1^\gamma \neq 0$. Thus ∂_1^γ is an epimorphism, so diagram (3.44) reduces to the diagram

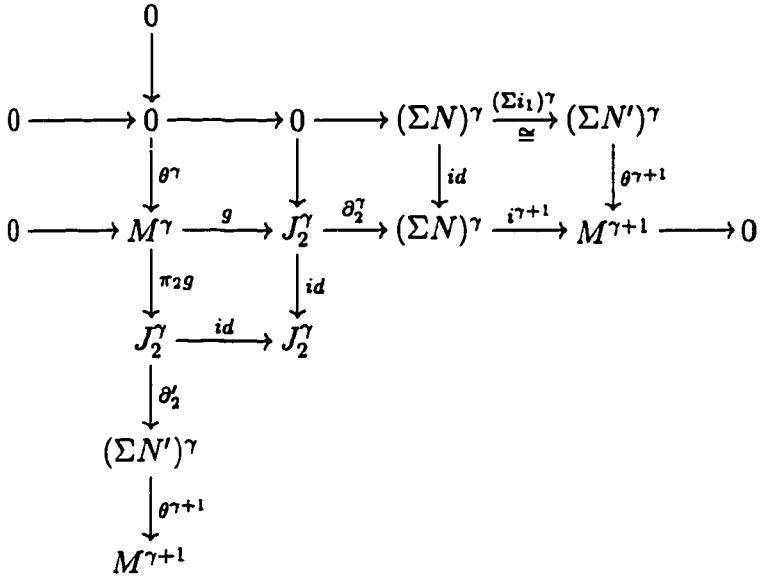


in which the middle column is exact. Since the two rows are short exact sequences,

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the Snake Lemma gives that the first column reduces to a short exact sequence in dimension γ.

(2b) Suppose $\partial_1^\gamma = 0$ and hence $\partial_2^\gamma \neq 0$. Note that $(N' \oplus J_1')^\gamma = J_1^\gamma = J^{\gamma+1} = 0$. Therefore, diagram (3.44) reduces to the diagram



with exact rows. Since $\partial_2^\gamma : J_2^\gamma \longrightarrow \langle \mathbb{Z}/p \rangle = (\Sigma N)^\gamma$ is nonzero, the exactness of the middle row gives that $i^{\gamma+1} = 0$ and hence $M^{\gamma+1} = 0$ and $\theta^{\gamma+1} = 0$. The map

$$(\partial_2')^\gamma = (\Sigma i_1)^\gamma \circ \partial_2^\gamma$$

is displayed in the first two rows as the map from J_2^γ over and up to $(\Sigma N')^\gamma$. In this case $(\partial_2')^\gamma \cong \partial_2^\gamma$, and the exactness of the column in dimension γ follows from the exactness of the second row and the fact that $\pi_2^\gamma = id$.

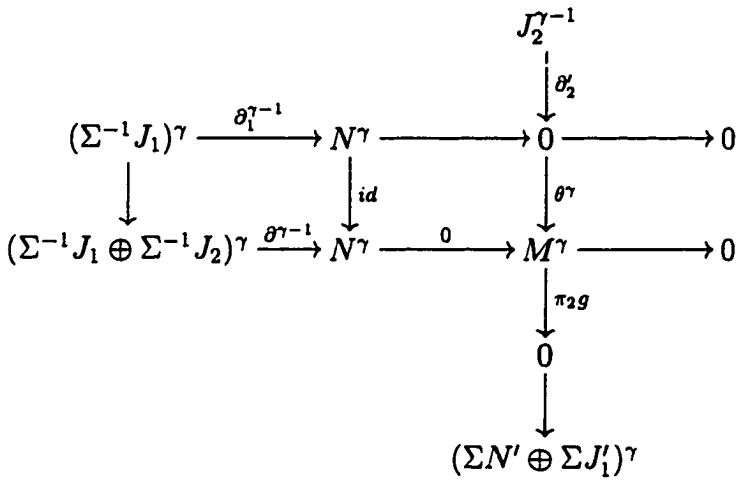
Case 3: $\partial^{\gamma-1} \neq 0$ and hence $\partial^\gamma = 0$.

Note that $(J_1')^\gamma = J_1^\gamma = J_2^\gamma = 0$.

(3a) Suppose $\partial_1^{\gamma-1} \neq 0$. In this case, $(N')^\gamma = 0$ and diagram (3.44) reduces to the

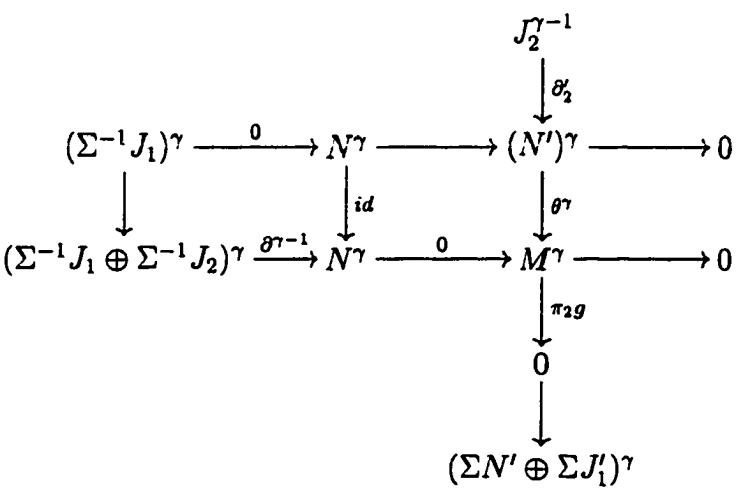
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diagram



with exact rows. The second row gives that $M^\gamma = 0$. Hence, the column is clearly exact in dimension γ.

(3b) Suppose $\partial_1^{\gamma-1} = 0$ and hence $\partial_2^{\gamma-1} \neq 0$. In this case, diagram (3.44) reduces to the diagram



with exact rows. The second row gives that $M^\gamma = 0$. This establishes exactness of the column at M^γ and J_2^γ. Exactness at $(N')^\gamma$ follows since $\theta^\gamma = 0$ and $\partial_2' : J_2^{\gamma-1} \longrightarrow \langle \mathbb{Z}/p \rangle$ is nonzero.

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3.4 A Non-free Example

In Theorem 3.15, the graded module M in the long exact sequence

$$\longrightarrow N \longrightarrow M \longrightarrow J \xrightarrow{\partial} \Sigma N \longrightarrow$$

turns out to be free. However, the additional hypothesis that M have the value A in a particular dimension is used in the proof. This value is certainly necessary for M to be free, but it is not clear that such a value does not follow from the long exact sequence containing M. To justify our need to assume the value of M in the key dimension, we construct a long exact sequence in T-Mod of the form given above in which M is not free.

Theorem 3.35. *There exists a long exact sequence*

$$\longrightarrow N \xrightarrow{f} M \xrightarrow{g} J \xrightarrow{\partial} \Sigma N \longrightarrow \tag{3.55}$$

in T-Mod in which N and J are both free with even-dimensional generators, and M is not free.

Proof. For our example, let $J = \Sigma^{-\omega}T$ and $N = \Sigma^{-V}T$ be monogenic with

$$|V| - |\omega| = |\omega^G| - |V^G| = 2.$$

The notation from the base case of the proof of Theorem 3.15 is used. Table 3.8 gives a plot of the most important locations.

Let $\partial : J \longrightarrow \Sigma N$ be any nonzero map. The module M is constructed from some other easily described modules. Let Z be the submodule of N defined by

$$Z^\alpha = \begin{cases} N^\alpha & \text{if } \alpha \text{ is in } [r+1, b+1] \\ 0 & \text{otherwise.} \end{cases}$$

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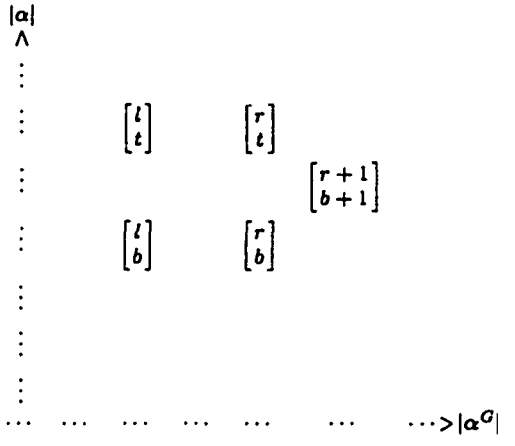


Table 3.8: Notation for Non-free Example

Let M_1 be the cokernel of the inclusion $i : Z \longrightarrow N$, and let $q : N \longrightarrow M_1$ be the associated quotient map. Let $M_2 = \mathrm{Ker}\,\partial$, and define M to be $M_1 \oplus M_2$. Let j denote the inclusion $M_2 \longrightarrow J$. We claim that N, M, and J fit into a long exact sequence of the form given by sequence (3.55) with $f = i_1 \circ q$ and $g = j \circ \pi_2$. Note that, for α in any location except $[r, b]$ and $[r+1, b+1]$,

$$\partial^\alpha = \partial^{\alpha-1} = 0, \quad j^\alpha = id, \quad \text{and} \quad q^\alpha = id.$$

Hence, sequence (3.55) reduces to the split short exact sequence

$$0 \longrightarrow N^\alpha \xrightarrow{i_1} M_1^\alpha \oplus M_2^\alpha \xrightarrow{\pi_2} J^\alpha \longrightarrow 0.$$

For α in $[r, b]$, and hence $\alpha + 1$ in $[r+1, b+1]$, sequence (3.55) reduces to

$$0 \longrightarrow M^\alpha \xrightarrow{j} A[d] \xrightarrow{\partial} \langle \mathbb{Z}/p \rangle \xrightarrow{q} 0 = M^{\alpha+1},$$

which is exact by definition of j^α and q^α. This establishes the claimed exactness of sequence (3.55).

Lemma A.5 lists the kernel of a nontrivial map $A[d] \longrightarrow \langle \mathbb{Z}/p \rangle$ as $L \oplus \langle \mathbb{Z} \rangle$. Hence, for α in $[r, b]$, $M^\alpha = L \oplus \langle \mathbb{Z} \rangle$. In Table 3.9, the values of M in the locations from

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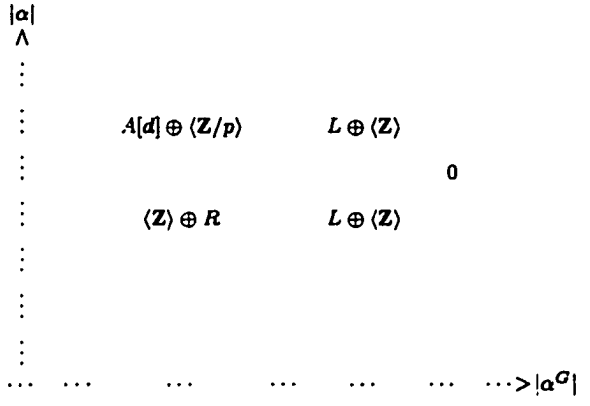


Table 3.9: Key Values for Non-free Example

Table 3.8 are plotted. All other values are easily computed from their associated split short exact sequences. It is easy to see that M is not free, since only one location could contain the Burnside ring A as a summand. Hence, if M were free, then it would have to be free on just one generator. Clearly, this is not the case. □

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Chapter 4

Isomorphic Free Modules

This chapter is devoted to characterizing exactly how much flexibility there is in the dimensions of the generators of a free module when only the coordinates of the generators are fixed. Even when there is only one generator, there is some flexibility.

Lemma 4.1. *Let α and β be in $RO(G)$ with $\gamma = \alpha - \beta$ in $RO_0(G)$. Then, $\Sigma^{-\alpha}T \cong \Sigma^{-\beta}T$ if and only if $d_\gamma \equiv \pm 1$ modulo p.*

Proof. First suppose that $d_\gamma \equiv \eta$ modulo p, where $\eta = \pm 1$. It must also be the case that $d_{-\gamma} \equiv \eta$ modulo p, so define integers n and m by the equations

$$1 = \eta d_\gamma + np = \eta d_{-\gamma} + mp.$$

Proposition 2.5 tells us that graded module maps

$$f : \Sigma^{-\alpha}T \longrightarrow \Sigma^{-\beta}T \quad \text{and} \quad g : \Sigma^{-\beta}T \longrightarrow \Sigma^{-\alpha}T$$

are completely determined by the Mackey functor maps

$$f^\alpha : A \longrightarrow A[d_\gamma] \quad \text{and} \quad g^\beta : A \longrightarrow A[d_{-\gamma}].$$

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Let $f^{\alpha}_{G}(\mu) = \eta\mu_{\gamma} + n\tau_{\gamma}$ and $g^{\beta}_{G}(\mu) = \eta\mu_{-\gamma} + m\tau_{-\gamma}$. It follows that $f^{\alpha}_{e} = id_{\mathbf{Z}}$, $f^{\alpha}_{G}(\tau) = \tau_{\gamma}$, $g^{\beta}_{e} = id_{\mathbf{Z}}$, and $g^{\beta}_{G}(\tau) = \tau_{-\gamma}$. The maps $f^{\beta} : A[d_{-\gamma}] \longrightarrow A$ and $g^{\alpha} : A[d_{\gamma}] \longrightarrow A$ are given by

$$\begin{aligned}
f^{\beta}_{G}(\mu_{-\gamma}) &= f^{\alpha}_{G}(\mu)\mu_{-\gamma} = \eta\mu + (-\eta b_{-\gamma} + nd_{-\gamma})\tau = \eta\mu - \eta m\tau, \\
f^{\beta}_{G}(\tau_{-\gamma}) &= f^{\alpha}_{G}(\mu)\tau_{-\gamma} = \tau, \\
f^{\beta}_{e} &= id_{\mathbf{Z}}, \\
g^{\alpha}_{G}(\mu_{\gamma}) &= g^{\beta}_{G}(\mu)\mu_{\gamma} = \eta\mu + (-\eta b_{\gamma} + md_{\gamma})\tau = \eta\mu - \eta n\tau, \\
g^{\alpha}_{G}(\tau_{\gamma}) &= g^{\beta}_{G}(\mu)\tau_{\gamma} = \tau, \quad \text{and} \\
g^{\alpha}_{e} &= id_{\mathbf{Z}},
\end{aligned}$$

as may be seen from diagram 2.3 and the rules for multiplying elements given in Theorem 4.9 of [9]. Here $b_{\gamma} = b_{-\gamma} = -(\frac{d_{\gamma}d_{-\gamma}-1}{p})$. It is easy to check that $g^{\alpha} = (f^{\alpha})^{-1}$ and $g^{\beta} = (f^{\beta})^{-1}$, and hence $g = f^{-1}$.

Conversely, suppose that $f : \Sigma^{-\alpha}T \longrightarrow \Sigma^{-\beta}T$ is an isomorphism. In particular, $f^{\alpha} : A \longrightarrow A[d_{\gamma}]$ is an isomorphism. Lemma 2.1 gives that $d_{\gamma} \equiv \pm 1$ modulo p. □

When two or more generators are involved, there is a chance for even greater flexibility in the generators. For example, in [9], Lewis shows that

$$A[d] \oplus \langle \mathbb{Z} \rangle \cong A \oplus \langle \mathbb{Z} \rangle \text{ whenever } p \nmid d. \tag{4.1}$$

This implies that, if α_1 and α_2 plot to distinct locations on the same vertical line and β_i plots to the same location as α_i, for $i = 1, 2$, then

$$(\Sigma^{-\alpha_1}T \oplus \Sigma^{-\alpha_2}T)^{\gamma} \cong (\Sigma^{-\beta_1}T \oplus \Sigma^{-\beta_2}T)^{\gamma} \text{ in every dimension } \gamma. \tag{4.2}$$

In fact, equation (4.1) is just one example of an isomorphism that masks the value of d in a sum $A[d] \oplus M$ for some Mackey functor M. The following Lemma characterizes

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the extent to which the value d is masked when $A[d]$ is added to any of the Mackey functors that appear in $\mathbf{H}^*_G(*)$.

Lemma 4.2. *Let d_1, d_2, d_3, and d_4 be integers prime to p. Then*

(a) $A[d_1] \oplus \langle \mathbb{Z} \rangle \cong A[d_2] \oplus \langle \mathbb{Z} \rangle$.

(b) $A[d_1] \oplus R \cong A[d_2] \oplus R$.

(c) $A[d_1] \oplus L \cong A[d_2] \oplus L$.

(d) $A[d_1] \oplus \langle \mathbb{Z}/p \rangle \cong A[d_2] \oplus \langle \mathbb{Z}/p \rangle$ *if and only if* $d_1 \equiv \pm d_2$ *modulo* p.

(e) $A[d_1] \oplus A[d_2] \cong A[d_3] \oplus A[d_4]$ *if and only if* $d_1 d_2 \equiv \pm d_3 d_4$ *modulo* p.

Proof. For each $i = 1, \ldots, 4$, let a_i and b_i be integers such that $a_i d_i + b_i p = 1$. Throughout the proof, elements of the Burnside ring A are written in terms of the standard $\{\mu, \tau, \iota\}$ basis. Also, the standard basis for $A[d_i]$ is denoted by $\{\mu_i, \tau_i, \iota_i\}$, for each $i = 1, \ldots, 4$. Note that, for the proofs of parts (a), (b), and (c), it suffices to consider the case in which $d_2 = 1$.

(a) This result is also established in Examples 1.1(d) of [9], but we present a simpler proof here. Let z and z_1 be generators for the copies of $\langle \mathbb{Z} \rangle(G/G)$ in $A \oplus \langle \mathbb{Z} \rangle$ and $A[d_1] \oplus \langle \mathbb{Z} \rangle$, respectively. Define $f : A \oplus \langle \mathbb{Z} \rangle \longrightarrow A[d_1] \oplus \langle \mathbb{Z} \rangle$ by

$$\begin{aligned} f_G(\mu) &= a_1\mu_1 + b_1\tau_1 - b_1 z_1, \\ f_G(\tau) &= \tau_1, \\ f_G(z) &= p\mu_1 - d_1\tau_1 + d_1 z_1, \quad \text{and} \\ f_e(\iota) &= \iota_1. \end{aligned}$$

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Define $g : A[d_1] \oplus \langle \mathbb{Z} \rangle \longrightarrow A \oplus \langle \mathbb{Z} \rangle$ by

$$\begin{aligned} g_G(\mu_1) &= d_1\mu + b_1 z, \\ g_G(\tau_1) &= \tau, \\ g_G(z_1) &= -p\mu + \tau + a_1 z, \quad \text{and} \\ g_e(\iota_1) &= \iota. \end{aligned}$$

It is easily checked that f and g are well defined and that $g = f^{-1}$.

(b) Let $\{\xi, i\}$ and $\{\xi_1, i_1\}$ denote the standard generators for the copies of R in $A \oplus R$ and $A[d_1] \oplus R$, respectively. Define $f : A \oplus R \longrightarrow A[d_1] \oplus R$ by

$$\begin{aligned} f_G(\mu) &= \mu_1 + \xi_1, \\ f_G(\tau) &= d_1\tau_1 + p\xi_1, \\ f_G(\xi) &= -b_1\tau_1 + a_1\xi_1, \\ f_e(\iota) &= d_1\iota_1 + i_1, \quad \text{and} \\ f_e(i) &= -pb_1\iota_1 + a_1 i_1. \end{aligned}$$

Define $g : A[d_1] \oplus R \longrightarrow A \oplus R$ by

$$\begin{aligned} g_G(\mu_1) &= \mu - b_1\tau - d_1\xi, \\ g_G(\tau_1) &= a_1\tau - p\xi, \\ g_G(\xi_1) &= b_1\tau + d_1\xi, \\ g_e(\iota_1) &= a_1\iota - i, \quad \text{and} \\ g_e(i_1) &= pb_1\iota + d_1 i. \end{aligned}$$

It is easily checked that f and g are well defined and that $g = f^{-1}$.

(c) Let $\{t, i\}$ and $\{t_1, i_1\}$ denote the standard generators for the copies of L in

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$A \oplus L$ and $A[d_1] \oplus L$, respectively. Define $f : A \oplus L \longrightarrow A[d_1] \oplus L$ by

$$\begin{aligned} f_G(\mu) &= \mu_1 - b_1 t_1, \\ f_G(\tau) &= d_1\tau_1 - pb_1 t_1, \\ f_G(t) &= \tau_1 + a_1 t_1, \\ f_e(\iota) &= d_1\iota_1 - pb_1 i_1, \quad \text{and} \\ f_e(i) &= \iota_1 + a_1 i_1. \end{aligned}$$

Define $g : A[d_1] \oplus L \longrightarrow A \oplus L$ by

$$\begin{aligned} g_G(\mu_1) &= \mu - b_1\tau + b_1 d_1 t, \\ g_G(\tau_1) &= a_1\tau + pb_1 t, \\ g_G(t_1) &= -\tau + d_1 t, \\ g_e(\iota_1) &= a_1\iota + pb_1 i, \quad \text{and} \\ g_e(i_1) &= -\iota + d_1 i. \end{aligned}$$

It is easily checked that f and g are well defined and that $g = f^{-1}$.

(d) Under the assumption that $d_1 \equiv \pm d_2$ modulo p, Lemma 2.1 can be used to form the desired isomorphism. Conversely, suppose that

$$f : A[d_1] \oplus \langle \mathbb{Z}/p \rangle \longrightarrow A[d_2] \oplus \langle \mathbb{Z}/p \rangle$$

is an isomorphism. Thus, the generators for $A[d_1]$ and $A[d_2]$ can be chosen so that $f_e(\iota_1) = \iota_2$, and hence $f_G(\tau_1) = \tau_2$. For some integers x and y and some z in $\mathbb{Z}/p$, $f_G(\mu_1) = x\mu_2 + y\tau_2 + z$. Since $f_G(\mu_1)$ and τ_2 must generate a maximal rank free summand of $A[d_2] \oplus \langle \mathbb{Z}/p \rangle$ at G/G, it follows that $x = \pm 1$. Since f commutes with restriction, $\pm d_2 + py = d_1$, i.e. $d_1 \equiv \pm d_2$ modulo p.

