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Class Group Relations from Burnside Ring Idempotents

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We show that for each finite cohomological Mackey functor on a finite group G there exist explicit relations in the category of finite abelian groups between the evaluations of the Mackey functor at all the subgroups, one for each conjugacy class of nonhypo-elementary subgroups of G. Furthermore, we show that the class groups of the intermediate fields of a Galois extension of number fields form such a Mackey functor on the Galois group, thereby obtaining class group relations by using the presence of the structure of a cohomological Mackey functor. © 1997 Academic Press

INTRODUCTION

The aim of this paper is to introduce explicit relations between the class groups $\mathrm{cl}(E)$ of intermediate fields E of a finite Galois extension L/K of number fields with Galois group G.

More precisely, we obtain for each subgroup H of G which is not p-hypo-elementary (i.e., has no normal Sylow p-subgroup with cyclic factor group) an explicit relation between the p-parts of all class groups of intermediate fields E with $\mathrm{Gal}(L/E) \leqslant H$. This relation comes from a primitive idempotent of the Burnside algebra $\mathbb{Q} \otimes \Omega(G)$ which is associated to H. In particular, the relation does not depend on the specific realization L/K of G as a Galois group, but only on G itself.

The methods we use are more group theoretic than number theoretic. In fact, the only thing we use about the class groups $\mathrm{cl}(E)$, $K \subseteq E \subseteq L$, is that they carry a rigid structure which has not been used systematically in the past, namely the structure of a cohomological Mackey functor. A Mackey functor is a structure which axiomatices the features of the character rings $R(H)$, $H \leqslant G$, namely the existence of conjugation, restriction, and induction maps between them satisfying a list of natural properties.

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In Section 1 we recall the definition of a Mackey functor and derive a general result about relations between the evaluations of a finite cohomological Mackey functor on the subgroups H of G (cf. Theorem 1.8). Crucial is the use of a result due to Yoshida (Theorem 1.3) which allows to consider cohomological Mackey functors as functors on the category of permutation modules and to use relations between permutation modules.

In Section 2 we show in an elementary way that the class groups form a cohomological Mackey functor and apply the results of Section 1 to this case.

Section 3 starts with a comparison of the results of Section 2 with classical results of Dirichlet in [8] and Brauer in [4]. We also give examples for the explicit relations of Section 2 in the case of small groups. The reader should have no problems to apply the same method to any group G she/he is interested in. Moreover, it is pointed out that if G is not p-hypo-elementary for any prime p, and if the Möbius coefficient $\mu(1, G)$ with respect to the partially ordered set (poset) of subgroups of G is nonzero, then the class group $\mathrm{cl}(L)$ is uniquely determined by the class groups of smaller fields in L/K. Interestingly, this is a topological property, since $\mu(1, G)$ is also the reduced Euler characteristic of the geometric realization of the poset of all nontrivial proper subgroups of G.

I am grateful to P. Webb for bringing the article [10] of Roggenkamp and Scott to my attention. This article uses similar ideas to obtain relations between class groups from relations between permutation modules, but without determining these relations explicitly and systematically. I would also like to mention P. Webb's article [16] which applies similar methods to group cohomology. Finally I am indebted to the referee for bringing the results listed in Remark 3.5 to my attention.

1. COHOMOLOGICAL MACKEY FUNCTORS

Throughout this section let G denote a finite group and k a commutative ring. For the reader's convenience we recall the definition of a Mackey functor, see for instance [13].

1.1. DEFINITIONS. A *k-Mackey functor* $M = (M, c, \mathrm{res}, \mathrm{ind})$ on G is quadruple consisting of

- a family of k-modules $M(H)$, where $H \leqslant G$,
- a family of k-module homomorphisms $c_{g,H}: M(H) \to M({}^gH)$, the *conjugation* maps, where $g \in G$, $H \leqslant G$, and ${}^gH = gHg^{-1}$,
- a family of k-module homomorphisms $\mathrm{res}^H_K: M(H) \to M(K)$, the *restriction* maps, where $K \leqslant H \leqslant G$, and

• a family of k-module homomorphisms $\mathrm{ind}_K^H: M(K) \to M(H)$, the *induction* maps, where $K \leqslant H \leqslant G$,

such that the following axioms are satisfied:

(M1) (Triviality) $c_{h,H} = \mathrm{res}_H^H = \mathrm{ind}_H^H = \mathrm{id}_{M(H)}$ for all $H \leqslant G$ and $h \in H$.