(e) Sufficiency of the congruence $d_1 d_2 \equiv \pm d_3 d_4$ modulo p is proven first. Then, that result is used to simplify the proof of necessity.

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For the proof of sufficiency, first let d' and d'' be integers relatively prime to p. It follows that there are integers a', a'', b', and b'' such that $a'd'+b'p = a''d''+b''p = 1$. Let $\{\mu', \tau', \iota'\}$ and $\{\mu'', \tau'', \iota''\}$ denote standard generators for $A[d']$ and $A[d'']$, respectively. Define $f : A[d'] \longrightarrow A[d'']$ by

$$\begin{aligned} f_G(\mu') &= a''d'\mu'' + b''d'\tau'', \\ f_G(\tau') &= \tau'', \quad \text{and} \\ f_e(\iota') &= \iota'', \end{aligned}$$

define $g : A \longrightarrow A[d'']$ by

$$\begin{aligned} g_G(\mu) &= \mu'', \\ g_G(\tau) &= d''\tau'', \quad \text{and} \\ g_e(\iota) &= d''\iota'', \end{aligned}$$

and let $h = (f, g) : A[d'] \oplus A \longrightarrow A[d'']$. Observe that h is an epimorphism,

$$\{\mu' - b''d'\tau' - a''d'\mu, d''\tau' - \tau\}$$

is a basis for $\operatorname{Ker} h_G$, and $d''\iota' - \iota$ generates $\operatorname{Ker} h_e$. By checking the restriction and transfer on these basis elements, it is easy to see that $\operatorname{Ker} h = A[a''d']$. Since $A[d'']$ is projective, the epimorphism h must split. It follows that $A[d'] \oplus A \cong A[d''] \oplus A[a''d']$.

Now suppose that $d_1d_2 \equiv \pm d_3d_4$ modulo p and hence $a_3d_1 \equiv \pm a_2d_4$ modulo p. Setting $d' = a_2d_1$ and $d'' = a_2d_3$ gives

$$\begin{aligned} A[a_2d_1] \oplus A &\cong A[a_2d_3] \oplus A[(a_2d_3)^{-1}a_2d_1] \\ &\cong A[a_2d_3] \oplus A[a_3d_1] \qquad (4.3) \\ &\cong A[a_2d_3] \oplus A[a_2d_4], \end{aligned}$$

where $?^{-1}$ denotes a multiplicative inverse modulo p. Applying $? \square A[d_2]$ to equation (4.3) gives $A[d_1] \oplus A[d_2] \cong A[d_3] \oplus A[d_4]$.

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Conversely, suppose that $A[d_1] \oplus A[d_2] \cong A[d_3] \oplus A[d_4]$. Applying the just established result to both sides of this equation gives $A[d_1 d_2] \oplus A \cong A[d_3 d_4] \oplus A$. Thus, it suffices to suppose that $A[d_1] \oplus A \cong A[d_2] \oplus A$ and to show that $d_1 \equiv \pm d_2$ modulo p.

By assumption, there is an epimorphism $h = (f, g) : A[d_1] \oplus A \longrightarrow A$ with kernel $A[d_2]$. Let $\{\mu, \tau, \iota\}$ and $\{\mu', \tau', \iota'\}$ denote the standard bases for the copies of A in the domain and range of g, respectively. There are integers x, y, α, z, w, and β such that

$$\begin{aligned} f_G(\mu_1) &= x\mu' + y\tau', \\ f_G(\tau_1) &= \alpha\tau', \\ f_e(\iota_1) &= \alpha\iota', \\ g_G(\mu) &= z\mu' + w\tau', \\ g_G(\tau) &= \beta\tau', \quad \text{and} \\ g_e(\iota) &= \beta\iota'. \end{aligned}$$

Since f and g are maps of Mackey functors, $\alpha d_1 = x + py$ and $\beta = z + pw$. Since h is an epimorphism, α and β must be relatively prime, and x and z must be relatively prime. Define $\tau_{ker} = \beta\tau_1 - \alpha\tau$ and $\iota_{ker} = \beta\iota_1 - \alpha\iota$. Since α and β are relatively prime, τ_{ker} and ι_{ker} are basis elements in $\operatorname{Ker} h$. Define $\mu_{ker} = z\mu_1 - x\mu + d_1 w\tau_1 - y\tau$, and observe that this provides another basis element in $\operatorname{Ker} h_G$. By counting rank, it is easy to see that $\{\mu_{ker}, \tau_{ker}, \iota_{ker}\}$ forms a basis for $\operatorname{Ker} h$. By checking the restriction and transfer of these elements, it is easy to see that $\operatorname{Ker} h \cong A[d_1]$. Thus $d_1 \equiv \pm d_2$ modulo p as desired. □

The importance of equation (4.1) is discussed in Remarks 3.2 of [9], where Lewis presents the following example.

Example 4.3. Let ζ, η, and ϕ be distinct irreducible complex representations, and let P be the projective space $P(\zeta \oplus \eta \oplus \phi)$. As Lewis explains, there are two different ways

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to build P as a generalized G-cell complex satisfying his Theorem 2.6. The ordering of the irreducibles (ζ, η, ϕ) displays $\mathbf{H}^*_G(P)$ as having generators in dimensions 0, $\eta^{-1}\zeta$, and $\phi^{-1}\zeta + \phi^{-1}\eta$, while the ordering (ϕ, ζ, η) gives generators in dimensions 0, $\zeta^{-1}\phi$, and $\eta^{-1}\phi + \eta^{-1}\zeta$.

In Remarks 3.2 of [9], Lewis explains how equation (4.1) can be used to see the isomorphism in every dimension between these two seemingly different values for $\mathbf{H}^*_G(P)$. The point being that $\alpha_1 = \eta^{-1}\zeta$, $\alpha_2 = \phi^{-1}\zeta + \phi^{-1}\eta$, $\beta_1 = \zeta^{-1}\phi$, and $\beta_2 = \eta^{-1}\phi + \eta^{-1}\zeta$ satisfy the conditions for the isomorphisms in equation (4.2). In fact, Lemma 4.2 now characterizes exactly what conditions on α_1, α_2, β_1, and β_2 provide the isomorphism in equation (4.2). However, more must be true. The two different sets of generators for $\mathbf{H}^*_G(P)$ in Example 4.3 must also generate isomorphic free T-modules. The following theorem characterizes the flexibility of the generators for a free T-module on two generators.

Theorem 4.4. *Let α_1, β_1, α_2, and β_2 be in $RO(G)$ with $(|\alpha_i^G|, |\alpha_i|) = (|\beta_i^G|, |\beta_i|)$, for $i = 1, 2$. Let $M = \Sigma^{-\alpha_1}T \oplus \Sigma^{-\alpha_2}T$, $N = \Sigma^{-\beta_1}T \oplus \Sigma^{-\beta_2}T$, and $\gamma_i = \alpha_i - \beta_i$, for $i = 1, 2$.*

(a) *If $|\alpha_1^G| = |\alpha_2^G|$ or $|\alpha_1| = |\alpha_2|$, then*
$M \cong N$ if and only if $d_{\gamma_1+\gamma_2} \equiv \pm 1$ modulo p.

(b) *If $|\alpha_1^G| \neq |\alpha_2^G|$ and $|\alpha_1| \neq |\alpha_2|$, then*
$M \cong N$ if and only if $d_{\gamma_i} \equiv \pm 1$ modulo p for $i = 1, 2$.

Remark 4.5. The precise explanation for the two seemingly different values of $\mathbf{H}^*_G(P)$ in Example 4.3 can now be given. In this case, the virtual representation $\gamma_1 + \gamma_2$ of Theorem 4.4(a) reduces to

$$\gamma_1 + \gamma_2 = (\phi^{-1}\zeta - \zeta^{-1}\phi) + (\phi^{-1}\eta - \eta^{-1}\phi).$$

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Since $d_?$ provides a homomorphism from $RSO_0(G)$ to $(\mathbb{Z}/p)^*$ such that $d_{\lambda_k - \lambda_{-k}} = -1$, the expression on the right-hand side gives $d_{\gamma_1+\gamma_2} \equiv 1$ modulo p. Hence, the two different sets of generators for $\mathbf{H}^*_G(P)$ in Example 4.3 provide isomorphic free T-modules, as expected.

Remark 4.6. If $p = 3$, then $d_\gamma \equiv \pm 1$ modulo p for all γ in $RO_0(G)$. Hence, Theorem 4.4 gives that a free T-module is characterized by a plot of its generators if $G = \mathbb{Z}/3$.

Proof of Theorem 4.4. For each part of the proof, the isomorphism $M \cong N$ is given by maps

$$F : M \longrightarrow N \quad \text{and} \quad G : N \longrightarrow M$$

with $G = F^{-1}$. Further, these maps are written as $F = (f_1, f_2)$ and $G = (g_1, g_2)$, where $f_i : \Sigma^{-\alpha_i} T \longrightarrow N$ and $g_i : \Sigma^{-\beta_i} T \longrightarrow M$, for $i = 1, 2$. Proposition 2.5 tells us that these module maps are completely determined by the Mackey functor maps $f_i^{\alpha_i} : A \longrightarrow N^{\alpha_i}$ and $g_i^{\beta_i} : A \longrightarrow M^{\beta_i}$, for $i = 1, 2$. Our argument focuses, in particular, on the Mackey functor maps G^{α_1}, G^{α_2}, F^{β_1}, and F^{β_2}, which need to be isomorphisms. Throughout the proof, these maps of Mackey functors are represented by matrices that only specify their value at the G/G-level. In each case, this information is sufficient to completely specify the map at the G/e-level. Also throughout the proof, whenever γ is in $RSO_0(G)$, b_γ is used to denote the unique integer such that $d_{-\gamma} d_\gamma + b_\gamma p = 1$. Of course $b_\gamma = b_{-\gamma}$. The proof of part (a) is divided into parts separating the different ways in which the hypotheses of the theorem can be satisfied.

(a1) Suppose that $|\alpha_1^G| = |\alpha_2^G|$ and $|\alpha_1| \neq |\alpha_2|$, say $|\alpha_1| > |\alpha_2|$. In this case, the maps G^{α_1}, G^{α_2}, F^{β_1}, and F^{β_2} are either of the form $A[d_1] \oplus \langle \mathbb{Z} \rangle \longrightarrow A[d_2] \oplus \langle \mathbb{Z} \rangle$ or $\langle \mathbb{Z} \rangle \oplus A[d_1] \longrightarrow \langle \mathbb{Z} \rangle \oplus A[d_2]$, for appropriate values d_1 and d_2. Note that the given ordering of the summands of M and N is used to determine the appropriate ordering

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for each instance. Throughout this portion of the proof, maps $A[d_1]\oplus\langle\mathbb{Z}\rangle\longrightarrow A[d_2]\oplus\langle\mathbb{Z}\rangle$ are displayed as 3×3 matrices using the ordered bases of the form $\{\mu,\tau,\epsilon\}$ for the domain and range, and maps $\langle\mathbb{Z}\rangle\oplus A[d_1]\longrightarrow\langle\mathbb{Z}\rangle\oplus A[d_2]$ are displayed as 3×3 matrices using the ordered bases $\{\epsilon^{-1}\kappa,\mu,\tau\}$ for the domain and range.

First suppose that $d_{\gamma_1+\gamma_2}\equiv\eta$ modulo p, where $\eta=\pm 1$. The congruences $d_{-\gamma_1}d_{-\gamma_2}\equiv d_{\gamma_1}d_{\gamma_2}\equiv\eta$ modulo p must also hold, so define integers m, m_1, and m_2 by the equations $d_{-\gamma_1}d_{-\gamma_2}+mp=\eta$, $d_{-\gamma_2}=\eta(d_{\gamma_1}+m_1p)$, and $d_{-\gamma_1}=\eta(d_{\gamma_2}+m_2p)$.

The module maps $f_i:\Sigma^{-\alpha_i}T\longrightarrow N$ and $g_i:\Sigma^{-\beta_i}T\longrightarrow M$, for $i=1,2$, are determined by the Mackey functor maps

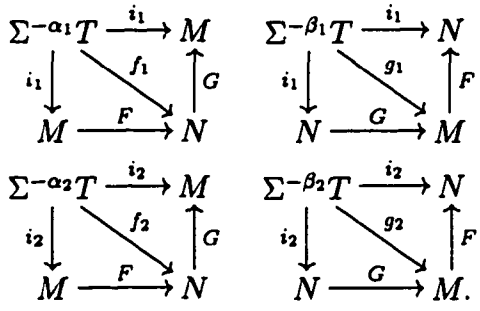
$$\begin{aligned}
f_1^{\alpha_1} &: A\longrightarrow A[d_{\gamma_1}]\oplus\langle\mathbb{Z}\rangle,\\
f_2^{\alpha_2} &: A\longrightarrow\langle\mathbb{Z}\rangle\oplus A[d_{\gamma_2}],\\
g_1^{\beta_1} &: A\longrightarrow A[d_{-\gamma_1}]\oplus\langle\mathbb{Z}\rangle,\quad\text{and}\\
g_2^{\beta_2} &: A\longrightarrow\langle\mathbb{Z}\rangle\oplus A[d_{-\gamma_2}],
\end{aligned}$$

which we define by

$$\begin{aligned}
f_1^{\alpha_1}(\mu) &= d_{-\gamma_1}\mu_{\gamma_1}+b_{\gamma_1}\tau_{\gamma_1}-m\epsilon_{\alpha_1-\beta_2},\\
f_2^{\alpha_2}(\mu) &= \epsilon^{-1}_{\beta_1-\alpha_2}\kappa_0+d_{-\gamma_2}\mu_{\gamma_2}+b_{\gamma_2}\tau_{\gamma_2},\\
g_1^{\beta_1}(\mu) &= \eta d_{-\gamma_2}\mu_{-\gamma_1}+\eta m\tau_{-\gamma_1}+\eta m\epsilon_{\beta_1-\alpha_2},\quad\text{and}\\
g_2^{\beta_2}(\mu) &= -\eta\epsilon^{-1}_{\alpha_1-\beta_2}\kappa_0+\eta d_{-\gamma_1}\mu_{-\gamma_2}+\eta m\tau_{-\gamma_2}.
\end{aligned}$$

The graded module map $F=(f_1,f_2):M\longrightarrow N$ is shown to be an isomorphism with inverse $G=(g_1,g_2):N\longrightarrow M$, by showing that the following four diagrams commute:

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(4.4)

Since, in each case, the module in the upper left-hand corner is a free module on a single generator, it suffices to verify the equations

$$\begin{aligned} G^{\alpha_1} f_1^{\alpha_1}(\mu) &= \mu, \\ G^{\alpha_2} f_2^{\alpha_2}(\mu) &= \mu, \\ F^{\beta_1} g_1^{\beta_1}(\mu) &= \mu, \quad \text{and} \\ F^{\beta_2} g_2^{\beta_2}(\mu) &= \mu. \end{aligned} \tag{4.5}$$

By using diagram 2.3 and the rules for multiplying elements given in Theorem 4.9 of [9], the maps F and G can be completely determined in any dimension. For example, if z is an element of $(\Sigma^{-\alpha_i}T)^{\beta_j}(G/G) = T^{\beta_j-\alpha_i}(G/G)$, then $f_i^{\beta_j}(z) = f_i^{\alpha_i}(\mu)z$. Straightforward computations thus yield the required maps.

$$G^{\alpha_1} = \begin{bmatrix} \eta d_{-\gamma_2} & 0 & -\eta p \\ -m_1 & 1 & \eta \\ \eta m & 0 & \eta d_{-\gamma_1} \end{bmatrix}$$

$$G^{\alpha_2} = \begin{bmatrix} \eta d_{-\gamma_2} & -\eta & 0 \\ \eta m p & \eta d_{-\gamma_1} & 0 \\ -\eta m & -m_2 & 1 \end{bmatrix}$$

$$F^{\beta_1} = \begin{bmatrix} d_{-\gamma_1} & 0 & p \\ 0 & 1 & -1 \\ -m & 0 & d_{-\gamma_2} \end{bmatrix}$$

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$$F^{\beta_2} = \begin{bmatrix} d_{-\gamma_1} & 1 & 0 \\ -mp & d_{-\gamma_2} & 0 \\ m & 0 & 1 \end{bmatrix}$$

Equations (4.5) can now be verified by using these matrices and the appropriate bases for each case. For example,

$$G^{\alpha_1} f_1^{\alpha_1}(\mu) = \begin{bmatrix} \eta d_{-\gamma_2} & 0 & -\eta p \\ -m_1 & 1 & \eta \\ \eta m & 0 & \eta d_{-\gamma_1} \end{bmatrix} \begin{bmatrix} d_{-\gamma_1} \\ b_{\gamma_1} \\ -m \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mu.$$

All of the entries resulting from the matrix multiplications are straightforward to verify. They all follow from the definitions of the values involved except the second entry of $G^{\alpha_1} f_1^{\alpha_1}(\mu)$ and the third entry of $G^{\alpha_2} f_2^{\alpha_2}(\mu)$. These values are seen to be zero by using the equations

$$m = \eta(b_{\gamma_1} - d_{-\gamma_1} m_1) = \eta(b_{\gamma_2} - d_{-\gamma_2} m_2),$$

which can be checked by plugging them into the equation defining m. Therefore, $F : M \cong N$.

Conversely, suppose that $F : M \cong N$, and say $f_1^{\alpha_1}(\mu) = a_1\mu_{\gamma_1} + b_1\tau_{\gamma_1} + x_1\epsilon_{\alpha_1-\beta_2}$ and $f_2^{\alpha_2}(\mu) = x_2\epsilon^{-1}_{\beta_1-\alpha_2}\kappa_0 + a_2\mu_{\gamma_2} + b_2\tau_{\gamma_2}$. Since $f_1^{\alpha_1}$ and $f_2^{\alpha_2}$ must be isomorphisms at the G/e-level, applying restrictions to the above equations shows that $a_i d_{\gamma_i} + b_i p = \pm 1$, for $i = 1, 2$. That is, $a_i \equiv \pm d_{-\gamma_i}$ modulo p, for $i = 1, 2$. As above, a straightforward computation yields

$$F^{\alpha_1} = \begin{bmatrix} a_1 & 0 & x_2 p \\ b_1 & \pm 1 & -x_2 d_{\gamma_1} \\ x_1 & 0 & a_2 \end{bmatrix}.$$

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Since this matrix must have determinant ± 1, the equation $a_1a_2 - x_1x_2p = \pm 1$ gives the congruence $d_{-\gamma_1}d_{-\gamma_2} \equiv \pm a_1a_2 \equiv \pm 1$ modulo p. Of course, this is equivalent to the congruence $d_{\gamma_1+\gamma_2} \equiv d_{\gamma_1}d_{\gamma_2} \equiv \pm 1$ modulo p, as desired.

(a2) Suppose that $|\alpha_1^G| \neq |\alpha_2^G|$ and $|\alpha_1| = |\alpha_2|$, say $|\alpha_1^G| < |\alpha_2^G|$. In this case, the maps G^{α_1}, G^{α_2}, F^{β_1}, and F^{β_2} are either of the form $L \oplus A[d_1] \longrightarrow L \oplus A[d_2]$ or $A[d_1] \oplus R \longrightarrow A[d_2] \oplus R$. Throughout this portion of the proof, t is used to denote the generator of $L(G/G)$ to distinguish it from the generator τ in $A[d](G/G)$. Also, maps $L \oplus A[d_1] \longrightarrow L \oplus A[d_2]$ are displayed as 3×3 matrices using the ordered bases $\{t, \mu, \tau\}$, for the domain and range, and maps $A[d_1] \oplus R \longrightarrow A[d_2] \oplus R$ are displayed as 3×3 matrices using the ordered bases $\{\mu, \tau, \xi\}$, for the domain and range.

First suppose that $d_{\gamma_1+\gamma_2} \equiv \eta$ modulo p, where $\eta = \pm 1$. Hence there is an integer m such that $d_{\gamma_1}d_{\gamma_2} + mp = \eta$. Further, define integers m_1 and m_2 by the equations $d_{\gamma_2} = \eta(d_{-\gamma_1} + m_1p)$ and $d_{\gamma_1} = \eta(d_{-\gamma_2} + m_2p)$, respectively.

The module maps $f_i : \Sigma^{-\alpha_i}T \longrightarrow N$ and $g_i : \Sigma^{-\beta_i}T \longrightarrow M$, for $i = 1, 2$, are determined by the Mackey functor maps

$$\begin{aligned} f_1^{\alpha_1} &: A \longrightarrow A[d_{\gamma_1}] \oplus R, \\ f_2^{\alpha_2} &: A \longrightarrow L \oplus A[d_{\gamma_2}], \\ g_1^{\beta_1} &: A \longrightarrow A[d_{-\gamma_1}] \oplus R, \quad \text{and} \\ g_2^{\beta_2} &: A \longrightarrow L \oplus A[d_{-\gamma_2}], \end{aligned}$$

which we define by

$$\begin{aligned} f_1^{\alpha_1}(\mu) &= \mu_{\gamma_1} + \xi_{\alpha_1-\beta_2}, \\ f_2^{\alpha_2}(\mu) &= -mt_{\alpha_2-\beta_1} + \mu_{\gamma_2}, \\ g_1^{\beta_1}(\mu) &= \mu_{-\gamma_1} + m_1\tau_{-\gamma_1} - \eta\xi_{\beta_1-\alpha_2}, \quad \text{and} \\ g_2^{\beta_2}(\mu) &= \eta m t_{\beta_2-\alpha_1} + \mu_{-\gamma_2} + m_2\tau_{-\gamma_2}. \end{aligned}$$

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As above, straightforward computations yield the required maps.

$$G^{\alpha_1} = \begin{bmatrix} 1 & 0 & 0 \\ m_1 d_{\gamma_1} - b_{\gamma_1} & \eta d_{\gamma_2} & \eta m \\ -\eta d_{\gamma_1} & -\eta p & \eta d_{\gamma_1} \end{bmatrix}$$

$$G^{\alpha_2} = \begin{bmatrix} \eta d_{\gamma_2} & \eta m d_{\gamma_2} & \eta m p \\ 0 & 1 & 0 \\ -\eta & m_2 d_{\gamma_2} - b_{\gamma_2} & \eta d_{\gamma_1} \end{bmatrix}$$

$$F^{\beta_1} = \begin{bmatrix} 1 & 0 & 0 \\ -b_{\gamma_1} & d_{\gamma_1} & -m \\ d_{-\gamma_1} & p & d_{\gamma_2} \end{bmatrix}$$

$$F^{\beta_2} = \begin{bmatrix} d_{\gamma_1} & -m d_{-\gamma_2} & -mp \\ 0 & 1 & 0 \\ 1 & -b_{\gamma_2} & d_{\gamma_2} \end{bmatrix}$$

Equations (4.5) can now be verified by using these matrices and the appropriate bases for each case. For example,

$$G^{\alpha_1} f_1^{\alpha_1}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ m_1 d_{\gamma_1} - b_{\gamma_1} & \eta d_{\gamma_2} & \eta m \\ -\eta d_{\gamma_1} & -\eta p & \eta d_{\gamma_1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mu.$$

The entries resulting from the matrix multiplications are straightforward to verify. They follow from the definitions of the values involved and the equations

$$m_1 = \eta(b_{\gamma_1} d_{\gamma_2} - m d_{-\gamma_1}) \quad \text{and} \quad m_2 = \eta(b_{\gamma_2} d_{\gamma_1} - m d_{-\gamma_2}).$$

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As before, this shows that $F : M \cong N$.

Conversely, suppose that $F : M \cong N$, and say $f_1^{\alpha_1}(\mu) = x\mu_{\gamma_1} + y\tau_{\gamma_1} + z\xi_{\alpha_1-\beta_2}$ and $f_2^{\alpha_2}(\mu) = ut_{\alpha_2-\beta_1} + v\mu_{\gamma_2} + w\tau_{\gamma_2}$. Straightforward computations yield

$$F^{\alpha_1} = \begin{bmatrix} x & 0 & 0 \\ y & xd_{\gamma_1}+py & u \\ z & pz & vd_{\gamma_2}+pw \end{bmatrix} \quad F^{\alpha_2} = \begin{bmatrix} xd_{\gamma_1}+py & u & pu \\ 0 & v & 0 \\ z & w & vd_{\gamma_2}+pw \end{bmatrix}.$$

Since F^{α_1} and F^{α_2} must be isomorphisms, and x and v are seen to be factors of $det(F^{\alpha_1})$ and $det(F^{\alpha_2})$, respectively, this forces $x = \pm 1$ and $v = \pm 1$. The fact that $det(F^{\alpha_1}) = \pm 1$ now displays the desired congruence $d_{\gamma_1}d_{\gamma_2} \equiv \pm 1$ modulo p.

(a3) Suppose that $|\alpha_1^G| = |\alpha_2^G|$ and $|\alpha_1| = |\alpha_2|$. In the proof in this case, unlike the previous two, the map G is not used. Instead we show directly that F is an isomorphism in every dimension.

First suppose that $d_{\gamma_1+\gamma_2} \equiv \eta$ modulo p, where $\eta = \pm 1$, and define m by the equation $d_{\gamma_1}d_{\gamma_2} + pm = \eta$. Let $\bar{s}$ and $\bar{u}$ be integers such that $\bar{s}d_{\gamma_2} - p\bar{u} = 1$, and define $s = \bar{s}m$ and $u = \bar{u}m$. Thus, $sd_{\gamma_2} - pu = m$.

The module maps $f_i : \Sigma^{-\alpha_i}T \longrightarrow N$, for $i = 1, 2$, are determined by the Mackey functor maps

$$f_1^{\alpha_1} : A \longrightarrow A[d_{\gamma_1}] \oplus A[d_{\alpha_1-\beta_2}], \quad \text{and}$$
$$f_2^{\alpha_2} : A \longrightarrow A[d_{\alpha_2-\beta_1}] \oplus A[d_{\gamma_2}],$$

which we define by

$$f_1^{\alpha_1}(\mu) = \mu_{\gamma_1} + s\tau_{\gamma_1} + u\tau_{\alpha_1-\beta_2}, \quad \text{and}$$
$$f_2^{\alpha_2}(\mu) = \tau_{\alpha_2-\beta_1} + \mu_{\gamma_2}.$$

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Straightforward computations yield a description of the map

$$F^{\alpha_1} : A \oplus A[d_{\alpha_1-\alpha_2}] \longrightarrow A[d_{\gamma_1}] \oplus A[d_{\alpha_1-\beta_2}].$$

Here, F^{α_1} is displayed as a 4×4 matrix using the ordered bases $\{\mu, \tau, \mu_{\alpha_1-\alpha_2}, \tau_{\alpha_1-\alpha_2}\}$ and $\{\mu_{\gamma_1}, \tau_{\gamma_1}, \mu_{\alpha_1-\beta_2}, \tau_{\alpha_1-\beta_2}\}$ for the domain and range, respectively.

$$F^{\alpha_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s & d_{\gamma_1} + ps & d_{\alpha_1-\alpha_2} & p \\ 0 & 0 & 1 & 0 \\ u & pu & \frac{d_{\gamma_2} d_{\alpha_1-\alpha_2} - d_{\alpha_1-\beta_2}}{p} & d_{\gamma_2} \end{bmatrix}$$

The determinant of this matrix is

$$(d_{\gamma_1} + ps)d_{\gamma_2} - p^2 u = d_{\gamma_1} d_{\gamma_2} + pm\bar{s}d_{\gamma_2} - p^2 m\bar{u} = d_{\gamma_1} d_{\gamma_2} + pm(\bar{s}d_{\gamma_2} - p\bar{u}) = \eta$$

Hence, F^{α_1} is an isomorphism.

Let γ be any element of $RO(G)$. By using diagram (2.2), F^{γ} can be written in terms of F^{α_1}.

$$\begin{array}{ccc} T^{\gamma-\alpha_1} \square M^{\alpha_1} & \xrightarrow[\cong]{1\square F^{\alpha_1}} & T^{\gamma-\alpha_1} \square N^{\alpha_1} \\ \downarrow & & \downarrow \\ M^{\gamma} & \xrightarrow{\quad F^{\gamma} \quad} & N^{\gamma} \end{array}$$

Note that $M^{\alpha_1} = T \oplus T^{\alpha_1-\alpha_2}$ and $N^{\alpha_1} = T^{\alpha_1-\beta_1} \oplus T^{\alpha_1-\beta_2}$. Since each of 0, $\alpha_1 - \alpha_2$, $\alpha_1 - \beta_1$, and $\alpha_1 - \beta_2$ are in $RO_0(G)$, Proposition C.1 gives that the vertical maps are isomorphisms. Hence, F^{γ} must be an isomorphism.

Conversely, suppose that $F : M \cong N$. Thus, the map

$$F^{\alpha_1} : A \oplus A[d_{\alpha_1-\alpha_2}] \longrightarrow A[d_{\gamma_1}] \oplus A[d_{\alpha_1-\beta_2}]$$

is an isomorphism. By Lemma 4.2, this is equivalent to the congruence

$$d_{\alpha_1-\alpha_2} \equiv \pm d_{\gamma_1} d_{\alpha_1-\beta_2} \text{ modulo } p.$$

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But $\gamma_1 + (\alpha_1 - \beta_2) = (\alpha_1 - \alpha_2) + (\gamma_1 + \gamma_2)$, so $d_{\alpha_1-\alpha_2} \equiv \pm d_{\alpha_1-\alpha_2} d_{\gamma_1+\gamma_2}$ modulo p. Therefore $d_{\gamma_1+\gamma_2} \equiv \pm 1$ modulo p, as desired.

(b) Suppose that $|\alpha_1^G| \neq |\alpha_2^G|$ and $|\alpha_1| \neq |\alpha_2|$. For any possible relationship between the coordinates of α_1 and α_2, Lemma 4.2 implies that $M \cong N$ if and only if $\Sigma^{-\alpha_i} T \cong \Sigma^{-\beta_i} T$, for $i = 1, 2$. Lemma 4.1 therefore gives the desired conclusions. □

In the case of $T(R)$-*Mod* and $T(\langle\mathbb{Z}\rangle)$-*Mod*, there is much more flexibility in the generators for free modules.

Lemma 4.7. *Let α and β be elements of $RO(G)$.*

(a) $\Sigma^{-\alpha} T(R) \cong \Sigma^{-\beta} T(R)$ *if and only if* $\alpha - \beta$ *is in* $RO_0(G)$.

(b) $\Sigma^{-\alpha} T(\langle\mathbb{Z}\rangle) \cong \Sigma^{-\beta} T(\langle\mathbb{Z}\rangle)$ *if and only if* $|\alpha^G - \beta^G| = 0$.

Proof. Since the proof in both cases is basically the same, let S denote either R or $\langle\mathbb{Z}\rangle$. Clearly the conditions on α and β are necessary, so our proof focuses on showing sufficiency. Since Σ^β can be applied to both sides of the desired equation, it suffices to consider the case in which $\beta = 0$. Define

$$f : T(S) \longrightarrow \Sigma^{-\alpha} T(S)$$

by taking $f^0 : S \longrightarrow S$ to be an isomorphism. Let γ be in $RO(G)$. By using diagram (2.2), f^γ can be written in terms of f^0.

$$\begin{array}{ccc} T(S)^\gamma \square T(S)^0 & \xrightarrow[\cong]{1\square f^0} & T(S)^\gamma \square T(S)^\alpha \\ \downarrow & & \downarrow \\ T(S)^\gamma & \xrightarrow{\quad f^\gamma \quad} & T(S)^{\gamma+\alpha} \end{array} \tag{4.6}$$

Remark B.17 describes the multiplicative structure of $T(S)$ and gives that the vertical maps are isomorphisms. This implies that f^γ is also an isomorphism. □

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Remark 4.8. The flexibility in the dimensions of the generators of free $T(S)$-modules for $S = R$ and $\langle\mathbb{Z}\rangle$ is now clear.

(a) A free $T(R)$-module is characterized by a plot of its generators.

(b) A free $T(\langle\mathbb{Z}\rangle)$-module is characterized by the number of generators in each column of its plot.

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Chapter 5

An Application of Theorem 3.4

In this chapter, we consider the examples of generalized G-cell complexes given by equivariant complex Grassmannians with linear $\mathbb{Z}/p$ actions. The first section describes how the well-known Schubert cells can be given the structure of representation cells. In the second section, we apply Theorem 3.4.

5.1 Complex Grassmannians

Given a complex G-representation V and a positive integer k, the Grassmannian $G(V,k)$ is the space of complex k-dimensional subspaces of V. Since the action of G on V permutes the k-spaces, $G(V,k)$ inherits a natural G-action. In fact, $G(V,k)$ has the structure of a generalized G-cell complex. This section is devoted to describing that cell structure.

The set of isomorphism classes of finite dimensional complex G-representations forms a ring, where multiplication is given by the tensor product over $\mathbb{C}$ and denoted by juxtaposition. In Section 2.3, we introduced the irreducible complex representations $\lambda_1, \ldots, \lambda_{p-1}$, where $\lambda_k(m) = e^{2\pi km i/p}$, for m in $\mathbb{Z}/p$. Additionally, let λ_0 denote

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the trivial one-dimensional complex representation. Of course, $\lambda_j \lambda_k = \lambda_{j+k}$ with subscripts read modulo p.

The G-cell structure for $G(V,k)$ comes from an equivariant version of the standard Schubert cells for Grassmannians, as discussed in [6, 7, 15]. Suppose that V is a finite dimensional complex representation and express $V = \bigoplus_{i=1}^{m} \phi_i$ as a sum of irreducibles. Moreover, fix an ordering of the irreducibles, $\phi_1, \ldots, \phi_m$, and hence fix a flag of subspaces

$$0 < V_1 < V_2 < \cdots < V_m = V,$$

where $V_j = \bigoplus_{i=1}^{j} \phi_i$ for each $1 \leq j \leq m$. In terms of this fixed flag, the standard Schubert cells can now be described as usual. Here, we most closely follow the notation used in [7].

Given a sequence of integers $0 \leq a_1 \leq \ldots \leq a_k \leq m-k$, define

$$\langle a_1, \ldots, a_k \rangle = \{X \in G(V,k) : dim_{\mathbb{C}}(X \cap V_{a_i+i}) = i \text{ for } 1 \leq i \leq k\}.$$

The cell $\langle a_1, \ldots, a_k \rangle$ is usually represented as a matrix

$$\begin{array}{c} \phi_1 \quad \cdots \quad \phi_{a_1} \quad \phi_{a_1+1} \quad \cdots \quad \phi_{a_2+2} \quad \cdots \quad \phi_{a_k+k} \quad \cdots \quad \phi_m \\ \left[\begin{array}{ccccccccccccccc} * & \cdots & * & 1 & 0 & \cdots & & & & & & & & & \\ * & \cdots & * & 0 & * & \cdots & * & 1 & 0 & \cdots & & & & & \\ \vdots & & & & & & & & & \ddots & & & & & \\ * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & 1 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

written in standard form. The symbol $*$ is used to denote an arbitrary complex number, and there are a_i of them in the i^{th} row, for each $1 \leq i \leq k$. The G-action on the cell $\langle a_1, \ldots, a_k \rangle$ comes from letting G act on each entry of the matrix according to the irreducible representations above the columns. Hence, equivariantly the cell

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$\langle a_1, \ldots, a_k \rangle$ is the interior of the representation cell DW where

$$W = \bigoplus_{i=1}^{k} \bigoplus_{\substack{j=1 \\ j \notin \{a_1+1, \ldots, a_{i-1}+i-1\}}}^{a_i+i-1} \phi_{a_i+i}^{-1} \phi_j.$$

Observe that the (real) dimension of $\langle a_1, \ldots, a_k \rangle$ is $|W| = 2\Sigma_{i=1}^{k} a_i$.

The set of cells $\langle a_1, \ldots, a_k \rangle$, for $0 \leq a_1 \leq \ldots \leq a_k \leq m-k$, forms a G-cell structure for $G(V,k)$. The basepoint $\langle 0, \ldots, 0 \rangle = V_k$ forms X_0, and $\langle a_1, \ldots, a_k \rangle$ can be attached after first building the subspace containing all cells $\langle b_1, \ldots, b_k \rangle$ with $b_i \leq a_i$ for each $1 \leq i \leq k$ and $b_j < a_j$ for some j. Typically, this is accomplished by taking $X_n - X_{n-1}$ to be $\{\langle a_1, \ldots, a_k \rangle : \Sigma_{i=1}^{k} a_i = n\}$, but there are other filtrations that are also useful. Let DW be the cell $\overline{\langle a_1, \ldots, a_k \rangle}$ and note that $DW \cong DW_1 \times \cdots \times DW_k$, where

$$W_i = \bigoplus_{\substack{j=1 \\ j \notin \{a_1+1, \ldots, a_{i-1}+i-1\}}}^{a_i+i-1} \phi_{a_i+i}^{-1} \phi_j$$

for each $1 \leq i \leq k$. The attaching map for DW is the restriction to the boundary of the map

$$DW_1 \times \cdots \times DW_k \longrightarrow \{\langle b_1, \ldots, b_k \rangle : b_i < a_i \text{ for } 1 \leq i \leq k\}$$

that sends the point

$$((z_{(1,1)}, \ldots, z_{(1,a_1)}), (z_{(2,1)}, \ldots, z_{(2,a_1)}, z_{(2,a_1+2)}, \ldots, z_{(2,a_2+1)}), \ldots (z_{(k,1)}, \ldots, z_{(k,a_k+k-1)}))$$

to the point

$$\begin{bmatrix}
z_{(1,1)} & \cdots & z_{(1,a_1)} & 1 - \sum_{j=1}^{a_1} |z_{(1,j)}^2| & 0 & \cdots & & & \\
z_{(2,1)} & \cdots & z_{(2,a_1)} & -\sum_{j=1}^{a_1} z_{(2,j)} \bar{z}_{(1,j)} & z_{(2,a_1+2)} & \cdots & z_{(2,a_2+1)} & 1 - \sum_{j=1, j \neq a_1+1}^{a_2+1} |z_{(2,j)}^2| & \cdots \\
\vdots & & & & & & & & \\
z_{(k,1)} & \cdots & z_{(k,a_1)} & -\sum_{j=1}^{a_1} z_{(k,j)} \bar{z}_{(1,j)} & z_{(k,a_1+2)} & \cdots & z_{(k,a_2+1)} & -\sum_{j=a_1+2}^{a_2+1} z_{(k,j)} \bar{z}_{(2,j)} & \cdots
\end{bmatrix}.$$

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If $V = V_1 \oplus V_2$ and $dim_{\mathbb{C}}(V_1) \geq k$, then it is easy to see that $G(V_1, k) \subseteq G(V, k)$ as a sub-G-cell complex. Hence, there is also an obvious G-cell structure for $G(V, k)$ if $dim_{\mathbb{C}}(V)$ is countably infinite.

Complex projective space $PV = G(V, 1)$ is considered in [9] using this Schubert cell structure. More specifically, Lewis shows that the cell structure induced by a particular ordering of the irreducible summands of V is most useful in that case. The particular ordering is called a *standard block ordering* for V. For each integer $j \geq 1$, let $V(j)$ be a set of representatives of the isomorphism classes of irreducible representations that appear in the decomposition of V at least j times. Give each nonempty set $V(j)$ the additional structure of a list by fixing some ordering. Concatenating the lists as

$$V(1), V(2), V(3), \ldots$$

gives a standard block odering for V. We make use of this ordering for Grassmannians in Section 5.2 and refer to the induced Schubert cell structure for $G(V, k)$ as a *standard block ordering cell structure.*

5.2 Applying Theorem 3.4

Theorem 5.1. *Let k be a positive integer, let V be a complex G-representation of countable (complex) dimension at least k, and let $X = G(V, k)$ the equivariant complex Grassmannian of k-planes in V. Then, the equivariant ordinary $RO(G)$-graded cohomology $\mathbf{H}^*_G(X)$ is free as a module over the cohomology of a point $\mathbf{H}^*_G(*)$.*

Proof. The cell structure described in Section 5.1 displays $G(V, k)$ as a generalized G-cell complex built with only even-dimensional representation cells. Hence, only the second finiteness condition in the hypotheses of Theorem 3.4 needs to be verified. If

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V is finite dimensional, then $G(V,k)$ has only finitely many cells, so the finiteness conditions are obvious. If V is infinite dimensional, then the finiteness conditions need to be checked.

For concreteness, we use a standard block ordering cell structure. For each n, let V_n be the sum of the first n irreducibles in a standard block ordering for V. The sequence of inclusions

$$* = G(V_k,k) \subseteq G(V_{k+1},k) \subseteq \cdots \subseteq G(V_n,k) \subseteq \cdots$$

provides a filtration for $X = G(V,k)$, where each stage involves adding only finitely many representation cells.

Let m be any fixed positive integer. We claim that all of the cells of $G(V,k)$ of fixed dimension m are contained in the finite subcomplex $G(V_{(m+k)p},k)$. So, let $\langle a_1,\ldots,a_k\rangle$ be any representation cell added beyond $G(V_{(m+k)p},k)$ in our filtration. Since, up to isomorphism, there are only p distinct irreducibles, the irreducible ϕ_{a_k+k} must appear at least $m+k$ times prior to column a_k+k. Also note that at most $k-1$ of those appearances can come from the set $\{\phi_{a_j+j} : a \le j \le k-1\}$. Hence, just by counting the fixed dimension contribution

$$\sum_{\substack{j=1 \\ j\notin\{a_1+1,\ldots,a_{k-1}+k-1\}}}^{a_k+k-1} |(\phi_{a_k+k}^{-1}\phi_j)^G|$$

from the bottom row of the matrix

$$\overbrace{\begin{bmatrix} * & \cdots & * & \cdots & & \\ * & \cdots & * & \cdots & & \\ \vdots & & \vdots & & \ddots & \\ * & \cdots & * & \cdots & & 1 \end{bmatrix}}^{V_{(m+k)p}} \quad \phi_{a_k+k},$$

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the fixed dimension associated with the cell $\langle a_1, \ldots, a_k \rangle$ must be greater than m. This establishes our claim. Moreover, since $G(V_{(m+k)p}, k)$ has only finitely many cells of fixed dimension m, so does $X = G(V, k)$. Theorem 3.4 therefore gives the desired conclusions for $\mathbf{H}^*_G(X)$. □

Remark 5.2. The shortcoming of Theorem 5.1 is that it does not specify the exact dimensions of the free generators. In fact, it does not even specify a plot of the dimensions of the generators. Of course, this shortcoming is inherited from Theorem 3.4. However, Theorem 5.1 is also clearly true for both R and $\langle \mathbb{Z} \rangle$-coefficients. In the R-coefficients case, again Theorem 5.1 does not specify a plot of the dimensions of the generators. However, in the $\langle \mathbb{Z} \rangle$-coefficients case, exact dimensions for the free generators are given directly from the cell structure. This is discussed in Remark 3.2 and follows from the fact that the boundary maps $\partial_{\langle \mathbb{Z} \rangle}$ are all zero.

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Appendix A

Mackey Functor Lemmas

In this appendix, we record some basic results about Mackey functors. In particular, we give properties of some maps and list solutions to some extension problems that appear frequently in Chapter 3. The results in this appendix are of the type that would be left to the reader if we were working in the category of abelian groups rather than the less familiar category of Mackey functors. The first two lemmas are easy consequences of Proposition 2.2, which characterizes maps out of L or into R.

Lemma A.1. *Let d be an integer prime to p.*

(a) *Suppose that $f : L \longrightarrow A[d]$ is an isomorphism at G/e. Then, $f_e(\iota) = \eta\iota$ and $f_G(\tau) = \eta\tau$, where $\eta = \pm 1$.*

(b) *Suppose that $f : L \longrightarrow R$ is an isomorphism at G/e. Then, $f_e(\iota) = \eta\iota$ and $f_G(\tau) = \eta p\xi$, where $\eta = \pm 1$.*

(c) *Suppose that $g : A[d] \longrightarrow R$ is an isomorphism at G/e. Then, $g_e(\iota) = \eta\iota$, $g_G(\mu) = \eta d\xi$, and $g_G(\tau) = \eta p\xi$, where $\eta = \pm 1$.*

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Lemma A.2. *If a short exact sequence of Mackey functors*

$$0 \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow 0$$

splits at G/e in the category of $\mathbb{Z}[G]$-modules, and either $D = R$ or $F = L$, then the sequence splits in Mack.