(M2) (Transitivity) $c_{g'g,H} = c_{g',{}^gH} \circ c_{g,H}$, $\mathrm{res}_L^K \circ \mathrm{res}_K^H = \mathrm{res}_L^H$, and $\mathrm{ind}_K^H \circ \mathrm{ind}_L^K = \mathrm{ind}_L^H$ for all $L \leqslant K \leqslant H \leqslant G$ and $g, g' \in G$.

(M3) (G-equivariance) $c_{g,K} \circ \mathrm{res}_K^H = \mathrm{res}_{{}^gK}^{{}^gH} \circ c_{g,H}$ and $c_{g,H} \circ \mathrm{ind}_K^H = \mathrm{ind}_{{}^gK}^{{}^gH} \circ c_{g,K}$ for all $K \leqslant H \leqslant G$ and $g \in G$.

(M4) (Mackey formula) For all $H \leqslant G$ and $U, K \leqslant H$ one has

$$\mathrm{res}_U^H \circ \mathrm{ind}_K^H = \sum_{h \in U\backslash H/K} \mathrm{ind}_{U \cap {}^hK}^U \circ \mathrm{res}_{U \cap {}^hK}^{{}^hK} \circ c_{h,K},$$

where $h \in H$ runs through a set of representatives for the double cosets $U\backslash H/K$.

A *morphism* $f: M \to N$ between k-Mackey functors on G is a family of k-module homomorphisms $f_H: M(H) \to N(H)$, $H \leqslant G$, which commute in the obvious way with conjugation, restriction, and induction maps.

A k-Mackey functor M on G is called *cohomological*, if the axiom

(M5) $\mathrm{ind}_K^H \circ \mathrm{res}_K^H = [H:K] \cdot \mathrm{id}_{M(H)}$, for all $K \leqslant H \leqslant G$,

holds.

The k-Mackey functors on G and the morphisms between them form a category k-Mack(G) which has the structure of a k-category; i.e., it is abelian and the morphism sets are k-modules such that the composition of morphisms is k-bilinear. In fact, we can define direct sums, kernels, cokernels, etc. in the usual way for each subgroup $H \leqslant G$ separately. The cohomological k-Mackey functors on G form a full k-subcategory k-Mack$^{\mathrm{coh}}(G)$ of k-Mack(G). In the sequel we will often write ${}^gx$ instead of $c_{g,H}(x)$ for $x \in M(H)$, $H \leqslant G$, and $M \in k$-Mack(G). Note that by the axioms (M1) and (M2) the conjugation maps are isomorphisms and provide $M(H)$, $H \leqslant G$, with a module structure for the group ring $k[N_G(H)/H]$. If $k = \mathbb{Z}$ we speak of a Mackey functor instead of a $\mathbb{Z}$-Mackey functor on G.

1.2. Examples. The standard example of a Mackey functor is the Burnside ring functor $\Omega(H)$, $H \leqslant G$, with the structure maps

$$c_{g,H}: \Omega(H) \to \Omega({}^gH), \qquad [H/U] \mapsto [{}^gH/{}^gU],$$

$$\mathrm{res}_K^H: \Omega(H) \to \Omega(K), \qquad [H/U] \mapsto \sum_{h \in K\backslash H/U} [K/K \cap {}^hU],$$

$$\mathrm{ind}_K^H: \Omega(K) \to \Omega(H), \qquad [K/V] \mapsto [H/V],$$

for $K \leqslant H \leqslant G$, $g \in G$, $U \leqslant H$, and $V \leqslant K$. The group $\Omega(G)$ is a commutative ring with unity $[G/G]$ under the multiplication $[G/H] \cdot [G/K] = \sum_{g \in H \backslash G/K} [G/H \cap {}^{g}K]$ for $H, K \leqslant G$. We refer to [7, Section 80] for the notation and basic results concerning the Burnside ring.

Another example is the character ring functor $R(H)$, $H \leqslant G$, with the usual structure maps.

For a kG-module V, the cohomology groups $H^n(H, V)$, $H \leqslant G$, with the usual conjugation and restriction maps, together with the corestrictions, form a cohomological k-Mackey functor on G.

We denote by kG-per the category of all finitely generated kG-permutation modules, i.e., kG-modules which have a finite G-stable k-basis. For a finite G-set S we denote the associated kG-module with k-basis S by $k[S]$.

By k-Funct(kG-per, k-Mod) we denote the category of k-functors from kG-per to k-Mod, i.e., functors which are k-linear on morphisms. The morphisms in this category are, as usual, the natural transformations between such functors. The following theorem is due to Yoshida, cf. [18, Theorem 4.3] or [12, Section 1].