Proposition 2.2 enables us to extend the splitting at G/e in Lemma A.2 to a splitting of the sequence as a whole. Hence, Lemma A.2 gives a family of extension problems that have only one solution. In the proof of Theorem 3.4, there are several extension problems to solve. The following lemma lists all possible solutions to the extension problems encountered there. Recall from Section 2.1 that $\{\tau\}$, $\{\xi\}$, and $\{\mu, \tau\}$ denote the standard bases for $L(G/G)$, $R(G/G)$, and $A[d](G/G)$, respectively. Also, the standard generator for

$$L(G/e) \cong R(G/e) \cong A[d](G/e) \cong \mathbb{Z}$$

is denoted by ι.

Lemma A.3. *Let E be a Mackey functor, and let m be a positive integer.*

(a) *Suppose that*

$$0 \longrightarrow L \longrightarrow E \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow 0$$

is a short exact sequence. Then

$$E = \begin{cases} L \oplus \langle \mathbb{Z} \rangle & \text{if the sequence splits,} \\ A[d] & \text{otherwise.} \end{cases}$$

(b) *Suppose that*

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow E \longrightarrow R \longrightarrow 0$$

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is a short exact sequence. Then

$$E = \begin{cases} \langle \mathbb{Z} \rangle \oplus R & \text{if the sequence splits,} \\ A[d] & \text{otherwise.} \end{cases}$$

(c) *Suppose that*

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow E \longrightarrow \langle \mathbb{Z}/p \rangle \longrightarrow 0$$

is a short exact sequence. Then

$$E = \begin{cases} \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle & \text{if the sequence splits,} \\ \langle \mathbb{Z} \rangle & \text{otherwise.} \end{cases}$$

(d) *Suppose that*

$$0 \longrightarrow L \longrightarrow E \longrightarrow \langle \mathbb{Z}/p \rangle \longrightarrow 0$$

is a short exact sequence. Then

$$E = \begin{cases} L \oplus \langle \mathbb{Z}/p \rangle & \text{if the sequence splits,} \\ R & \text{otherwise.} \end{cases}$$

(e) *Suppose that*

$$0 \longrightarrow A[d] \longrightarrow E \longrightarrow \langle \mathbb{Z}/p \rangle^m \longrightarrow 0$$

is a short exact sequence. Then

$$E = \begin{cases} A[d] \oplus \langle \mathbb{Z}/p \rangle^m & \text{if the sequence splits,} \\ R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{m-1} & \text{otherwise.} \end{cases}$$

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(f) *Suppose that*

$$0 \longrightarrow L \longrightarrow E \longrightarrow \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^m \longrightarrow 0$$

is a short exact sequence. Then

$$E = \begin{cases} L \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^m & \text{if the sequence splits,} \\ R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{m-1} & \text{if the sequence doesn't split, but a } \langle \mathbb{Z} \rangle \text{ splits off,} \\ A[d] \oplus \langle \mathbb{Z}/p \rangle^m & \text{otherwise.} \end{cases}$$

Proof. In each case, let f denote the monomorphism into E, and let g denote the epimorphism out of E. Also, let ρ and τ denote the restriction and transfer maps, respectively, for E.

For A.3(a), the short exact sequence of abelian groups

$$0 \longrightarrow L(G/H) \longrightarrow E(G/H) \longrightarrow \langle \mathbb{Z} \rangle (G/H) \longrightarrow 0$$

clearly splits, for $H = e, G$. Hence $E(G/G) = \mathbb{Z} \oplus \mathbb{Z}$ and $E(G/e) = \mathbb{Z}$. Fix a basis $\{u, v\}$ for $E(G/G)$ where $f_G(\tau) = v$ and $g_G(u) = 1$. Let $\iota' = f_e(\iota)$. Clearly, ι' generates $E(G/e)$. In terms of this basis for E, $\tau(\iota') = v$ and $\rho(v) = p\iota'$. Hence, the short exact sequence has the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{f_G = i_2} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{g_G = \pi_1} & \mathbb{Z} & \longrightarrow & 0 \\ & & {\scriptstyle p}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle (d,p)}\downarrow\uparrow{\scriptstyle (0,1)} & & \downarrow\uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[f_e = id]{} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

The integer d is defined by the equation $\rho(u) = d\iota'$. It follows that $E = L \oplus \langle \mathbb{Z} \rangle$ if $p|d$, and $E = A[d]$ otherwise.

For A.3(b), again the sequence clearly splits at each level. Define s and k by the equations $g_G(s) = \xi$ and $k = f_G(1)$. Hence $\{s, k\}$ forms a basis for $E(G/G) = \mathbb{Z} \oplus \mathbb{Z}$.

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Let ι' be the generator of $E(G/e)$ such that $g_e(\iota') = \iota$. Note that $\rho(s) = \iota'$, $\rho(k) = 0$, and $\tau(\iota') = ps - ak$, for some $a \in \mathbb{Z}$. In terms of our basis for E, the short exact sequence has the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{f_G=i_2} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{g_G=\pi_1} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow\uparrow & & \downarrow\uparrow{\scriptstyle (1,0)} \quad \downarrow\uparrow{\scriptstyle (p,-a)} & & \downarrow\uparrow{\scriptstyle 1} \quad \downarrow\uparrow{\scriptstyle p} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[g_e=id]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

If $p \nmid a$ then $E = A[d]$ where $ad \equiv 1 \bmod p$. This can easily be seen in terms of the alternative $\{\sigma, \kappa\}$ basis for $A[d](G/G)$. Otherwise $E = R \oplus \langle \mathbb{Z} \rangle$, as can be shown by the isomorphism $h : R \oplus \langle \mathbb{Z} \rangle \to E$ defined by $h_G(\xi) = s - ck$ and $h_G(1) = k$, where $pc = a$.

For A.3(c) and A.3(d), clearly $E(G/G)$ is either $\mathbb{Z} \oplus \mathbb{Z}/p$ or $\mathbb{Z}$. This gives the asserted conclusions in A.3(c), so focus now on A.3(d). As usual, take $f_e = id$. If $E(G/G) = \mathbb{Z} \oplus \mathbb{Z}/p$, then choose a basis for $E(G/G)$ such that $f_G = i_1$ and $g_G = \pi_2$. It is easy to see that $\rho = p\pi_1$ and $\tau = i_1$. Hence, $E = L \oplus \langle \mathbb{Z}/p \rangle$. If $E(G/G) = \mathbb{Z}$, then choose a generator for $E(G/G)$ such that f_G is multiplication by p. It is easy to see that $\rho = id$ and τ is multiplication by p. Hence, $E = R$.

For A.3(e), let a be a multiplicative inverse of d modulo p, and use the $\{\sigma, \kappa\}$ basis for $A[d]$. Let $x_1 = f_G(\sigma)$ and $y = f_G(\kappa)$. Taking $f_e = id$ gives $\rho(x_1) = \iota$ and $\rho(y) = 0$. Clearly, $E(G/G)$ must have free rank 2. Further, since ρ maps onto $E(G/e) = \mathbb{Z}$, $E(G/G)$ must split as $\mathbb{Z} \oplus \operatorname{Ker}\rho$. Hence it must be that $\operatorname{Ker}\rho = \mathbb{Z} \oplus P$ where P is torsion. Select x_2 as a generator of a rank 1 summand of $\operatorname{Ker}\rho$. Since $\rho(y) = 0$, it follows that $y = cx_2 + w$ for some $w \in P$ and integer c. Since px_2 is clearly in $\operatorname{Ker} g = \operatorname{Image} f$, it follows that $px_2 = jx_1 + ky$ for some integers j and k. However, $0 = \rho(px_2) = j\iota$, so $px_2 = ky = kcx_2 + kw$. Hence, $(kc - p)x_2 + kw = 0$. This gives that $kc = p$.

We claim that the value of E depends on whether or not $p|c$. If $c = \pm p$, and hence

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$k = \pm 1$, then $w = 0$. Therefore $y = cx_2$, and it is easy to check that the elements $x_1 \mp ax_2$ and ι generate a copy of R in E. Further x_2 must generate a copy of $\langle \mathbb{Z} \rangle$, and clearly $\{x_1 - ax_2, x_2\}$ forms a basis for a rank 2 free summand of $E(G/G)$. Since $\text{Coker}\, f = \langle \mathbb{Z}/p \rangle^m$ and $f_G(\kappa)$ is divisible by p, it must be that $P = \langle \mathbb{Z}/p \rangle^{m-1}$. Hence, $E = R \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^{m-1}$. If $c = \pm 1$, then $\{x_1, y\}$ generates a rank 2 summand of $E(G/G)$. Further, $\tau(\iota) = px_1 - ay$, and hence $A[d]$ splits off of E. Therefore, $E = A[d] \oplus \langle \mathbb{Z}/p \rangle^m$.

For A.3(f), t and ι are used to denote the standard basis elements for L. Clearly, $E(G/G)$ must have free rank 2. Let $\iota' = f_e(\iota)$ and $t' = f_G(t)$. Hence, $\rho(t') = p\iota'$. In particular, $\rho \neq 0$. Hence, $\text{Ker}\, \rho = \mathbb{Z} \oplus P$, where P is torsion. Select a generator z for a rank 1 free summand of $\text{Ker}\, \rho$.

We claim that our sequence splits if and only if ρ is not an epimorphism. Clearly, $\text{Image}\, \rho$ is either $p\mathbb{Z}$ or $\mathbb{Z}$. Assume first that ρ is not an epimorphism, and hence $\text{Image}\, \rho = p\mathbb{Z}$. In this case, t' must generate a free summand of $E(G/G)$. Since $\tau(\iota') = t'$, the image of f is a copy of L. Hence, the sequence splits and $E = L \oplus \langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle^m$.

We now assume that ρ is an epimorphism and show that this leads to two non-split possibilities. Let y be an element of $E(G/G)$ with $\rho(y) = \iota'$. Since $\rho(z) = 0$, it is easy to see that $\{y, z\}$ generates a rank 2 free summand of $E(G/G)$. Hence $t' = cy - az + w$ for some $w \in P$ and integers c and a. Further, since $p\iota' = \rho(t') = c\iota'$, it follows that $c = p$. We claim that w can be taken to be zero. Applying g_G to our equation for t' gives $0 = pg_G(y) - ag_G(z) + g_G(w)$. Of the terms on the right-hand side, $pg_G(y)$ cannot be nonzero torsion, and $g_G(w)$ must be torsion. If $p|a$, then this forces $g_G(w) = 0$. Since $\text{Ker}\, g$ is free, it follows that $w = 0$. If $p \nmid a$, then there is some $w' \in P$ with $-aw' = w$. Hence, $t' = py - a(z + w')$. If we replace z by $z + w'$, then $t' = py - az$ and $\{y, z\}$ still generates a rank 2 free summand of $E(G/G)$. In

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terms of the new basis elements y and z, our sequence has the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{f_G} & \mathbb{Z}\oplus(\mathbb{Z}\oplus P) & \xrightarrow{g_G} & \mathbb{Z}\oplus(\mathbb{Z}/p)^m & \longrightarrow & 0 \\ & & {\scriptstyle p}\downarrow\uparrow{\scriptstyle 1} & & {\scriptstyle (1,0,0)}\downarrow\uparrow{\scriptstyle (p,-a,0)} & & \downarrow\uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\cong]{f_e} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

If $p|a$, say $a = pb$, then $t' = p(y - bz)$. By using $\{y - bz, z\}$ as a basis for the rank 2 free summand of $E(G/G)$, we see that $R \oplus \langle\mathbb{Z}\rangle$ is a summand of E. Moreover, $f_G(t) = p(y-bz)$, and $g_G(z)$ must generate a rank 1 free summand of $\mathbb{Z}\oplus(\mathbb{Z}/p)^m$. Since $f_G(t)$ is divisible by p, it must be that $P = \langle\mathbb{Z}/p\rangle^{m-1}$. Hence $E = R\oplus\langle\mathbb{Z}\rangle\oplus\langle\mathbb{Z}/p\rangle^{m-1}$. If $p \nmid a$, then the elements y, z, and ι' generate a summand $A[d]$ of E, where d is an inverse of a modulo p. Hence, $E = A[d] \oplus \langle\mathbb{Z}/p\rangle^m$ □

For some of the short exact sequences in Lemma A.3, the precise structure of the maps must be understood.

Lemma A.4. *Let d be an integer prime to p. Let $\phi : L \longrightarrow A[d]$, $\theta : A[d] \longrightarrow R$, and $\psi : A[d] \longrightarrow \langle\mathbb{Z}\rangle$ be the maps such that $\phi_e(\iota) = \iota, \phi_G(\tau) = \tau$, $\theta_e(\iota) = \iota$, $\theta_G(\mu) = d\xi$, $\theta_G(\tau) = p\xi$, $\psi_e(\iota) = 0$, $\psi_G(\mu) = \epsilon$, and $\psi_G(\tau) = 0$.*

(a) *Suppose that*

$$0 \longrightarrow L \xrightarrow{f} A[d] \xrightarrow{g} \langle\mathbb{Z}\rangle \longrightarrow 0$$

is a short exact sequence. Then, $\eta_1 = \pm 1$ and $\eta_2 = \pm 1$ can be selected so that $f = \eta_1\phi$ and $g = \eta_2\psi$.

(b) *Suppose that*

$$0 \longrightarrow \langle\mathbb{Z}\rangle \xrightarrow{f} A[d] \xrightarrow{g} R \longrightarrow 0$$

is a short exact sequence. Then, $\eta_1 = \pm 1$ and $\eta_2 = \pm 1$ can be selected so that $f_G(\epsilon) = \eta_1\kappa$ and $g = \eta_2\theta$.

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(c) *Suppose that*

$$0 \longrightarrow A[d] \xrightarrow{f} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{g} \langle \mathbb{Z}/p \rangle \longrightarrow 0$$

is a short exact sequence. Write $f = (f_1, f_2)$ *and* $g = (g_1, g_2)$, *where* $f_1 : A[d] \longrightarrow R$, $f_2 : A[d] \longrightarrow \langle \mathbb{Z} \rangle$, $g_1 : R \longrightarrow \langle \mathbb{Z}/p \rangle$, *and* $g_2 : \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z}/p \rangle$. *Then,* $\eta_1 = \pm 1$ *and* $\eta_2 = \pm 1$ *can be selected so that* $f_1 = \eta_1 \theta$, $f_2 = \eta_2 \psi$, *and* $(g_2)_G(\epsilon) = -d\eta_1\eta_2(g_1)_G(\xi)$.

(d) *Suppose that*

$$0 \longrightarrow \langle \mathbb{Z} \rangle \oplus L \xrightarrow{g} A[d] \xrightarrow{\partial} \langle \mathbb{Z}/p \rangle \longrightarrow 0$$

is an exact sequence. Write $g = (g_1, g_2)$, *where* $g_1 : \langle \mathbb{Z} \rangle \longrightarrow A[d]$ *and* $g_2 : L \longrightarrow A[d]$. *Then,* $\eta_1 = \pm 1$ *and* $\eta_2 = \pm 1$ *can be selected so that* $(g_1)_G(\epsilon) = \eta_1 \kappa$ *and* $g_2 = \eta_2 \phi$.

In addition to solving some extension problems, we also need to compute the kernels of some maps onto $\langle \mathbb{Z}/p \rangle$.

Lemma A.5. **(a)** *If* $R \xrightarrow{\partial} \langle \mathbb{Z}/p \rangle$ *is nonzero, then* $\operatorname{Ker} \partial = L$.

(b) *If* $A[d] \xrightarrow{\partial} \langle \mathbb{Z}/p \rangle$ *is nonzero, then* $\operatorname{Ker} \partial = L \oplus \langle \mathbb{Z} \rangle$.

(c) *If* $R \oplus \langle \mathbb{Z} \rangle \xrightarrow{(\partial_1, \partial_2)} \langle \mathbb{Z}/p \rangle$ *is a map such that both* ∂_1 *and* ∂_2 *are nonzero, then* $\operatorname{Ker} \partial = A[d]$, *where* d *is defined by* $\partial_2(\epsilon) = d\partial_1(\xi)$.

(d) *If* $\langle \mathbb{Z} \rangle \oplus \langle \mathbb{Z}/p \rangle \xrightarrow{(\partial_1, \partial_2)} \langle \mathbb{Z}/p \rangle$ *is a map such that both* ∂_1 *and* ∂_2 *are nonzero, then* $\operatorname{Ker} \partial = \langle \mathbb{Z} \rangle$.

Proof. For A.5(a), the kernel at G/G is generated by $p\xi$ and the kernel at G/e is generated by ι. Together, these elements generate an L.

For A.5(b), clearly τ and $p\mu$ form a basis for $\operatorname{Ker} \partial_G$, and hence so do τ and $p\mu - d\tau$. Also, ι clearly generates $\operatorname{Ker} \partial_e$. The elements τ and ι generate a summand L, and the element $p\mu - d\tau$ generates a summand $\langle \mathbb{Z} \rangle$ in $\operatorname{Ker} \partial$.

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For A.5(c), let ϵ denote a generator for the $\langle\mathbb{Z}\rangle$ summand in the domain of ∂. Define the integer d by the equation $\partial_2(\epsilon) = d\partial_1(\xi)$. Hence $d\xi - \epsilon$ and $p\xi$ form a basis for $\operatorname{Ker}\partial_G$. Clearly ι generates $\operatorname{Ker}\partial_e$. Since $\rho(d\xi - \epsilon) = d\iota$, $\rho(p\xi) = p\iota$, and $\tau(\iota) = p\xi$, it follows that $\operatorname{Ker}\partial = A[d]$.

For A.5(d), let ϵ be a generator for $\mathbb{Z}$. Choose a generator ζ for $\mathbb{Z}/p$ such that $\partial_1(\epsilon) + \partial_2(\zeta) = 0$. It is easy to see that $\operatorname{Ker}\partial = \langle C\rangle$, where $C \cong \mathbb{Z}$ is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}/p$ generated by $\epsilon + \zeta$. □

The next lemma lists several examples of right exact sequences that must, in fact, be exact.

Lemma A.6. *In each of the following exact sequences:*

(a) $\langle\mathbb{Z}\rangle \xrightarrow{f} \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \longrightarrow 0$

(b) $L \xrightarrow{f} R \longrightarrow \langle\mathbb{Z}/p\rangle \longrightarrow 0$

(c) $R \xrightarrow{f} R \oplus \langle\mathbb{Z}/p\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \longrightarrow 0$

(d) $\langle\mathbb{Z}\rangle \xrightarrow{f} \langle\mathbb{Z}\rangle \oplus \langle\mathbb{Z}/p\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \longrightarrow 0$

(e) $\langle\mathbb{Z}/p\rangle \xrightarrow{f} \langle\mathbb{Z}/p\rangle \oplus \langle\mathbb{Z}/p\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \longrightarrow 0$

the map f must be a monomorphism.

Proof. In A.6(a) and A.6(b), since $\operatorname{Coker} f_G = \mathbb{Z}/p$, the map $f_G : \mathbb{Z} \longrightarrow \mathbb{Z}$ must be multiplication by p. It is also clear that f_e must be an isomorphism in both cases.

For A.6(c), since $\operatorname{Coker} f = \langle\mathbb{Z}/p\rangle$, it follows that $\pi_1 \circ f$ is an epimorphism. Hence, $\pi_1 \circ f$ is an isomorphism. Of course, it follows that f is a monomorphism.

For A.6(d), write $f = (f_1, f_2)$. Since $\operatorname{Coker} f = \langle\mathbb{Z}/p\rangle$, clearly $f_1 \neq 0$. Hence f is a monomorphism.

For A.6(e), it is obvious that f must be a monomorphism. □

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Throughout Chapter 3 there are computations that involve determining the value d in a Mackey functor $A[d]$. The following lemma describes certain maps between two twisted Burnside rings.

Lemma A.7. *If $h : A[d] \longrightarrow A$ is a map that is an isomorphism at G/e, then h is a monomorphism with cokernel $\langle \mathbb{Z}/x \rangle$ for some $x \equiv d$ modulo p.*

Proof. Let $\{\mu', \tau', \iota'\}$ and $\{\mu, \tau, \iota\}$ be standard bases for the domain and range of h, respectively. Without loss of generality, we can take $h_e(\iota') = \iota$. Since h commutes with transfer, $h_G(\tau') = \tau$. Since h commutes with restriction, it follows that $h_G(\mu') = x\mu + y\tau$ for some integers x and y with $x + py = d$. Hence, Coker $h = \langle \mathbb{Z}/x \rangle$ and $x \equiv d$ modulo p. □

The following lemma is used, in particular, to give constraints on the boundary map $\mathbf{H}^*_G(?; R) \xrightarrow{\partial} \Sigma \mathbf{H}^*_G(?; \langle \mathbb{Z} \rangle)$ induced by the short exact sequence

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \longrightarrow 0$$

of coefficients.

Lemma A.8. *Let M be a Mackey functor module over R, let C be an abelian group, and let $f : M \longrightarrow \langle C \rangle$ be a Mackey functor map. Then, f factors through the inclusion $\langle C' \rangle \longrightarrow \langle C \rangle$, where $C' = \{c \in C : pc = 0\}$. Thus, if C is torsion free, then $f = 0$.*

Proof. Write M as a quotient of copies of R and R_G. Then it suffices to prove the result for $M = R$ and $M = R_G$. Since $Mack(R_G, \langle C \rangle) = 0$ and $f : R \longrightarrow \langle C \rangle$ has to kill the image of the transfer in R, the result is immediate. □

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Appendix B

Computation of $\mathbf{H}^*_G(*;N)$ for arbitrary N

In this Appendix, the computation of $\mathbf{H}^*_G(*)$ from Appendix A in [9] is generalized to arbitrary coefficients. The original computation of the additive structure of $\mathbf{H}^*_G(*)$ is unpublished work of Robert Stong. The results in this appendix are the product of joint work with Gaunce Lewis and are included here with his permission. We extended his computation for Burnside ring coefficients to arbitrary coefficients, because the more general result is needed here. The arguments for the general case are mainly the obvious generalization of the arguments given in Appendix A in [9].