1.3. Theorem. *There is an en equivalence of k-categories*

$$\Psi: k\text{-Mack}^{\text{coh}}(G) \mapsto k\text{-Funct}(kG\text{-per}, k\text{-Mod})$$

with the property that

$$(\Psi(M))(k[G/H]) \cong M(H) \tag{1.3.a}$$

in k-Mod *for each* $M \in k\text{-Mack}^{\text{coh}}(G)$ *and each* $H \leqslant G$.

In what follows we do not need the fact that Ψ is an equivalence, but only that there exists a functor with the property (1.3.a). Since a functor maps isomorphic objects to isomorphic objects, we have the following immediate corollary.

1.4. Corollary. *Let $H_1, ..., H_m, U_1, ..., U_n$ be subgroups of G such that*

$$\bigoplus_{i=1}^{m} k[G/H_i] \cong \bigoplus_{i=1}^{n} k[G/U_i] \tag{1.4.a}$$

in kG-per. Then

$$\bigoplus_{i=1}^{m} M(H_i) \cong \bigoplus_{i=1}^{n} M(U_i)$$

in k-Mod *for each* $M \in k\text{-Mack}^{\text{coh}}(G)$.

1.5. The hypothesis (1.4.a) in the above corollary suggests to study the functor

$$k[-]: G\text{-set} \to kG\text{-per}, \qquad S \mapsto k[S],$$

on the category of finite G-sets. If we denote by $P_k(G)$ the Grothendieck group of kG-per with respect to direct sums, this functor induces a surjective ring homomorphism

$$i_G: \Omega(G) \mapsto P_k(G), \qquad [G/H] \mapsto [k[G/H]], \tag{1.5.a}$$

where the ring structure on $P_k(G)$ comes from taking the tensor product $M \otimes_k N$ for $M, N \in kG$-per with the diagonal G-action.

Note that for $k = \mathbb{Z}_p$, the p-adic completion of $\mathbb{Z}$, we know by the Krull–Schmidt–Azumaya theorem, cf. [6, Theorem 6.12], that $M \cong N$ in kG-per, if and only if $[M] = [N]$ in $P_k(G)$. In fact, this becomes apparent, if one passes further to the category $\mathbb{Z}_pG$-lat of $\mathbb{Z}_pG$-lattices which contains $\mathbb{Z}_pG$-per as a full subcategory, and to the Grothendieck ring $A_{\mathbb{Z}_p}(G)$ of $\mathbb{Z}_pG$-lat with respect to direct sums, for which we have a canonical injective ring homomorphism $P_{\mathbb{Z}_p}(G) \to A_{\mathbb{Z}_p}(G)$.

Therefore, for $k = \mathbb{Z}_p$, it suffices to determine the kernel of i_G in (1.5.a) in order to obtain all relations of the form (1.4.a). Actually it suffices to determine the kernel of

$$i_G: \mathbb{Q} \otimes \Omega(G) \to \mathbb{Q} \otimes P_{\mathbb{Z}_p}(G) \tag{1.5.b}$$

which is easier, since the ring structure of $\mathbb{Q} \otimes \Omega(G)$ is more convenient than the one of $\Omega(G)$.

1.6. Let $\mathscr{S}_G$ denote the poset of subgroups of G, ordered by inclusion and considered as a G-set by the conjugation action. Let $\mathscr{R}_G \subseteq \mathscr{S}_G$ be a set of representatives for the G-orbits, i.e., conjugacy classes of subgroups of G. Then for each $H \in \mathscr{R}_G$ there is a ring homomorphism $\phi_H^{(G)}: \Omega(G) \to \mathbb{Z}$ which maps the class $[S]$ of a finite G-set S to the number $|S^H|$ of H-fixed points. The collection of these maps $\phi_H^{(G)}$, $H \in \mathscr{R}_G$, induces a $\mathbb{Q}$-algebra isomorphism

$$\rho_G: \mathbb{Q} \otimes \Omega(G) \to \prod_{H \in \mathscr{R}_G} \mathbb{Q},$$

cf. [7, 80.13], and we have a set of mutually orthogonal primitive idempotents $\varepsilon_H^{(G)} \in \mathbb{Q} \otimes \Omega(G)$, $H \in \mathscr{R}_G$, whose sum is 1 and which are defined by

$\phi_K^{(G)}(\varepsilon_H^{(G)}) = \delta_{K,H}$ for $K, H \in \mathscr{R}_G$. There is an explicit formula, due to Gluck (cf. [9]),

$$\varepsilon_H^{(G)} = \frac{1}{|N_G(H)|} \sum_{K \leqslant H} |K|\, \mu(K, H)[G/K], \tag{1.6.a}$$

where $\mu(K, H)$ denotes the Möbius function on $\mathscr{S}_G$, i.e.,

$$\mu(K, H) = \sum_{K = H_0 < \cdots < H_n = H} (-1)^n \tag{1.6.b}$$

(cf. [11]), where the sum runs over all strictly increasing chains in $\mathscr{S}_G$ connecting K to H.

A finite group H is called *p-hypo-elementary* for a prime p, if $H/O_p(H)$ is cyclic, where $O_p(H)$ denotes the biggest normal p-subgroup of H.

1.7. Proposition. *Let p be a prime. Then the kernel of the map*

$$i_G: \mathbb{Q} \otimes \Omega(G) \to \mathbb{Q} \otimes P_{\mathbb{Z}_p}(G), \qquad [G/H] \mapsto [\mathbb{Z}_p[G/H]],$$

is the $\mathbb{Q}$-subspace with basis $\varepsilon_H^{(G)}$, where H runs through all $H \in \mathscr{R}_G$ which are not p-hypo-elementary.

Proof. There are full embeddings of categories,

$$\mathbb{Z}_p G\text{-per} \subseteq \mathbb{Z}_p G\text{-triv} \subseteq \mathbb{Z}_p G\text{-lat},$$

where the category $\mathbb{Z}_p G$-triv consists of those $\mathbb{Z}_p G$-lattices which are permutation modules when restricted to a Sylow p-subgroup of G. These modules are called p-permutation modules and also trivial-source modules (cf. [7, Section 81B; 5]). If $PP_{\mathbb{Z}_p}(G)$ denotes the Grothendieck ring of $\mathbb{Z}_p G$-triv with respect to direct sums we obtain natural subring inclusions

$$P_{\mathbb{Z}_p}(G) \subseteq PP_{\mathbb{Z}_p}(G) \subseteq A_{\mathbb{Z}_p}(G).$$

Since $i_G: \mathbb{Q} \otimes \Omega(G) \to \mathbb{Q} \otimes P_{\mathbb{Z}_p}(G)$ is a $\mathbb{Q}$-algebra homomorphism, its kernel is an ideal, and therefore spanned over $\mathbb{Q}$ by those primitive idempotents $\varepsilon_H^{(G)}$, $H \in \mathscr{R}_G$, with $i_G(\varepsilon_H^{(G)}) = 0$. Now Lemma 81.28 in [7] states that $\varepsilon_H^{(G)}$, $H \in \mathscr{R}_G$, is in the kernel of the map

$$\mathbb{C} \otimes \Omega(G) \xrightarrow{i_G} \mathbb{C} \otimes P_{\mathbb{Z}_p}(G) \subseteq \mathbb{C} \otimes PP_{\mathbb{Z}_p}(G),$$

if and only if H is not p-hypo-elementary. This completes the proof. ∎

Let $\mathrm{K}_0(\mathrm{ab})$ denote the Grothendieck group of the category ab of finite abelian groups with respect to direct sums. For $A \in \mathrm{ab}$ let $[A]$ denote the associated element in $\mathrm{K}_0(\mathrm{ab})$. Note that two finite abelian groups A and B

are isomorphic, if and only if $[A]=[B]$ in $\mathrm{K}_0(\mathrm{ab})$, since each finite abelian group has a unique decomposition (up to isomorphism and ordering) into a direct sum of indecomposable groups (i.e., cyclic groups of prime power order). For $A \in \mathrm{ab}$ and a prime p we write A_p for the Sylow p-subgroup of A. For $A \in \mathrm{ab}$ and $n \in \mathbb{N}$ we write A^n for the n-fold direct sum $A \oplus \cdots \oplus A$.

We call a k-Mackey functor on G *finite*, if $M(H)$ is finite for all $H \leqslant G$. A finite group H is called *hypo-elementary*, if it is p-hypo-elementary for some prime p.