Let N be a Mackey functor with restriction $\rho : N(G/G) \longrightarrow N(G/e)$ and transfer $\tau : N(G/e) \longrightarrow N(G/G)$. The restriction map induces a map $\widetilde{\rho} : N(G/G) \longrightarrow N(G/e)^G$, and the transfer map induces a map $\widetilde{\tau} : N(G/e)/G \longrightarrow N(G/G)$. Given any $\mathbb{Z}[G]$-module B, the trace map $b \mapsto \Sigma_{g\in G} gb$ induces maps $B \longrightarrow B^G$ and $B/G \longrightarrow B$, which are both denoted by tr. In [9], Lewis defines the following two Mackey functors determined by B:

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$$\begin{array}{cc} L(B) & R(B) \\ B/G & B^G \\ {\scriptstyle tr}\downarrow\uparrow{\scriptstyle \pi} & {\scriptstyle i}\downarrow\uparrow{\scriptstyle tr} \\ B & B \\ \theta\circlearrowleft & \theta\circlearrowleft \end{array}$$

The given action on B is denoted by θ. When we write $L(N(G/e))$, the given G-action on $N(G/e)$ is assumed. Similarly, the permutation action on $N(G/e)^p$ is assumed for $L(N(G/e)^p)$. The same conventions are used for $R(?)$. This notation allows us to describe the cohomology of a point $\mathbf{H}^*_G(*;N)$ in terms of N.

Theorem B.1. *Let $\alpha \in RO(G)$. Then,* $\mathbf{H}^\alpha_G(*;N) =$

$$\begin{cases} N\square A[d_\alpha] & \text{if } |\alpha| = |\alpha^G| = 0, \\ R(N(G/e)) & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0, \\ L(N(G/e)) & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| > 0, \\ \langle \mathrm{Coker}\,(\tau : N(G/e) \longrightarrow N(G/G))\rangle & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = 0, \\ \langle \mathrm{Ker}\,(\rho : N(G/G) \longrightarrow N(G/e))\rangle & \text{if } |\alpha| < 0 \text{ and } |\alpha^G| = 0, \\ \langle \mathrm{Ker}\,(\widetilde{\tau} : N(G/e)/G \longrightarrow N(G/G))\rangle & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = -1, \\ \langle \mathrm{Coker}\,(\widetilde{\rho} : N(G/G) \longrightarrow N(G/e)^G)\rangle & \text{if } |\alpha| < 0 \text{ and } |\alpha^G| = 1, \\ \langle \mathrm{Coker}\,(tr : N(G/e) \longrightarrow N(G/e)^G)\rangle & \begin{cases} \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even}, \\ \text{or } |\alpha| < 0,\ |\alpha^G| > 1, \text{ and } \alpha \text{ is odd}, \end{cases} \\ \langle \mathrm{Ker}\,(tr : N(G/e)/G \longrightarrow N(G/e))\rangle & \begin{cases} \text{if } |\alpha| < 0,\ |\alpha^G| > 0, \text{ and } \alpha \text{ is even}, \\ \text{or } |\alpha| > 0,\ |\alpha^G| < -1, \text{ and } \alpha \text{ is odd}, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

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Moreover, if $\beta \in RO_0(G)$, then the product map

$$\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*;A) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$$

induced by the isomorphism $N \square A \longrightarrow N$ is an isomorphism.

Denote $N \square A[d_\alpha]$ by $N[d_\alpha]$. Also, adopt the notation

$$\begin{aligned}
C_\tau(N) &\equiv \text{Coker}\,(\tau : N(G/e) \longrightarrow N(G/G)),\\
K_\rho(N) &\equiv \text{Ker}\,(\rho : N(G/G) \longrightarrow N(G/e)),\\
K_{\widetilde{\tau}}(N) &\equiv \text{Ker}\,(\widetilde{\tau} : N(G/e)/G \longrightarrow N(G/G)),\\
C_{\widetilde{\rho}}(N) &\equiv \text{Coker}\,(\widetilde{\rho} : N(G/G) \longrightarrow N(G/e)^G),\\
C_{tr}(N) &\equiv \text{Coker}\,(tr : N(G/e) \longrightarrow N(G/e)^G), \text{and}\\
K_{tr}(N) &\equiv \text{Ker}\,(tr : N(G/e)/G \longrightarrow N(G/e))
\end{aligned}$$

for the abelian groups appearing in Theorem B.1. This notation is used in Table B.1 to give a plot for $\mathbf{H}^*_G(*;N)$ similar to the plot given in Table 2.2.

The proof of Theorem B.1 is accomplished by a sequence of lemmas. The outline of our proof is roughly the same as that used in [9] for the case in which $N = A$. However, some issues arise for general N that did not arise for A. The proofs given here focus mainly on these new issues.

Let η be a nontrivial irreducible complex representation. Our computations depend heavily on the two cofibre sequences

$$G_+ \xrightarrow{h} S\eta_+ \xrightarrow{l} \Sigma G_+ \tag{B.1}$$

$$S\eta_+ \xrightarrow{f} S^0 \xrightarrow{\epsilon_\eta} S^\eta \tag{B.2}$$

from [9]. Note that the composite fh is the collapse map $c : G_+ \longrightarrow S^0$. The resulting maps in cohomology are also denoted with the same letters.

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			$\vert\alpha\vert$					
			$\wedge$					
			$\vdots$					
	$\langle C_{tr}(N)\rangle$		$\langle C_\tau(N)\rangle$					
$\langle K_{tr}(N)\rangle$		$\langle K_{\bar{\tau}}(N)\rangle$	$\vdots$					
	$\langle C_{tr}(N)\rangle$		$\langle C_\tau(N)\rangle$					
$\langle K_{tr}(N)\rangle$		$\langle K_{\bar{\tau}}(N)\rangle$	$\vdots$					
$\cdots$	$R(N(G/e))$	$\cdots$	$N[d_\alpha]$	$\cdots$	$L(N(G/e))$	$\cdots>$	$\vert\alpha^G\vert$	
			$\vdots$	$\langle C_{\tilde{\rho}}(N)\rangle$		$\langle C_{tr}(N)\rangle$		
			$\langle K_\rho(N)\rangle$		$\langle K_{tr}(N)\rangle$			
			$\vdots$	$\langle C_{\tilde{\rho}}(N)\rangle$		$\langle C_{tr}(N)\rangle$		
			$\langle K_\rho(N)\rangle$		$\langle K_{tr}(N)\rangle$			
			$\vdots$					

Table B.1: $\mathbf{H}^*_G(*)$

The proof of Theorem B.1 makes use of both cohomology and homology, and starts by computing these for the free orbit G/e.

Lemma B.2. *Let $\alpha \in RO(G)$.*

$$\mathbf{H}^\alpha_G(G;N) = \mathbf{H}^G_\alpha(G;N) = \begin{cases} N_G & \text{if } |\alpha| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $\beta \in RO_0(G)$, the product maps

$$\mathbf{H}^\alpha_G(G;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(G;N) \quad \text{and} \quad \mathbf{H}^G_\alpha(G;N) \square \mathbf{H}^G_\beta(*) \longrightarrow \mathbf{H}^G_{\alpha+\beta}(G;N)$$

are isomorphisms.

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Proof. The dimension axiom gives

$$\mathbf{H}^\alpha_G(G;N) = \mathbf{H}^\alpha_G(*;N)_G = \begin{cases} N_G = R(N(G/e)^p) & \text{if } |\alpha| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The product map $\mathbf{H}^\alpha_G(G;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(G;N)$ can be written as

$$\begin{cases} R(N(G/e)^p) = R(N(G/e)^p) \square A[d_\beta] \longrightarrow R(N(G/e)^p) & \text{if } |\alpha| = 0, \\ 0 \longrightarrow 0 & \text{otherwise.} \end{cases}$$

It follows that this product map is determined by its value at G/e. Since the nonequivariant product map

$$H^{|\alpha|}(G;N(G/e)) \otimes H^{|\beta|}(*) \longrightarrow H^{|\alpha+\beta|}(G;N(G/e))$$

is an isomorphism, the equivariant product must also be an isomorphism. A similar argument works for homology. □

The following lemma is a generalization of Lemma A.1 from [9].

Lemma B.3. *Let η be a nontrivial irreducible complex G-representation, and let α be in $RO(G)$. Then,*

$$\mathbf{H}^\alpha_G(S\eta;N) = \begin{cases} R(N(G/e)) & \text{if } |\alpha| = 0, \\ L(N(G/e)) & \text{if } |\alpha| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mathbf{H}^G_\alpha(S\eta;N) = \begin{cases} L(N(G/e)) & \text{if } |\alpha| = 0, \\ R(N(G/e)) & \text{if } |\alpha| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Moreover, for $\beta \in RO_0(G)$, the product maps

$$\mathbf{H}^\alpha_G(S\eta;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(S\eta;N) \quad \textit{and} \quad \mathbf{H}^G_\alpha(S\eta;N) \square \mathbf{H}^G_\beta(*) \longrightarrow \mathbf{H}^G_{\alpha+\beta}(S\eta;N)$$

are isomorphisms.

Proof. Note that $\mathbf{H}^\beta_G(*) = A[d_\beta]$ is projective and hence flat. By applying the product map $?^* \square \mathbf{H}^\beta_G(*) \longrightarrow ?^{*+\beta}$ to the long exact sequence

$$\mathbf{H}^{\alpha-1}_G(G;N) \longrightarrow \mathbf{H}^{\alpha-1}_G(G;N) \xrightarrow{l} \mathbf{H}^\alpha_G(S\eta;N) \xrightarrow{h} \mathbf{H}^\alpha_G(G;N) \longrightarrow \mathbf{H}^\alpha_G(G;N), \qquad \text{(B.3)}$$

it follows from Lemma B.2 and a 5-Lemma argument that the product map

$$\mathbf{H}^\alpha_G(S\eta;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(S\eta;N)$$

must be an isomorphism. As in [9], sequence (B.3) reduces to the short exact sequence

$$0 \longrightarrow L(\mathbf{H}^{\alpha-1}_G(*;N)(G/e)) \longrightarrow \mathbf{H}^\alpha_G(S\eta;N) \longrightarrow R(\mathbf{H}^\alpha_G(*;N)(G/e)) \longrightarrow 0.$$

This, together with the dimension axiom, gives the desired values of $\mathbf{H}^\alpha_G(S\eta;N)$. A similar argument works for homology. □

By inserting the values given by Lemma B.3 into the long exact sequence induced by cofibre sequence (B.2), the proof of the following generalization of Lemma A.2 from [9] is immediate.

Lemma B.4. *Let α be in $RO(G)$ and η be a nontrivial complex irreducible G-representation. The map $\epsilon : \mathbf{H}^{\alpha-\eta}_G(*;N) \longrightarrow \mathbf{H}^\alpha_G(*;N)$ induced by the inclusion $S^0 \subset S^\eta$ is*

$$\begin{cases} \textit{mono} & \textit{for } |\alpha| \neq 1,2; \\ \textit{epi} & \textit{for } |\alpha| \neq 0,1; \\ \textit{iso} & \textit{for } |\alpha| \neq 0,1,2. \end{cases}$$

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Similarly, the map $\epsilon : \mathbf{H}^G_\alpha(*;N) \longrightarrow \mathbf{H}^G_{\alpha-\eta}(*;N)$ *is*

$$\begin{cases} \textit{mono} & \textit{for } |\alpha| \neq 0,1; \\ \textit{epi} & \textit{for } |\alpha| \neq 1,2; \\ \textit{iso} & \textit{for } |\alpha| \neq 0,1,2. \end{cases}$$

Moreover, if $N(G/e) = 0$, *then the map* ϵ *in cohomology and homology is an isomorphism in every dimension.*

It follows that Lemma A.3 from [9] generalizes to arbitrary coefficients.

Lemma B.5. *Let* $\alpha \in RO(G)$. *If* $|\alpha|$ *and* $|\alpha^G|$ *are either both positive or both negative, then* $\mathbf{H}^\alpha_G(*;N) = 0$. *Moreover, if* $N(G/e) = 0$, *then* $\mathbf{H}^\alpha_G(*;N) = 0$ *for all* α *with* $|\alpha^G| \neq 0$.

Proof. First, consider the case in which $|\alpha| = |\alpha^G| = n \neq 0$. If $n > 0$, then there are actual complex representations V and W with $|V^G| = n$, $|W^G| = 0$, and $\alpha = V - W$. Write $V = n + \eta_1 + \ldots + \eta_k$ and $W = \phi_1 + \ldots + \phi_k$, where η_i and ϕ_i are nontrivial complex irreducible representations, for each $1 \leq i \leq k$. The dimension axiom gives $\mathbf{H}^n_G(*;N) = 0$, and Lemma B.4 gives the sequences of isomorphisms

$$0 = \mathbf{H}^n_G(*;N) \xrightarrow{\epsilon_{\eta_1}} \mathbf{H}^{n+\eta_1}_G(*;N) \xrightarrow{\epsilon_{\eta_2}} \cdots \xrightarrow{\epsilon_{\eta_k}} \mathbf{H}^V_G(*;N)$$

and

$$\mathbf{H}^\alpha_G(*;N) = \mathbf{H}^{V-W}_G(*;N) \xrightarrow{\epsilon_{\phi_1}} \mathbf{H}^{V-W+\phi_1}_G(*;N) \xrightarrow{\epsilon_{\phi_2}} \cdots \xrightarrow{\epsilon_{\phi_k}} \mathbf{H}^V_G(*;N) = 0.$$

A similar argument works for $n < 0$ and the cases in which $|\alpha| \neq |\alpha^G|$. □

It follows from Lemmas B.3 and B.5 that Lemma A.5 from [9] can be generalized to arbitrary coefficients.

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Lemma B.6. *Let $\alpha \in RO(G)$ with $|\alpha| = 0$ and $|\alpha^G| \neq 0$. Then*

$$\mathbf{H}^\alpha_G(*;N) = \begin{cases} R(N(G/e)) & \text{if } |\alpha^G| < 0, \\ L(N(G/e)) & \text{if } |\alpha^G| > 0. \end{cases}$$

Moreover, for $\beta \in RO_0(G)$, the product map

$$\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$$

is an isomorphism.

Proof. For $|\alpha^G| < 0$, consider the exact sequence

$$\mathbf{H}^{\alpha-\eta}_G(*;N) \xrightarrow{\epsilon} \mathbf{H}^\alpha_G(*;N) \longrightarrow \mathbf{H}^\alpha_G(S\eta;N) \longrightarrow \mathbf{H}^{\alpha+1-\eta}_G(*;N),$$

and, for $|\alpha^G| > 0$, consider the exact sequence

$$\mathbf{H}^{\alpha+\eta-1}_G(*;N) \longrightarrow \mathbf{H}^{\alpha+\eta-1}_G(S\eta;N) \longrightarrow \mathbf{H}^\alpha_G(*;N) \xrightarrow{\epsilon} \mathbf{H}^{\alpha+\eta}_G(*;N).$$

In each case, Lemma B.5 gives that the ends are zero. Since the middle maps are isomorphisms, Lemma B.3 gives both the desired values and the desired conclusions about the product map. □

The following generalization of Lemma A.4 from [9] completes our computations of the values of $\mathbf{H}^\alpha_G(*;N)$ for all even α with $(|\alpha^G|, |\alpha|) \neq (0,0)$.

Lemma B.7. *Let $\alpha \in RO(G)$ such that α is even and $|\alpha| \neq 0$. Then*

$$\mathbf{H}^\alpha_G(*;N) = \begin{cases} \langle \mathrm{Ker}(tr : N(G/e)/G \longrightarrow N(G/e)) \rangle & \text{if } |\alpha| \leq -2 \text{ and } |\alpha^G| > 0, \\ \langle \mathrm{Ker}(\rho : N(G/G) \longrightarrow N(G/e)) \rangle & \text{if } |\alpha| \leq -2 \text{ and } |\alpha^G| = 0, \\ \langle \mathrm{Coker}(tr : N(G/e) \longrightarrow N(G/e)^G) \rangle & \text{if } |\alpha| \geq 2 \text{ and } |\alpha^G| < 0, \\ \langle \mathrm{Coker}(\tau : N(G/e) \longrightarrow N(G/G)) \rangle & \text{if } |\alpha| \geq 2 \text{ and } |\alpha^G| = 0. \end{cases}$$

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Moreover, for $\beta \in RO_0(G)$, the product map

$$\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$$

is an isomorphism.

Proof. By Lemma B.4, it suffices to compute the values for $|\alpha| = \pm 2$. First, consider the cases in which $|\alpha| = -2$ and $|\alpha^G| \geq 0$. The arguments in the proof of Lemma A.4 in [9] give the exact sequence

$$0 \longrightarrow \mathbf{H}^\alpha_G(*;N) \xrightarrow{\epsilon} \mathbf{H}^{\alpha+\eta}_G(*;N) \xrightarrow{c} \mathbf{H}^{\alpha+\eta}_G(G;N) = \mathbf{H}^{\alpha+\eta}_G(*;N)_G.$$

For $|\alpha^G| > 0$, the value of $\mathbf{H}^\alpha_G(*;N)$ follows from the observation that c is the map $\overline{tr} : L(N(G/e)) \longrightarrow L(N(G/e))_G$. Since $? \square \mathbf{H}^\beta_G(*)$ preserves kernels, the product $\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$ is seen to be an isomorphism as a consequence of our earlier lemmas. For $|\alpha^G| = 0$, arguments like those used in the proof of Lemma B.5 give that $\mathbf{H}^\alpha_G(*;N) = \mathbf{H}^{-\eta}_G(*;N)$. Hence, $\mathbf{H}^\alpha_G(*;N)$ is the kernel of the map $\bar{\rho} : \mathbf{H}^0_G(*;N) = N \longrightarrow \mathbf{H}^0_G(*;N)_G = N_G$. It also follows that the product $\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$ is an isomorphism.

The cases in which $|\alpha| = 2$ and $|\alpha^G| \leq 0$ are handled by using the isomorphism $\mathbf{H}^\alpha_G(*;N) \cong \mathbf{H}^G_{-\alpha}(*;N)$. Here, the relevant exact sequence is

$$\mathbf{H}^G_{\eta-\alpha}(*;N)_G = \mathbf{H}^G_{\eta-\alpha}(G;N) \xrightarrow{c} \mathbf{H}^G_{\eta-\alpha}(*;N) \xrightarrow{\epsilon} \mathbf{H}^G_{-\alpha}(*;N) \longrightarrow 0.$$

Arguments analogous to those given above give the desired results in these cases. □

The following generalization of Lemmas A.8 and A.9 of [9] gives the values of $\mathbf{H}^\alpha_G(*;N)$ in all odd dimensions α.

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Lemma B.8. *Let $\alpha \in RO(G)$ such that α is odd. Then,*

$$\mathbf{H}^\alpha_G(*;N) = \begin{cases} \langle \mathrm{Ker}(tr : N(G/e)/G \longrightarrow N(G/e)) \rangle & \text{if } |\alpha| \geq 1 \text{ and } |\alpha^G| < -1, \\ \langle \mathrm{Ker}(\widetilde{\tau} : N(G/e)/G \longrightarrow N(G/G)) \rangle & \text{if } |\alpha| \geq 1 \text{ and } |\alpha^G| = -1, \\ \langle \mathrm{Coker}(tr : N(G/e) \longrightarrow N(G/e)^G) \rangle & \text{if } |\alpha| \leq -1 \text{ and } |\alpha^G| > 1, \\ \langle \mathrm{Coker}(\widetilde{\rho} : N(G/G) \longrightarrow N(G/e)^G) \rangle & \text{if } |\alpha| \leq -1 \text{ and } |\alpha^G| = 1. \end{cases}$$

Moreover, for $\beta \in RO_0(G)$, the product map

$$\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$$

is an isomorphism.

Proof. By Lemma B.4, it suffices to compute the values for $|\alpha| = \pm 1$. For $|\alpha| = 1$, the arguments in the proof of Lemma A.8 in [9] give the exact sequence

$$0 \longrightarrow \mathbf{H}^\alpha_G(*;N) \xrightarrow{f} \mathbf{H}^\alpha_G(S\eta;N) \xrightarrow{\delta} \mathbf{H}^{\alpha+1-\eta}_G(*;N) \xrightarrow{\epsilon} \mathbf{H}^{\alpha+1}_G(*;N) \longrightarrow .$$

From this sequence and the dimension axiom, it is easy to see that δ_e is an isomorphism. It is also easy to see that the product must be an isomorphism as asserted. If $|\alpha^G| < -1$, then the computation of $\mathbf{H}^\alpha_G(*;N) = \mathrm{Ker}\,\delta$ follows from the observation that $\delta : L(N(G/e)) \longrightarrow R(N(G/e))$ is the unique map that is an isomorphism at G/e. If $|\alpha^G| = -1$, then an argument similar to the one used in the proof of Lemma B.5 gives $\mathbf{H}^\alpha_G(*;N) = \mathbf{H}^{\eta-1}_G(*;N)$. Hence, $\mathbf{H}^\alpha_G(*;N)$ is the kernel of the unique map $\delta : L(N(G/e)) \longrightarrow N$ that is an isomorphism at G/e.

For $|\alpha| = -1$, the relevant exact sequence is

$$\longrightarrow \mathbf{H}^{\alpha-1}_G(*;N) \xrightarrow{\epsilon} \mathbf{H}^{\alpha+\eta-1}_G(*;N) \xrightarrow{f} \mathbf{H}^{\alpha+\eta-1}_G(S\eta;N) \longrightarrow \mathbf{H}^\alpha_G(*;N) \longrightarrow 0.$$

Arguments similar to those given above give the desired results. □

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Finally, the values of $\mathbf{H}^\alpha_G(*;N)$ for $\alpha \in RO_0(G)$ are computed.

Lemma B.9. *Let $\alpha \in RO_0(G)$. Then,*

$$\mathbf{H}^\alpha_G(*;N) = N \square A[d_\alpha].$$

Moreover, for $\beta \in RO_0(G)$, the product map

$$\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$$

is an isomorphism.

Proof. In the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathbf{H}^{-\eta}_G(*;N) \square \mathbf{H}^\alpha_G(*) & \longrightarrow & \mathbf{H}^0_G(*;N) \square \mathbf{H}^\alpha_G(*) & \longrightarrow & \mathbf{H}^0_G(S\eta;N) \square \mathbf{H}^\alpha_G(*) \\
& \downarrow \cong & & \downarrow & & \downarrow \cong \\
0 \longrightarrow & \mathbf{H}^{\alpha-\eta}_G(*;N) & \longrightarrow & \mathbf{H}^\alpha_G(*;N) & \longrightarrow & \mathbf{H}^\alpha_G(S\eta;N)
\end{array}$$

the rows are exact, and the left and right vertical maps are isomorphims by previous lemmas. Moreover, in the obvious rightward extension of the diagram, the next vertical arrow is also an isomorphism. Of course, it follows that

$$\mathbf{H}^0_G(*;N) \square \mathbf{H}^\alpha_G(*) \longrightarrow \mathbf{H}^\alpha_G(*;N)$$

is also an isomorphism. This gives the desired value of $\mathbf{H}^\alpha_G(*;N)$, and a similar argument gives the desired conclusion for the product $\mathbf{H}^\alpha_G(*;N) \square \mathbf{H}^\beta_G(*) \longrightarrow \mathbf{H}^{\alpha+\beta}_G(*;N)$. □

Combining Lemmas B.2 through B.9 gives a proof of Theorem B.1. Hence, for any Mackey functor N and any α in $RO(G)$, Theorem B.1 can be used to write $\mathbf{H}^\alpha_G(*;N)$ in terms of N. However, our main interest is in the particular cases of Theorem B.1 in which $N = R$, L, or $\langle\mathbb{Z}\rangle$. For easy reference, these particular cases are recorded here.

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Corollary B.10. *Let $\alpha \in RO(G)$. Then*

$$\mathbf{H}^\alpha_G(*;R) = \begin{cases} R & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \leq 0; \\ L & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| > 0; \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^G| \leq 0, \text{ and } \alpha \text{ is even;} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^G| > 1, \text{ and } \alpha \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

								$\vert\alpha\vert$	
								$\wedge$	
								$\vdots$	
$\cdots$	$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$			$\langle \mathbb{Z}/p \rangle$	
$\cdots$	$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$		$\langle \mathbb{Z}/p \rangle$			$\langle \mathbb{Z}/p \rangle$	
$\cdots$	R		R		R	L		L	$\cdots > \vert\alpha^G\vert$
						0		$\langle \mathbb{Z}/p \rangle$	$\cdots$
					0				
						0		$\langle \mathbb{Z}/p \rangle$	$\cdots$
					0				
					$\vdots$	$\vdots$		$\vdots$	

Table B.2: $\mathbf{H}^*_G(*;R)$

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Corollary B.11. *Let $\alpha \in RO(G)$. Then*

$$\mathbf{H}^\alpha_G(*;L) = \begin{cases} R & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0; \\ L & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \geq 0; \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even;} \\ \langle \mathbb{Z}/p \rangle & \text{if } |\alpha| < 0,\ |\alpha^G| \geq 1, \text{ and } \alpha \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases}$$

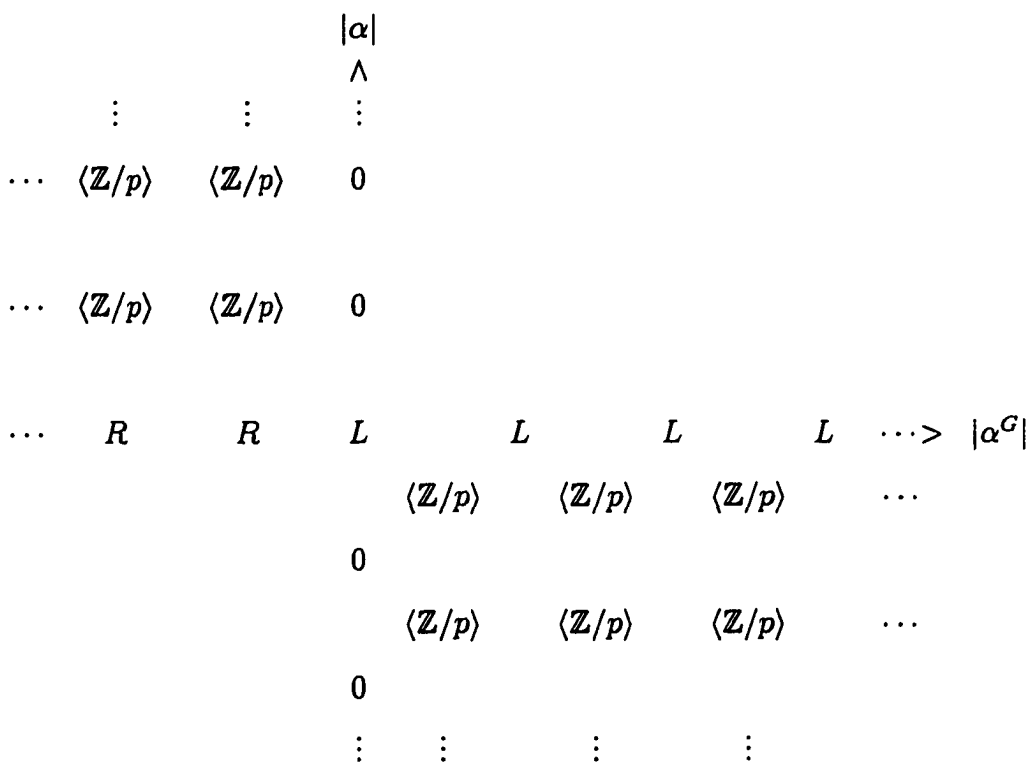


Table B.3: $\mathbf{H}^*_G(*;L)$

Corollary B.12. *Let C be an abelian group, and let $\alpha \in RO(G)$. Then*

$$\mathbf{H}^\alpha_G(*;\langle C \rangle) = \begin{cases} \langle C \rangle & \text{if } |\alpha^G| = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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A table for $\mathbf{H}^*_G(*; \langle\mathbb{Z}\rangle)$ is not displayed because it is so simple, taking the value $\langle\mathbb{Z}\rangle$ on the vertical axis and zero elsewhere.

A comparison of the Tables for $\mathbf{H}^*_G(*; L)$ and $\mathbf{H}^*_G(*; R)$ shows that, for any virtual representation λ with $|\lambda^G| = 0$ and $|\lambda| = 2$, $\mathbf{H}^\alpha_G(*; L) \cong \mathbf{H}^{\alpha+2-\lambda}_G(*; R)$, for all α in $RO(G)$. Moreover, a comparison of the homotopy groups of the Eilenberg-Mac Lane spectra $K(L, 0)$ and $K(R, 0)$ that represent $\mathbf{H}^*_G(?; L)$ and $\mathbf{H}^*_G(?; R)$, respectively, shows that $K(L, 0) \cong \Sigma^{2-\lambda}K(R, 0)$. Of course, $\Sigma^{2-\lambda}K(R, 0)$ represents $\Sigma^{2-\lambda}\mathbf{H}^*_G(?; R)$. The map $\mathbf{H}^*_G(?; L) \longrightarrow \mathbf{H}^*_G(?; R)$ induced by the canonical coefficient map $L \longrightarrow R$ is closely related to the map $\Sigma^{2-\lambda}\mathbf{H}^*_G(?; R) \longrightarrow \mathbf{H}^*_G(?; R)$ given by multiplication by ξ in $\mathbf{H}^{\lambda-2}_G(*)$.

Proposition B.13. *Let λ be in $RO(G)$ with $|\lambda^G| = 0$ and $|\lambda| = 2$, and let X be a G-space. Then, there is an X-natural isomorphism*

$$c_* : \mathbf{H}^*_G(X; L) \longrightarrow \Sigma^{2-\lambda}\mathbf{H}^*_G(X; R)$$

making the diagram

$$\begin{array}{ccc} \mathbf{H}^*_G(X; L) & \xrightarrow{i_*} & \mathbf{H}^*_G(X; R) \\ {\scriptstyle c_*}\downarrow{\scriptstyle \cong} & \nearrow_{\xi_*} & \\ \Sigma^{2-\lambda}\mathbf{H}^*_G(X; R) & & \end{array}$$

commute. Here, the map i_* is the map induced by the canonical map of coefficients $L \longrightarrow R$, and the map ξ_* is the map given by multiplicaton by ξ in $T^{\lambda-2}$.

Proof. By Proposition 3.5 of [11] and Proposition 2.2, the composites

$$[K(L, 0), K(R, 0)]_G \xrightarrow{\pi_0^G} Mack(L, R) \xrightarrow{\text{eval at } G/e} Hom_{\mathbf{Z}}(\mathbb{Z}, \mathbb{Z}) \qquad \text{(B.4)}$$

$$[K(L, 0), \Sigma^{2-\lambda}K(R, 0)]_G \xrightarrow{\pi_0^G} Mack(L, L) \xrightarrow{\text{eval at } G/e} Hom_{\mathbf{Z}}(\mathbb{Z}, \mathbb{Z}) \qquad \text{(B.5)}$$

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$$[\Sigma^{2-\lambda}K(R,0),K(R,0)]_G \xrightarrow{\pi_0^G} Mack(L,R) \xrightarrow{\text{eval at } G/e} Hom_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \qquad \text{(B.6)}$$

are isomorphisms. Under isomorphism (B.4), the map $i : K(L,0) \longrightarrow K(R,0)$ induced by the canonical map $L \longrightarrow R$ goes to $id : \mathbb{Z} \longrightarrow \mathbb{Z}$. By our observations prior to the proposition, there is an isomorphism

$$c\colon K(L,0) \longrightarrow \Sigma^{2-\lambda}K(R,0)$$

that goes to $id : \mathbb{Z} \longrightarrow \mathbb{Z}$ under isomorphism (B.5). Since

$$\xi \in \tilde{\mathbf{H}}_G^{\lambda-2}(S^0)(G/G) \cong [S^0, \Sigma^{\lambda-2}K(A,0)]_G,$$

the composite

$$\Sigma^{2-\lambda}K(R,0) \cong \Sigma^{2-\lambda}K(R,0) \wedge S^0 \xrightarrow{1\wedge\xi} \Sigma^{2-\lambda}K(R,0) \wedge \Sigma^{\lambda-2}K(A,0) \xrightarrow{mult} K(R,0)$$

is the map $\xi : \Sigma^{2-\lambda}K(R,0) \longrightarrow K(R,0)$, which induces the multiplication map

$$\xi_* : \Sigma^{2-\lambda}\mathbf{H}^*_G(X;R) \longrightarrow \mathbf{H}^*_G(X;R).$$

Since $R(G/e) = \mathbb{Z}$, the equivariant spectrum $K(R,0)$ restricts to the nonequivariant spectrum $K(\mathbb{Z},0)$. If we forget the G-action, then λ is equivalent to the trivial representation on $\mathbb{R}^2$. Hence, the spectrum $\Sigma^{2-\lambda}K(R,0)$ also restricts to $K(\mathbb{Z},0)$, Since $\rho(\xi) = 1 \in \mathbb{Z} = [S^0, K(\mathbb{Z},0)]_e$, the map $\xi : \Sigma^{2-\lambda}K(R,0) \longrightarrow K(R,0)$ goes to $id : \mathbb{Z} \longrightarrow \mathbb{Z}$ under isomorphism (B.6). It follows that the diagram

$$\begin{array}{ccc} K(L,0) & \xrightarrow{i} & K(R,0) \\ {\scriptstyle c}\downarrow{\scriptstyle \cong} & \nearrow_{\xi} & \\ \Sigma^{2-\lambda}K(R,0) & & \end{array}$$

commutes. Thus, the induced diagram in cohomology must also commute. □

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The usefulness of alternative coefficients in the proof of Theorm 3.4 comes from the long exact sequences in cohomology induced by short exact sequences of coefficients. The next four propositions follow from the relevant values of the cohomology of a point, the cohomology of a free orbit G/e, and the long exact cohomology sequence induced by the given short exact coefficient sequence.

Proposition B.14. *The short exact sequence of Mackey functors*

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \longrightarrow R \longrightarrow 0$$

induces a short exact sequence in the cohomology of a point

$$0 \longrightarrow \mathbf{H}^*_G(*;\langle \mathbb{Z} \rangle) \xrightarrow{f} \mathbf{H}^*_G(*) \xrightarrow{g} \mathbf{H}^*_G(*;R) \longrightarrow 0$$

Moreover, $f^\alpha : \mathbf{H}^\alpha_G(*;\langle \mathbb{Z} \rangle) \longrightarrow \mathbf{H}^\alpha_G(*)$ *is*

$$\begin{cases} \langle \mathbb{Z} \rangle \longrightarrow A[d_\alpha] \text{ by } 1 \mapsto \pm\kappa_\alpha & \text{if } |\alpha| = |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \text{ by } 1 \mapsto \pm p & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \text{ by } 1 \mapsto \pm 1 & \text{if } |\alpha| < 0 \text{ and } |\alpha^G| = 0, \\ 0 & \text{otherwise;} \end{cases}$$

and $g^\alpha : \mathbf{H}^\alpha_G(*) \longrightarrow \mathbf{H}^\alpha_G(*;R)$ *is*

$$\begin{cases} A[d_\alpha] \longrightarrow R \text{ by } \mu \mapsto \pm\xi & \text{if } |\alpha| = |\alpha^G| = 0, \\ \text{an epimorphism} & \text{if } |\alpha| \neq 0 \text{ and } |\alpha^G| = 0, \\ \text{an isomorphism} & \text{otherwise.} \end{cases}$$

Proposition B.15. *The short exact sequence of Mackey functors*

$$0 \longrightarrow L \longrightarrow A \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow 0$$

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induces a long exact sequence in the cohomology of a point

$$\longrightarrow \mathbf{H}^*_G(*;L) \xrightarrow{f} \mathbf{H}^*_G(*) \xrightarrow{g} \mathbf{H}^*_G(*;\langle\mathbb{Z}\rangle) \xrightarrow{h} \Sigma\mathbf{H}^*_G(*;L) \longrightarrow$$

such that $f^\alpha : \mathbf{H}^\alpha_G(*;L) \longrightarrow \mathbf{H}^\alpha_G(*)$ *is*

$$\begin{cases} L \longrightarrow A[d_\alpha] \text{ by } \tau \mapsto \pm\tau & \text{if } |\alpha| = |\alpha^G| = 0, \\ \text{an isomorphism} & \text{if } |\alpha^G| \leq -2 \text{ or } |\alpha^G| \geq 2, \\ 0 & \text{otherwise;} \end{cases}$$

$g^\alpha : \mathbf{H}^\alpha_G(*) \longrightarrow \mathbf{H}^\alpha_G(*;\langle\mathbb{Z}\rangle)$ *is*

$$\begin{cases} A[d_\alpha] \longrightarrow \langle\mathbb{Z}\rangle \text{ by } \mu \mapsto \pm 1 & \text{if } |\alpha| = |\alpha^G| = 0, \\ \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}\rangle \text{ by } 1 \mapsto \pm 1 & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = 0, \\ \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}\rangle \text{ by } 1 \mapsto \pm p & \text{if } |\alpha| < 0 \text{ and } |\alpha^G| = 0, \\ 0 & \text{otherwise;} \end{cases}$$

and $h^\alpha : \mathbf{H}^\alpha_G(*;\langle\mathbb{Z}\rangle) \longrightarrow \mathbf{H}^{\alpha+1}_G(*;L)$ *is*

$$\begin{cases} \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \text{ by } 1 \mapsto \pm\nu & \text{if } |\alpha^G| = 0 \text{ and } |\alpha| < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition B.16. $\mathbf{H}^*_G(G;\langle\mathbb{Z}\rangle) = 0$ *and the maps* $A \longrightarrow R$ *and* $L \longrightarrow A$ *induce isomorphisms* $\mathbf{H}^*_G(G) \xrightarrow{\cong} \mathbf{H}^*_G(G;R)$ *and* $\mathbf{H}^*_G(G;L) \xrightarrow{\cong} \mathbf{H}^*_G(G)$.

Remark B.17. (a) The multiplicative structure of $T(\langle\mathbb{Z}\rangle)$ is determined in Lemma B.4. For α in $RO(G)$ with $|\alpha^G| = 0$ and $|\alpha| > 0$, multiplication by the generator ϵ_α in $T^\alpha = T(\langle\mathbb{Z}\rangle)^\alpha$ is an isomorphism in $T(\langle\mathbb{Z}\rangle)$.

(b) Let $g : \mathbf{H}^*_G(*) \longrightarrow \mathbf{H}^*_G(*;R)$ be the map induced by the canonical coefficient map $A \longrightarrow R$. By using the results in Proposition B.14, the multiplicative structure of

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$T(R)$ can be determined. For α in $RO_0(G)$, define $\xi_\alpha \in T(R)^\alpha$ to be $g^\alpha(\mu_\alpha)$. Observe that $\xi_\alpha \epsilon_\beta = g^{\alpha+\beta}(\epsilon_{\alpha+\beta})$, for β in $RO(G)$ with $|\beta| > 0$ and $|\beta^G| = 0$. In all remaining cases, the multiplication in $T(R)$ is easily determined from the multiplication in T given by Theorem 4.9 of [9].

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Appendix C

Constraints on Some Module Maps

In this appendix, we gather some useful properties of the cup product and standard shift maps used in Chapter 3. In each case, the properties are consequences of either Proposition 2.5 or the defining properties of maps in T-mod, such as diagram (2.2).

Proposition C.1. *Let M be a T-module. Let β be in $RO(G)$, and let α be in $RO_0(G)$. Then, the cup product maps*

(a) $T^\alpha \square T^\beta \longrightarrow T^{\alpha+\beta}$ *and*

(b) $T^\alpha \square M^\beta \longrightarrow M^{\alpha+\beta}$

are isomorphisms.

Proof. (a) This result is discussed after Remarks 4.4 in [9]. Briefly, the stable Hurewicz map $h : \pi^G_{-\alpha}(S^0) \longrightarrow \tilde{\mathbf{H}}^\alpha_G(S^0)$ is shown in [10] to be an isomorphism. Hence, Proposition C.1(a) follows by an argument similar to the one used in [2] to show that the multiplication map in stable homotopy $\boldsymbol{\pi}^G_{-\alpha}(S^0) \square \boldsymbol{\pi}^G_{-\beta}(S^0) \longrightarrow \boldsymbol{\pi}^G_{-(\alpha+\beta)}(S^0)$ is an isomorphism.

(b) This result follows from part (a) and the defining properties of a module M. Denote the multiplication in T by ϕ, and denote the action of T on M by ψ.

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Associativity of scalar multiplication gives the diagram

$$\begin{array}{ccc} T^\alpha \square T^{-\alpha} \square M^{\alpha+\beta} & \xrightarrow[\cong]{\phi \square id} & T^0 \square M^{\alpha+\beta} \\ \downarrow{\scriptstyle id \square \psi_{-\alpha}^{\alpha+\beta}} & & \downarrow{\scriptstyle \psi}{\scriptstyle \cong} \\ T^\alpha \square M^\beta & \xrightarrow{\psi_\alpha^\beta} & M^{\alpha+\beta}, \end{array}$$

which shows that ψ_α^β is an epimorphism and $id \square \psi_{-\alpha}^{\alpha+\beta}$ is a monomorphism. Similarly the diagram

$$\begin{array}{ccc} T^{-\alpha} \square T^\alpha \square M^\beta & \xrightarrow[\cong]{\phi \square id} & T^0 \square M^\beta \\ \downarrow{\scriptstyle id \square \psi_\alpha^\beta} & & \downarrow{\scriptstyle \psi}{\scriptstyle \cong} \\ T^{-\alpha} \square M^{\alpha+\beta} & \xrightarrow{\psi_{-\alpha}^{\alpha+\beta}} & M^\beta, \end{array}$$

gives that $\psi_{-\alpha}^{\alpha+\beta}$ is an epimorphism and $id \square \psi_\alpha^\beta$ is a monomorphism.

Again, associativity gives the diagram

$$\begin{array}{ccccc} T^\alpha \square T^{-\alpha} \square T^\alpha \square M^\beta & \xrightarrow[\cong]{\phi \square id \square id} & T^0 \square T^\alpha \square M^\beta & \xrightarrow[\cong]{\phi \square id} & T^\alpha \square M^\beta \\ \downarrow{\scriptstyle id \square (id \square \psi_\alpha^\beta)} & & \downarrow{\scriptstyle id \square \psi_\alpha^\beta} & & \downarrow{\scriptstyle \psi_\alpha^\beta} \\ T^\alpha \square T^{-\alpha} \square M^{\alpha+\beta} & \xrightarrow[\cong]{\phi \square id} & T^0 \square M^{\alpha+\beta} & \xrightarrow[\cong]{\psi} & M^{\alpha+\beta}. \end{array}$$

Since $T^\alpha = A[d_\alpha]$ is flat, it follows that $id \square (id \square \psi_\alpha^\beta)$ is a monomorphism, and hence so is ψ_α^β. Therefore ψ_α^β is an isomorphism as desired. □

Corollary C.2. *Let M be a T-module and let β be in $RO(G)$. If $M^\beta = A[d]$ for some d prime to p, then there is some γ in $RO(G)$ with $(|\gamma^G|, |\gamma|) = (|\beta^G|, |\beta|)$ such that $M^\gamma = A$.*

Proof. Suppose that $M^\beta = A[d]$ for some d. Letting $\alpha = \lambda_1 - \lambda_d$ gives

$$d_\alpha \equiv \pm d^{-1} \text{ modulo } p.$$

Since α is in $RO_0(G)$, Proposition C.1 tells us that the cup product map

$$T^\alpha \square M^\beta \longrightarrow M^{\alpha+\beta}$$

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is an isomorphism. We therefore have

$$M^{\alpha+\beta} \cong T^{\alpha} \square M^{\beta} \cong A[d_{\alpha}] \square A[d] \cong A.$$

Let $\gamma = \alpha + \beta$. □

Since the standard shift maps of Definition 2.7 are maps between free T-modules, Proposition 2.5 and the multiplication rules for T in Theorem 4.9 of [9] can be used to describe these maps in every dimension. Recall from [9] that $\langle\beta\rangle$ is an element of $RO(G)$ such that $\beta - \langle\beta\rangle$ is in $RO_0(G)$.

Proposition C.3. *Let α be in $RO(G)$.*

(a) *Let β be in $RO(G)$ such that $|\beta| = 0$ and $|\beta^G| < 0$, and let $h : T \longrightarrow \Sigma^{\beta}T$ be a horizontal shift map defined by $h^0(\mu) = \xi_{\beta}$. Then, $h^{\alpha} : T^{\alpha} \longrightarrow T^{\alpha+\beta}$ is*

$$\begin{cases} A[d_{\alpha}] \longrightarrow R \text{ by } \mu_{\alpha} \mapsto d_{\alpha}\xi_{\alpha+\beta} & \text{if } |\alpha| = |\alpha^G| = 0, \\ R \longrightarrow R \text{ by } \xi_{\alpha} \mapsto \xi_{\alpha+\beta} & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0, \\ L \longrightarrow R \text{ by } \tau_{\alpha} \mapsto p\xi_{\alpha+\beta} & \text{if } |\alpha| = 0 \text{ and } 0 < |\alpha^G| < -|\beta^G|, \\ L \longrightarrow A[d_{\alpha+\beta}] \text{ by } \tau_{\alpha} \mapsto \tau_{\alpha+\beta} & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| = -|\beta^G|, \\ L \longrightarrow L \text{ by } \tau_{\alpha} \mapsto \tau_{\alpha+\beta} & \text{if } |\alpha| = 0 \text{ and } -|\beta^G| < |\alpha^G|, \\ \langle\mathbb{Z}\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \text{ by } \epsilon_{\alpha} \mapsto \epsilon_{\alpha}\xi_{\beta} & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = 0, \\ \langle\mathbb{Z}/p\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \text{ by } \epsilon_{\alpha-\gamma}\xi_{\gamma} \mapsto \epsilon_{\alpha-\gamma}\xi_{\gamma+\beta} & \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even}, \\ \langle\mathbb{Z}/p\rangle \longrightarrow \langle\mathbb{Z}/p\rangle \text{ by } \nu_{\alpha} \mapsto d_{\langle\beta\rangle-\beta}\nu_{\alpha+\beta} & \text{if } |\alpha| < 0,\ |\alpha^G| > 1 - |\beta^G|, \text{ and } \alpha \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

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(b) *Let γ be in $RO(G)$ such that $|\gamma| > 0$ and $|\gamma^G| = 0$, and let $v : T \longrightarrow \Sigma^\gamma T$ be a vertical shift map defined by $v^0(\mu) = \epsilon_\gamma$. Then, $v^\alpha : T^\alpha \longrightarrow T^{\alpha+\gamma}$ is*

$$\begin{cases} A[d_\alpha] \longrightarrow \langle \mathbb{Z} \rangle \text{ by } \mu_\alpha \mapsto \epsilon_{\alpha+\gamma} & \text{if } |\alpha| = |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \text{ by } \epsilon_\alpha \mapsto \epsilon_{\alpha+\gamma} & \text{if } |\alpha| > 0 \text{ and } |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \text{ by } \epsilon_{-\alpha}^{-1}\kappa_0 \mapsto p\epsilon_{\alpha+\gamma} & \text{if } -|\gamma| < |\alpha| < 0 \text{ and } |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow A[d_{\alpha+\gamma}] \text{ by } \epsilon_{-\alpha}^{-1}\kappa_0 \mapsto \kappa_{\alpha+\gamma} & \text{if } |\alpha| = -|\gamma| \text{ and } |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z} \rangle \text{ by } \epsilon_{-\alpha}^{-1}\kappa_0 \mapsto \epsilon_{-\alpha-\gamma}^{-1}\kappa_0 & \text{if } |\alpha| < -|\gamma| \text{ and } |\alpha^G| = 0, \\ \langle \mathbb{Z}/p \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \epsilon_\beta\xi_{\alpha-\beta} \mapsto \epsilon_{\gamma+\beta}\xi_{\alpha-\beta} & \text{if } |\alpha| > 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even}, \\ R \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \xi_\alpha \mapsto \xi_\alpha\epsilon_\gamma & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| < 0, \\ \langle \mathbb{Z}/p \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \nu_\alpha \mapsto \nu_{\alpha+\gamma} & \text{if } |\alpha| < -|\gamma|,\ |\alpha^G| > 1, \text{ and } \alpha \text{ is odd}, \\ 0 & \text{otherwise .} \end{cases}$$

(c) *Let a be an integer prime to p, and let η be in $RO(G)$ such that $|\eta| < 0$, $|\eta^G| > 1$, and η odd. Let $\partial : T \longrightarrow \Sigma^\eta T$ be a diagonal shift map defined by $\partial^0(\mu) = a\nu_\eta$. Then, $\partial^\alpha : T^\alpha \longrightarrow T^{\alpha+\eta}$ is*

$$\begin{cases} A[d_\alpha] \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \mu_\alpha \mapsto a\nu_{\alpha+\eta} & \text{if } |\alpha| = |\alpha^G| = 0, \\ \langle \mathbb{Z} \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \epsilon_\alpha \mapsto a\nu_{\alpha+\eta} & \text{if } |\alpha^G| = 0 \text{ and } 0 < |\alpha| \leq -1 - |\eta|, \\ R \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \xi_\alpha \mapsto ad_{\langle\alpha\rangle-\alpha}\nu_{\alpha+\eta} & \text{if } 3 - |\eta^G| \leq |\alpha^G| < 0 \text{ and } |\alpha| = 0, \\ \langle \mathbb{Z}/p \rangle \longrightarrow \langle \mathbb{Z}/p \rangle \text{ by } \epsilon_{\alpha-\beta}\xi_\beta \mapsto ad_{\langle\beta\rangle-\beta}\nu_{\alpha+\eta} & \begin{cases} \text{if } \alpha \text{ is even, } 0 < |\alpha| \leq -1 - |\eta|, \\ \text{and } 3 - |\eta^G| \leq |\alpha^G| < 0, \end{cases} \\ 0 & \text{otherwise .} \end{cases}$$

By using Proposition C.3 and the characterization of maps out of box products given in section 2.1, the multiplication maps for the shift maps can also be described.

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Corollary C.4. *Let α be in $RO(G)$.*

(a) *Let β be in $RO(G)$ such that $|\beta| = 0$ and $|\beta^G| < 0$. Then, the cup product map $T^\beta \square T^\alpha \longrightarrow T^{\beta+\alpha}$ is*

$$\begin{cases} \text{an isomorphism} & \begin{cases} \text{if } |\alpha| = 0 \text{ and } |\alpha^G| \leq 0, \\ \text{or } |\alpha| = 0 \text{ and } -|\beta^G| < |\alpha^G|, \\ \text{or } |\alpha| > 0,\ |\alpha^G| \leq 0, \text{ and } \alpha \text{ is even}, \\ \text{or } |\alpha| < 0,\ |\alpha^G| > 1 - |\beta^G|, \text{ and } \alpha \text{ is odd}, \end{cases} \\ L \longrightarrow R \text{ by } \tau \mapsto \pm p\xi & \text{if } |\alpha| = 0 \text{ and } 0 < |\alpha^G| < -|\beta^G|, \\ L \longrightarrow A[d_{\beta+\alpha}] \text{ by } \tau \mapsto \pm\tau & \text{if } |\alpha| = 0 \text{ and } |\alpha^G| = -|\beta^G|, \\ 0 & \text{otherwise}. \end{cases}$$

(b) *Let γ be in $RO(G)$ such that $|\gamma| > 0$ and $|\gamma^G| = 0$. Then, the cup product map $T^\gamma \square T^\alpha \longrightarrow T^{\gamma+\alpha}$ is*

$$\begin{cases} \text{an isomorphism} & \begin{cases} \text{if } |\alpha| \geq 0 \text{ and } |\alpha^G| = 0, \\ \text{or } |\alpha| < -|\gamma| \text{ and } |\alpha^G| = 0, \\ \text{or } |\alpha| \geq 0,\ |\alpha^G| < 0, \text{ and } \alpha \text{ is even}, \\ \text{or } |\alpha| < -|\gamma|,\ |\alpha^G| > 1, \text{ and } \alpha \text{ is odd}, \end{cases} \\ \langle\mathbb{Z}\rangle \xrightarrow{p} \langle\mathbb{Z}\rangle & \text{if } -|\gamma| < |\alpha| < 0 \text{ and } |\alpha^G| = 0, \\ \langle\mathbb{Z}\rangle \longrightarrow A[d_{\gamma+\alpha}] \text{ by } 1 \mapsto \pm\kappa & \text{if } |\alpha| = -|\gamma| \text{ and } |\alpha^G| = 0, \\ 0 & \text{otherwise}. \end{cases}$$

(c) *Let η be in $RO(G)$ such that $|\eta| < 0$, $|\eta^G| > 1$, and η odd. Then, the cup*

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product map $T^\eta \square T^\alpha \longrightarrow T^{\eta+\alpha}$ *is*

$$\begin{cases} \text{an isomorphism} & \text{if } 3 - |\eta^G| \leq |\alpha^G| \leq 0,\ 0 \leq |\alpha| \leq -1 - |\eta|, \text{ and } \alpha \text{ is even,} \\ 0 & \text{otherwise .} \end{cases}$$

In Chapter 3, we study a long exact sequence with a nontrivial boundary map $\partial : J \longrightarrow \Sigma^{1-V}T$ out of a free module J. In the case in which $J = \Sigma^{-\omega}T$ is monogenic, $\partial : \Sigma^{-\omega}T \longrightarrow \Sigma^{1-V}T$ is determined by the map of Mackey functors $\partial^\omega : A \longrightarrow T^{\omega-V+1}$. In fact, since $\partial^\omega \neq 0$, it must be the case that $\omega - V + 1$ is odd and $T^{\omega-V+1} = \langle \mathbb{Z}/p \rangle$. This second corollary of Proposition C.3 states that, even though $\partial \neq 0$, it still must be zero in most dimensions. Note that, for $T(R)$, the appropriate version of Proposition C.3 follows easily from Remark B.17.

Corollary C.5. *Let S be either A or R, let ω and V be even elements of $RO(G)$, and let $\eta = \omega - V + 1$. Suppose that $|\eta| < 0$ and $|\eta^G| > 1$. If $\partial : \Sigma^{-\omega}T(S) \longrightarrow \Sigma^{1-V}T(S)$ is nonzero, then $\partial^\alpha : T(S)^{\alpha-\omega} \longrightarrow T(S)^{\alpha+1-V}$ is nonzero if and only if α is even with $2 + |V^G| \leq |\alpha^G| \leq |\omega^G|$ and $|\omega| \leq |\alpha| \leq |V| - 2$.*

Proof. Consider the map $f = \Sigma^\omega \partial : T(S) \longrightarrow \Sigma^\eta T(S)$. By Proposition C.3, $f^\alpha \neq 0$ if and only if α is even, $3 - |\eta^G| \leq |\alpha^G| \leq 0$, and $0 \leq |\alpha| \leq -1 - |\eta|$. Substituting $\eta = \omega - V + 1$ gives the desired result. □

Theorem 3.15 tells us that the module M from long exact sequence (3.35)

$$\longrightarrow N \xrightarrow{f} M \xrightarrow{g} J \xrightarrow{\partial} \Sigma N \longrightarrow .$$

can turn out to be free, even though $\partial \neq 0$. For the proof of Theorem 3.4, it is useful to understand the structure of sequence (3.20) in the simplest possible case in which both $J = \Sigma^{-\omega}T$ and $N = \Sigma^{-V}T$ have just one generator. In this case, as shown in Theorem 3.15, it must be that $M \cong N' \oplus J'$, where $N' = \Sigma^{-V'}T$ and

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$J' = \Sigma^{-\omega'}T$. The dimension V' comes from shifting V to the right $|\omega^G - V^G|$ units and the dimension ω' comes from shifting ω to the left by the same amount. A switch of this type is pictured in Table 3.5. More precisely, this switch is reflected in the long exact sequence in Theorem C.6.

To organize our computations in the proof of Theorem C.6, the notation in Table 3.4 is used. As in Table 3.4, we similarly denote $l = |V^G|$, $r = |\omega^G|$, $b = |\omega|$, and $t = |V|$. For the proof of Theorem C.6, we need more notation. In the case that $r - l > 2$, denote an arbitrary even integer between l and r by m. Similarly, denote even integers between b and t by m'. Let j and k denote arbitrary positive integers. The locations pictured in Table C.1 should now be clear. For example, $[r + 1 + 2j, b - 1 - 2k]$ represents a family of locations that fills all the odd lattice points of a quadrant of the plane.

Theorem C.6. *Let V, V', ω, and ω' be elements of $RO(G)$ such that*

$$|V| = |V'| > |\omega| = |\omega'|, \quad |V^G| = |(\omega')^G| < |(V')^G| = |\omega^G|,$$

and $V - \omega$ is even. Define $N = \Sigma^{-V}T$, $J = \Sigma^{-\omega}T$, $N' = \Sigma^{-V'}T$, and $J' = \Sigma^{-\omega'}T$.

If $\partial : J \longrightarrow \Sigma N$ is nonzero, then the sequence

$$\longrightarrow N \xrightarrow{i} N' \oplus J' \xrightarrow{r} J \xrightarrow{\partial} \Sigma N \longrightarrow \tag{C.1}$$

in T-Mod is exact if and only if the sequence

$$0 \longrightarrow N^V \xrightarrow{i^V} (N' \oplus J')^V \xrightarrow{r^V} J^V \longrightarrow 0 \tag{C.2}$$

in Mack is exact and $r = (r_1, r_2)$, where $r_1 : N' \longrightarrow J$ and $r_2 : J' \longrightarrow J$ are standard shift maps. Moreover, the exactness of sequence (C.2) implies that $i = (i_1, i_2)$, where $i_1 : N \longrightarrow N'$ and $i_2 : N \longrightarrow J'$ are standard shift maps.

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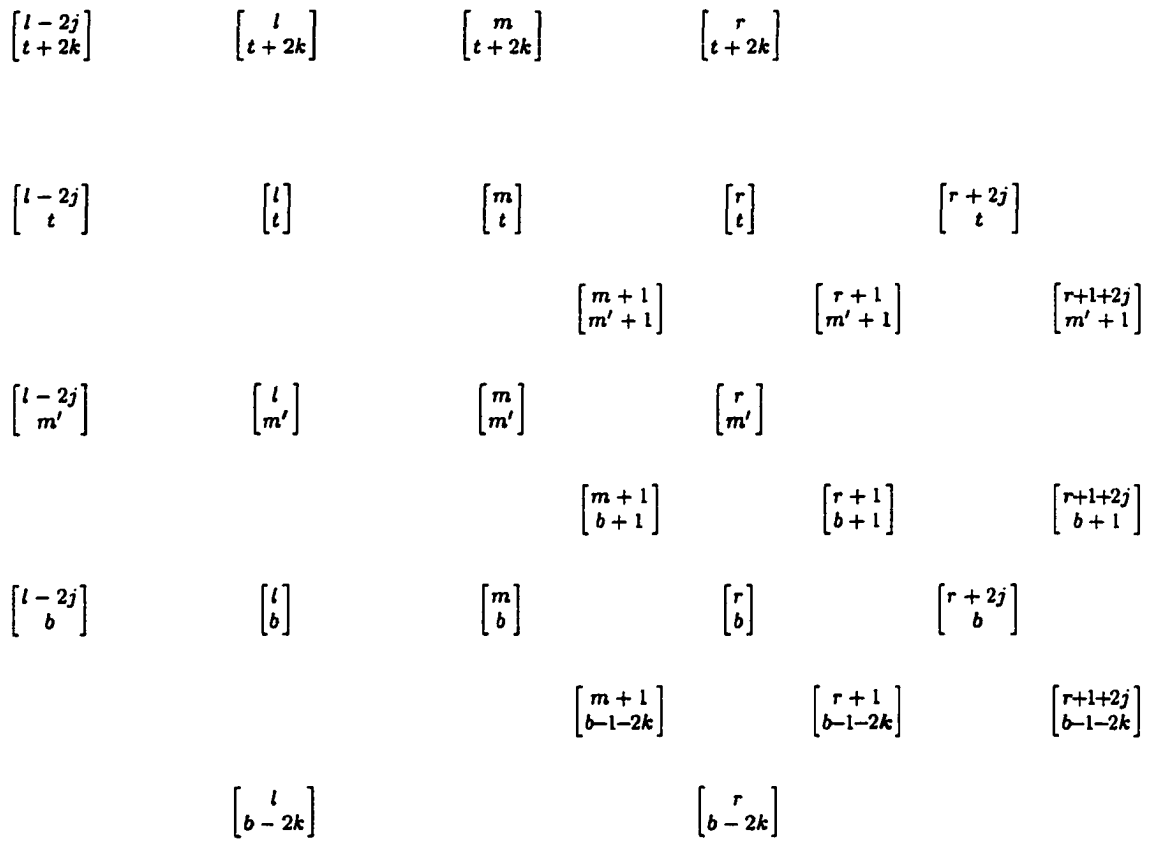


Table C.1: Notation for the Base Case

Proof. Let $\beta_1 = V - V'$, $\beta_2 = \omega' - \omega$, $\gamma_1 = V - \omega'$, and $\gamma_2 = V' - \omega$. Observe that $\beta_2 - \beta_1 = \gamma_2 - \gamma_1$ is in $RO_0(G)$.

First, suppose that sequence (C.1) is exact. The exactness of sequence (C.2) follows by evaluating sequence (C.1) in dimension V. Moreover, Proposition A.4 specifies the structure of the maps in sequence (C.2) and gives that $i^V = (i_1^V, i_2^V)$, where ${i_1}^V(\mu) = a_1\xi_{\beta_1}$ and ${i_2}^V(\mu) = b_1\epsilon_{\gamma_1}$ for integers $a_1 = \pm 1$ and $b_1 = \pm 1$. This shows that the exactness of sequence (C.2) implies that $i = (i_1, i_2)$, where $i_1 : N \longrightarrow N'$ and $i_2 : N \longrightarrow J'$ are standard shift maps. Evaluating sequence (C.1) in dimensions V' and ω' gives the short exact sequences

$$0 \longrightarrow L \longrightarrow A \xrightarrow{r_1^{V'}} \langle \mathbb{Z} \rangle \longrightarrow 0$$

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and

$$0 \longrightarrow \langle \mathbb{Z} \rangle \longrightarrow A \xrightarrow{r_2^{\omega'}} R \longrightarrow 0,$$

respectively. Proposition A.4 also specifies the structure of the maps in these sequences and gives that ${r_1}^{V'}(\mu) = b_2 \epsilon_{\gamma_2}$ and ${r_2}^{\omega'}(\mu) = a_2 \xi_{\beta_2}$, for integers $b_2 = \pm 1$ and $a_2 = \pm 1$. Hence, $r_1 : N' \longrightarrow J$ and $r_2 : J' \longrightarrow J$ are standard shift maps.

Conversely, suppose that sequence (C.2) is exact and $r = (r_1, r_2)$, where

$$r_1 : N' \longrightarrow J \quad \text{and} \quad r_2 : J' \longrightarrow J$$

are standard shift maps. Our observations above give that $i = (i_1, i_2)$, where

$$i_1 : N \longrightarrow N' \quad \text{and} \quad i_2 : N \longrightarrow J'$$

are standard shift maps. Hence, there are integers $a_1 = \pm 1$, $b_1 = \pm 1$, $b_2 = \pm 1$, and $a_2 = \pm 1$ such that

$${i_1}^{V}(\mu) = a_1 \xi_{\beta_1}, \quad {i_2}^{V}(\mu) = b_1 \epsilon_{\gamma_1}, \quad {r_1}^{V'}(\mu) = b_2 \epsilon_{\gamma_2}, \quad \text{and} \quad {r_2}^{\omega'}(\mu) = a_2 \xi_{\beta_2}.$$

The composite $r \circ i$ is zero, since $r^V \circ i^V = 0$ in sequence (C.2). Similary,

$$\partial \circ r = (\partial r_1, \partial r_2) = 0,$$

since $\partial^{V'} = 0 = \partial^{\omega'}$ and hence

$$\partial^{V'} \circ r_1^{V'} = 0 = \partial^{\omega'} r_2^{\omega'}.$$

Also, $(\Sigma i) \circ \partial = 0$, since $(\Sigma i)^{\omega} = 0$. This shows that the composites in sequence (C.1) are all zero. The exactness of sequence (C.1) therefore follows by showing that $\operatorname{Ker} r \subseteq \operatorname{Image} i$, $\operatorname{Ker} \partial \subseteq \operatorname{Image} r$, and $\operatorname{Ker} i \subseteq \operatorname{Image} \Sigma^{-1} \partial$. This is checked in every dimension.

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First, consider the dimensions α such that $\partial^\alpha \neq 0$. Corollary C.5 tells us that α is therefore in $[r,b] \cup [r,m'] \cup [m,b] \cup [m,m']$. In these cases, $N^\alpha = (N' \oplus J')^{\alpha+1} = 0$ and hence sequence (C.1) takes the form

$$0 \longrightarrow (N' \oplus J')^\alpha \xrightarrow{(r_1^\alpha, r_2^\alpha)} J^\alpha \xrightarrow{\partial^\alpha} N^{\alpha+1} \longrightarrow 0. \tag{C.3}$$

For α in $[r,b]$ and hence $\alpha+1$ in $[r+1, b+1]$, sequence (C.3) becomes

$$0 \longrightarrow \langle \mathbb{Z} \rangle \oplus L \xrightarrow{(r_1^\alpha, r_2^\alpha)} A[d] \xrightarrow{\partial^\alpha} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[r,m']$ and hence $\alpha+1$ in $[r+1, m'+1]$, sequence (C.3) becomes

$$0 \longrightarrow \langle \mathbb{Z} \rangle \xrightarrow{r_1^\alpha} \langle \mathbb{Z} \rangle \xrightarrow{\partial^\alpha} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[m,b]$ and hence $\alpha+1$ in $[m+1, b+1]$, sequence (C.3) becomes

$$0 \longrightarrow L \xrightarrow{r_2^\alpha} R \xrightarrow{\partial^\alpha} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[m,m']$ and hence $\alpha+1$ in $[m+1, m'+1]$, sequence (C.3) becomes

$$0 \longrightarrow 0 \longrightarrow \langle \mathbb{Z}/p \rangle \xrightarrow{\partial^\alpha} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

In each case, Proposition C.3 specifies the maps r_1^α, r_2^α, and ∂^α, and exactness is easily verified. In fact, by writing the map (r_1^α, r_2^α) at the G/G-level as a matrix, it is easy to see that it is an injection with cokernel $\mathbb{Z}/p$. Exactness is trivial at the G/e-level.

Now consider the cases in which $\partial^\alpha = 0$. If $\partial^{\alpha-1} \neq 0$, then $(N' \oplus J')^\alpha = 0$, and the exactness of the portion

$$J^{\alpha-1} \xrightarrow{\partial^{\alpha-1}} \langle \mathbb{Z}/p \rangle \longrightarrow 0 \longrightarrow J^\alpha \xrightarrow{0} N^{\alpha+1}$$

of sequence (C.1) in dimension α is easily checked. If $\partial^{\alpha-1} = 0$, then exactness of sequence (C.1) in dimension α is equivalent to the exactness of the sequence

$$0 \longrightarrow N^\alpha \xrightarrow{(i_1^\alpha, i_2^\alpha)} (N' \oplus J')^\alpha \xrightarrow{(r_1^\alpha, r_2^\alpha)} J^\alpha \longrightarrow 0. \tag{C.4}$$

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In most such dimensions α, one of N^α, $(N')^\alpha$, $(J')^\alpha$, or J^α is zero, and sequence (C.4) is easily seen to be exact by Proposition C.3. This leaves the locations $[l,t]$, $[r+1+2j, b-1-2k]$, $[l-2j,t]$, $[l-2j,t+2k]$, and $[l,t+2k]$, where all the entries are nonzero.

For α in $[l,t]$, sequence (C.4) becomes

$$0 \longrightarrow A[d_{\alpha-V}] \xrightarrow{(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{(b_2\epsilon_{\gamma_2}, a_2\xi_{\beta_2})} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[r+1+2j, b-1-2k]$, sequence (C.4) becomes

$$0 \longrightarrow \langle \mathbb{Z}/p \rangle \xrightarrow{(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})} \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \xrightarrow{(b_2\epsilon_{\gamma_2}, a_2\xi_{\beta_2})} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[l-2j,t]$, sequence (C.4) becomes

$$0 \longrightarrow R \xrightarrow{(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})} R \oplus \langle \mathbb{Z}/p \rangle \xrightarrow{(b_2\epsilon_{\gamma_2}, a_2\xi_{\beta_2})} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[l-2j,t+2k]$, sequence (C.4) becomes

$$0 \longrightarrow \langle \mathbb{Z}/p \rangle \xrightarrow{(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})} \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z}/p \rangle \xrightarrow{(b_2\epsilon_{\gamma_2}, a_2\xi_{\beta_2})} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

For α in $[l,t+2k]$, sequence (C.4) becomes

$$0 \longrightarrow \langle \mathbb{Z} \rangle \xrightarrow{(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})} \langle \mathbb{Z}/p \rangle \oplus \langle \mathbb{Z} \rangle \xrightarrow{(b_2\epsilon_{\gamma_2}, a_2\xi_{\beta_2})} \langle \mathbb{Z}/p \rangle \longrightarrow 0.$$

In each of these final cases, it is straightforward to check that the map $(a_1\xi_{\beta_1}, b_1\epsilon_{\gamma_1})$ is an injection with cokernel $\langle \mathbb{Z}/p \rangle$. □

Corollary C.7. *Let V, V', ω, and ω' be elements of $RO(G)$ such that*

$$|V| = |V'| > |\omega| = |\omega'|, \quad |V^G| = |(\omega')^G| < |(V')^G| = |\omega^G|,$$

and $V-\omega$ is even. Define $N = \Sigma^{-V}T$, $J = \Sigma^{-\omega}T$, $N' = \Sigma^{-V'}T$, and $J' = \Sigma^{-\omega'}T$.

If $\partial : J \longrightarrow \Sigma N$ is nonzero, then there exists a long exact sequence

$$\longrightarrow N \longrightarrow N' \oplus J' \longrightarrow J \xrightarrow{\partial} \Sigma N \longrightarrow$$

in T-Mod if and only if $d_{\omega'-\omega+V'-V} \equiv \pm 1$ modulo p.

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Proof. Let $\beta_1 = V - V'$, $\beta_2 = \omega' - \omega$, $\gamma_1 = V - \omega'$, and $\gamma_2 = V' - \omega$. Suppose that $d_{\beta_2-\beta_1} \equiv \eta$ modulo p, where $\eta = \pm 1$. Define $i = (i_1, i_2) : N \longrightarrow N' \oplus J'$ and $r = (r_1, r_2) : N' \oplus J' \longrightarrow J$ by

$$i_1{}^V(\mu) = \xi_{\beta_1}, \quad i_2{}^V(\mu) = \epsilon_{\gamma_1}, \quad r_1{}^{V'}(\mu) = \epsilon_{\gamma_2}, \quad \text{and} \quad r_2{}^{\omega'}(\mu) = -\eta\xi_{\beta_2}.$$

Observe that $d_{\beta_2-\beta_1} - \eta \equiv 0$ modulo p. Theorem 4.9 of [9] gives the relation

$$\epsilon_{\gamma_2}\xi_{\beta_1} = d_{\beta_2-\beta_1}\epsilon_{\gamma_1}\xi_{\beta_2},$$

and hence

$$r^V i^V(\mu) = \epsilon_{\gamma_2}\xi_{\beta_1} - \eta\epsilon_{\gamma_1}\xi_{\beta_2} = (d_{\beta_2-\beta_1} - \eta)\epsilon_{\gamma_1}\xi_{\beta_2} = 0$$

That is, in the sequence

$$0 \longrightarrow A \xrightarrow{i^V} R \oplus \langle \mathbb{Z} \rangle \xrightarrow{r^V} \langle \mathbb{Z}/p \rangle \longrightarrow 0,$$

the composite $r^V \circ i^V$ is zero. In terms of the $\{\mu, \tau\}$ basis for the domain and the $\{\xi_{\beta_1}, \epsilon_{\gamma_1}\}$ basis for the range, the map i_G^V is given by the 2×2 matrix $\left[\begin{smallmatrix} 1 & p \\ 1 & 0 \end{smallmatrix}\right]$. It is easy to see that $\operatorname{Coker} i^V = \langle \mathbb{Z}/p \rangle$, and hence the the sequence

$$0 \longrightarrow N^V \xrightarrow{i^V} (N' \oplus J')^V \xrightarrow{r^V} J^V \longrightarrow 0$$

is exact. Since $r_1 : N' \longrightarrow J$ and $r_2 : J' \longrightarrow J$ are standard shift maps, Theorem C.6 gives that the sequence

$$\longrightarrow N \xrightarrow{i} N' \oplus J' \xrightarrow{r} J \xrightarrow{\partial} \Sigma N \longrightarrow$$

is exact.

Conversely, suppose that

$$\longrightarrow N \xrightarrow{i} N' \oplus J' \xrightarrow{r} J \xrightarrow{\partial} \Sigma N \longrightarrow$$

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is a long exact sequence. Theorem C.6 gives that $i = (i_1, i_2)$ and $r = (r_1, r_2)$, where $i_1 : N \longrightarrow N'$, $i_2 : N \longrightarrow J'$, $r_1 : N' \longrightarrow J$ and $r_2 : J' \longrightarrow J$ are standard shift maps. Define a_1, b_1, b_2, and a_2 by ${i_1}^V(\mu) = a_1\xi_{\beta_1}$, ${i_2}^V(\mu) = b_1\epsilon_{\gamma_1}$, ${r_1}^{V'}(\mu) = b_2\epsilon_{\gamma_2}$, and ${r_2}^{\omega'}(\mu) = a_2\xi_{\beta_2}$. In dimension V, the composite $r^V \circ i^V$ is zero, and hence

$$0 = r^V i^V(\mu) = (a_1 b_2 d_{\beta_2 - \beta_1} + a_2 b_1)\epsilon_{\gamma_1}\xi_{\beta_2}.$$

Thus $a_1 b_2 d_{\beta_2-\beta_1} + a_2 b_1 \equiv 0$ modulo p. Multiplying both sides of this equation by $a_1 b_2$ gives that $d_{\beta_2-\beta_1} \equiv -a_1 a_2 b_1 b_2 \equiv \pm 1$ modulo p. □

In our final two lemmas, we give results than enable us construct maps between T-modules of the form appearing throughout the proof of Theorem 3.4. The notation introduced in Chapter 3, which associates a functor to a projective T-module Q, is used in the statements of these lemmas.

Lemma C.8. *Let Q be a sum of a standard projective T-module and a free T-module. Let $q : Q \longrightarrow Q(R)$ be the change of coefficients map induced by the map $A \longrightarrow R$ sending μ to ξ. Let H be a T-module and $f : Q \longrightarrow H$ be a T-module map. Then, there exists at most one T-module map $\tilde{f} : Q(R) \longrightarrow H$ such that the diagram*

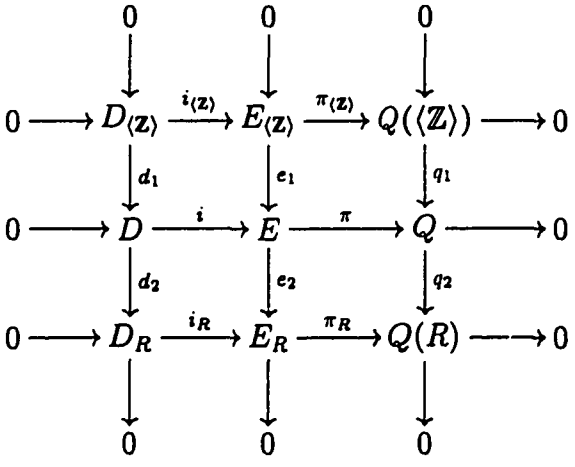
$$\begin{array}{ccc} Q & \xrightarrow{q} & Q(R) \\ & {\scriptstyle f}\searrow & \swarrow{\scriptstyle \tilde{f}} \\ & H & \end{array}$$

commutes. Moreover, if H is a $T(R)$-module regarded as a T-module via the projection $T \longrightarrow T(R)$, then $\tilde{f}$ exists and is a $T(R)$-module map.

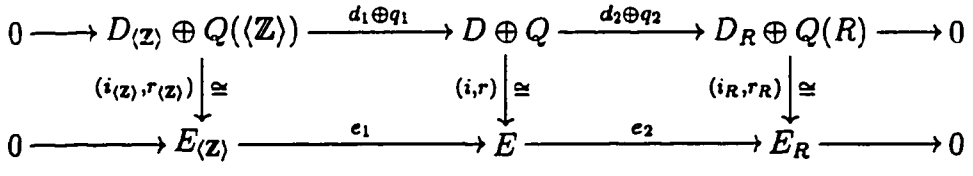
Proof. Since q is surjective, there can exist at most one map $\tilde{f}$. If H is a $T(R)$-module, then the existence of $\tilde{f}$ follows by noting that the generators of $Q(R)$ as a $T(R)$-module are the images of the generators of Q as a T-module. Thus, one knows how to define $\tilde{f}$. Since f and $\tilde{f} \circ q$ do the same thing on the generators of Q, the diagram must commute. □

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Lemma C.9. *Let Q be a T-module that is the sum of a standard projective T-module and a free T-module. Assume that the diagram*

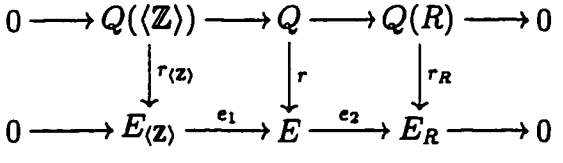


in T-Mod commutes and has exact rows and columns. Here, the right column is the canonical short exact sequence of change of coefficients maps. Suppose further that the bottom row is a sequence in $T(R)$-Mod. Then, there is a splitting $r : Q \longrightarrow E$ for π, and the induced maps $r_{\langle \mathbb{Z} \rangle} : Q(\langle \mathbb{Z} \rangle) \longrightarrow E_{\langle \mathbb{Z} \rangle}$ and $r_R : Q(R) \longrightarrow E_R$ yield the isomorphism



of short exact sequences.

Proof. Since Q is projective, there is a splitting $r : Q \longrightarrow E$ such that $r \circ \pi = id_Q$. Lemma 3.3 then gives us a unique map of short exact sequences



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determined by r. Since $\pi \circ r = id_Q$, the uniqueness implies that this map of sequences splits the map of seqences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{\langle \mathbb{Z} \rangle} & \xrightarrow{e_1} & E & \xrightarrow{e_2} & E_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\langle \mathbb{Z} \rangle}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_R} & & \\ 0 & \longrightarrow & Q(\langle \mathbb{Z} \rangle) & \longrightarrow & Q & \longrightarrow & Q(R) & \longrightarrow & 0 \end{array}$$

For $S = A$, R, and $\langle \mathbb{Z} \rangle$, the isomorphisms $(i_S, r_S) : D_S \oplus Q(S) \longrightarrow E_S$ fit into the asserted diagram. □

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Appendix D

Limits of Projectives

This appendix is devoted to a proposition that enables us to compute the limits appearing in the the proof of Theorem 3.4. Before stating the proposition, we first lay out the notation and assumptions that are required. Note that an indexed set refers to a collection that may contain the same element more than once.

Let $\mathcal{M}$ be an indexed set of nonnegative integers such that each integer appears only finitely many times in $\mathcal{M}$. Suppose that $\mathcal{M}$ has a filtration

$$\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}_k \subseteq \ldots$$

such that $\mathcal{M} = \bigcup_{k\geq 0} \mathcal{M}_k$. Let $\mathcal{W}$ be an indexed set of representation-like elements of $RO(G)$ such that, for each integer $m \geq 0$, there are only finitely many ω in $\mathcal{W}$ with $|\omega^G| = m$. Suppose that $\mathcal{W}$ has a filtration

$$\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \ldots \subseteq \mathcal{W}_k \subseteq \ldots$$

such that $\mathcal{W} = \bigcup_{k\geq 0} \mathcal{W}_k$. Let $m_0 \leq m_1 \leq \ldots \leq m_k \leq \ldots \leq \infty$ be a sequence of nonnegative integers such that $\lim_{k\to\infty} m_k = \infty$. Note that $m_k = \infty$ is allowed. For each $k \geq 0$, let $\mathcal{W}'_k$ be a set of representation-like elements of $RO(G)$ such that each $\omega \in \mathcal{W}'_k$ satisfies $|\omega^G| \geq m_k$, and define $\overline{\mathcal{W}}_k = \mathcal{W}_k \cup \mathcal{W}'_k$.

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For each $k \geq 0$, let

$$Q_k = \left(\bigoplus_{m \in \mathcal{M}_k} \Sigma^{-m} T_G \right) \oplus \left(\bigoplus_{\omega \in \mathcal{W}_k} \Sigma^{-\omega} T \right),$$

$$Q'_k = \bigoplus_{\omega \in \mathcal{W}'_k} \Sigma^{-\omega} T,$$

and $\overline{Q}_k = Q_k \oplus Q'_k$. Further, let $\pi_k : \overline{Q}_k \longrightarrow Q_k$ and $q_{k+1} : Q_{k+1} \longrightarrow Q_k$ be the obvious projections onto a summand. Suppose that the modules $\overline{Q}_k$ fit into a sequence

$$\longrightarrow \overline{Q}_{k+1} \xrightarrow{\overline{q}_{k+1}} \overline{Q}_k \longrightarrow \cdots \longrightarrow \overline{Q}_0$$

such that, for each $k \geq 0$, the diagram

$$\begin{array}{ccc} \overline{Q}_{k+1} & \xrightarrow{\overline{q}_{k+1}} & \overline{Q}_k \\ {\scriptstyle \pi_{k+1}}\downarrow & & \downarrow{\scriptstyle \pi_k} \\ Q_{k+1} & \xrightarrow{q_{k+1}} & Q_k \end{array}$$

commutes.

Let N be a Mackey functor, and let $N_1 \longrightarrow N_2$ be a map of Mackey functors. Note that the diagrams

$$\begin{array}{ccccccc} \longrightarrow \overline{Q}_{k+1}(N) & \xrightarrow{\overline{q}_{k+1}(N)} & \overline{Q}_k(N) & \longrightarrow \cdots \longrightarrow & \overline{Q}_0(N) \\ {\scriptstyle \pi_{k+1}(N)}\downarrow & & \downarrow{\scriptstyle \pi_k(N)} & & \downarrow{\scriptstyle \pi_0(N)} \\ \longrightarrow Q_{k+1}(N) & \xrightarrow{q_{k+1}(N)} & Q_k(N) & \longrightarrow \cdots \longrightarrow & Q_0(N), \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow \overline{Q}_{k+1}(N_1) & \xrightarrow{\overline{q}_{k+1}(N_1)} & \overline{Q}_k(N_1) & \longrightarrow \cdots \longrightarrow & \overline{Q}_0(N_1) \\ \downarrow & & \downarrow & & \downarrow \\ \longrightarrow \overline{Q}_{k+1}(N_2) & \xrightarrow{\overline{q}_{k+1}(N_2)} & \overline{Q}_k(N_2) & \longrightarrow \cdots \longrightarrow & \overline{Q}_0(N_2), \end{array}$$

and

$$\begin{array}{ccccccc} \longrightarrow Q_{k+1}(N_1) & \xrightarrow{q_{k+1}(N_1)} & Q_k(N_1) & \longrightarrow \cdots \longrightarrow & Q_0(N_1) \\ \downarrow & & \downarrow & & \downarrow \\ \longrightarrow Q_{k+1}(N_2) & \xrightarrow{q_{k+1}(N_2)} & Q_k(N_2) & \longrightarrow \cdots \longrightarrow & Q_0(N_2) \end{array}$$

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commute, where the unlabeled vertical maps are the associated change of coefficient maps.

Define

$$Q = \left(\bigoplus_{m\in\mathcal{M}} \Sigma^{-m}T_G\right) \oplus \left(\bigoplus_{\omega\in\mathcal{W}} \Sigma^{-\omega}T\right)$$

and

$$\widetilde{Q} = \left(\prod_{m\in\mathcal{M}} \Sigma^{-m}T_G\right) \times \left(\prod_{\omega\in\mathcal{W}} \Sigma^{-\omega}T\right).$$

Proposition D.1. **(a)** $\lim_k \overline{Q}_k(N) \cong Q(N)$.

(b) *Under the isomorphisms from part (a), the natural map*

$$\lim_k \overline{Q}_k(N_1) \longrightarrow \lim_k \overline{Q}_k(N_2)$$

is identified with the change of coefficients map $Q(N_1) \longrightarrow Q(N_2)$.

(c) $\lim_k{}^1 \overline{Q}_k(N) = 0$.

The proof of Proposition D.1 is carried out in a sequence of three lemmas.

Lemma D.2. **(a)** *The natural map* $Q(N) \longrightarrow \widetilde{Q}(N)$ *is an isomorphism.*

(b) $\lim_k Q_k(N) \cong Q(N)$.

(c) *Under the isomorphisms from part (b), the natural map*

$$\lim_k Q_k(N_1) \longrightarrow \lim_k Q_k(N_2)$$

is identified with the change of coefficients map $Q(N_1) \longrightarrow Q(N_2)$.

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Proof. (a) Since coproducts and products are computed dimensionwise, it suffices to fix α in $RO(G)$ and show that $Q^\alpha(N) = \widetilde{Q}^\alpha(N)$. Let $a = \max\{|\alpha^G|, |\alpha|\}$. Let j be large enough so that $m_j > a$ and, for all m in $\mathcal{M} - \mathcal{M}_j$, $m > a$. For each β in $(\mathcal{M} - \mathcal{M}_j) \cup (\mathcal{W} - \mathcal{W}_j)$, $(\Sigma^{-\beta} T_G(N))^\alpha = (\Sigma^{-\beta} T(N))^\alpha = 0$. Since $\mathcal{M}_j$ and $\mathcal{W}_j$ are finite, the sum and product defining $Q^\alpha(N)$ and $\widetilde{Q}^\alpha(N)$, respectively, are equal.

(b) Since, for each $k \geq 0$, the map $Q_{k+1}(N) \longrightarrow Q_k(N)$ is projection onto a summand, it follows that

$$\lim_k Q_k(N) \cong \widetilde{Q}(N).$$

Hence, part (a) gives that $\lim_k Q_k(N) \cong Q(N)$.

(c) The map $Q(N_1) \cong \lim_k Q_k(N_1) \longrightarrow \lim_k Q_k(N_2) \cong Q(N_2)$ must clearly have the asserted form as a consequence of part (b). □

Lemma D.3. *Let α be in $RO(G)$. There is a nonnegative integer j such that, for each integer $k \geq j$, $\overline{Q}_k^\alpha(N) = Q_k^\alpha(N)$.*

Proof. Let j be large enough so that $m_j > \max\{|\alpha^G|, |\alpha|\}$. For $k \geq j$, Q'_k has generators in dimensions β with $|\beta| \geq |\beta^G| \geq m_j$, and hence $(Q'_k)^\alpha = 0$. □

Lemma D.4. **(a)** $\lim_k \overline{Q}_k(N) \cong \lim_k Q_k(N)$.

(b) $\lim_k{}^1 \overline{Q}_k(N) = \lim_k{}^1 Q_k(N) = 0$.

Proof. Clearly, the maps $Q_{k+1}(N) \longrightarrow Q_k(N)$ are surjective in every dimension. The sequence $\{Q_k(N)\}$ therefore satisfies the Mittag-Leffler condition. Hence,

$$\lim_k{}^1 Q_k(N) = 0.$$

The remaining equalities asserted in the lemma follow from the previous lemma, since both $\lim_k \overline{Q}_k(N)$ and $\lim_k{}^1 \overline{Q}_k(N)$ are computed dimension-wise. □

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