1.8. THEOREM. *Let M be a finite cohomological Mackey functor on G.*

(i) *Let p be a prime and let $H \leqslant G$ be not p-hypo-elementary. Then the relation*

$$\sum_{K \leqslant H} |K|\, \mu(K, H)[M(K)_p] = 0 \tag{1.8.a}$$

holds in $\mathrm{K}_0(\mathrm{ab})$. *More precisely*,

$$\bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ even}}} M(H_0)_p^{|H_0|} \cong \bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ odd}}} M(H_0)_p^{|H_0|} \tag{1.8.b}$$

in ab, *where the sums run over all strictly increasing chains in $\mathscr{S}_G$ which end in H and which have even (resp. odd) length.*

(ii) *Let $H \leqslant G$ be not hypo-elementary. Then the relation*

$$\sum_{K \leqslant H} |K|\, \mu(K, H)[M(K)] = 0 \tag{1.8.c}$$

holds in $\mathrm{K}_0(\mathrm{ab})$. *More precisely*,

$$\bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ even}}} M(H_0)^{|H_0|} \cong \bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ odd}}} M(H_0)^{|H_0|} \tag{1.8.d}$$

in ab.

Proof. (i) Note that $M_p(H) := M(H)_p$ for $H \leqslant G$ defines a cohomological $\mathbb{Z}_p$-Mackey functor M_p on G. Hence, the isomorphism (1.8.b) in ab (or equivalently in $\mathbb{Z}_p$-Mod) follows from Corollary 1.4 and Proposition 1.7 by taking into account that two objects in $\mathbb{Z}_p G$-per are isomorphic, if and only if their classes in $P_{\mathbb{Z}_p}(G)$ coincide, and by using the explicit formulae (1.6.a) and (1.6.b). Equation (1.8.a) then follows immediately.

(ii) This follows from part (a), since $A \cong \bigoplus_p A_p$ for any finite abelian group A, where the sum runs over all primes dividing the order of A. ∎

2. CLASS GROUP RELATIONS

Throughout this section let L/K denote a Galois extension of number fields with Galois group G. For $H \leqslant G$ we denote the associated fixed field by L^H.

For any number field F, we denote by $\mathcal{O}_F$ the ring of integers of F, by I_F the group of fractional ideals of F and by P_F the subgroup of principal fractional ideals of F. Then the class group $\mathrm{cl}(F)$ of F is the factor group I_F/P_F.

2.1. For $U \leqslant H \leqslant G$ and $g \in G$ we have group homomorphisms

$$\begin{aligned} c_{g,H}&: I_{L^H} \to I_{gL^H}, & \mathfrak{a} &\mapsto g(\mathfrak{a}),\\ \mathrm{res}^H_U&: I_{L^H} \to I_{L^U}, & \mathfrak{a} &\mapsto \mathfrak{a}\mathcal{O}_{L^U},\\ \mathrm{ind}^H_U&: I_{L^U} \to I_{L^H}, & \mathfrak{b} &\mapsto \mathcal{N}_{L^U/L^H}(\mathfrak{b}), \end{aligned} \tag{2.1.a}$$

for $\mathfrak{a} \in I_{L^H}$ and $\mathfrak{b} \in I_{L^U}$, where $\mathcal{N}_{L^U/L^H}(\mathfrak{b}) \in I_{L^H}$ is the $\mathcal{O}_{L^H}$-module generated by the elements $N_{L^U/L^H}(b) \in L^H$, $b \in \mathfrak{b}$. For a nontrivial prime ideal $\mathfrak{P}$ of $\mathcal{O}_{L^U}$ one has $\mathcal{N}_{L^U/L^H}(\mathfrak{P}) = \mathfrak{p}^{f_{\mathfrak{P}/\mathfrak{p}}}$ with $\mathfrak{p} = \mathcal{O}_{L^H} \cap \mathfrak{P}$ and $f_{\mathfrak{P}/\mathfrak{p}}$ the residue degree of $\mathfrak{P}/\mathfrak{p}$. Note that res^H_U is injective. It is clear that the subgroups P_{L^H}, $H \leqslant G$, of I_{L^H} are stable under these three types of maps ($\mathcal{N}_{L^U/L^H}(x\mathcal{O}_{L^U}) = N_{L^U/L^H}(x)\,\mathcal{O}_{L^H}$ for $x \in L^U$), and therefore, we obtain induced maps

$$\begin{aligned} c_{g,H}&: \mathrm{cl}(L^H) \to \mathrm{cl}(L^{gH}),\\ \mathrm{res}^H_U&: \mathrm{cl}(L^H) \to \mathrm{cl}(L^U),\\ \mathrm{ind}^H_U&: \mathrm{cl}(L^U) \to \mathrm{cl}(L^H), \end{aligned} \tag{2.1.b}$$

for $U \leqslant H \leqslant G$ and $g \in G$.

2.2. Proposition. *The groups I_{L^H}, $H \leqslant G$, together with the maps in* (2.1.a) *form a cohomological Mackey functor on G. In particular, the class groups* $\mathrm{cl}(L^H)$, $H \leqslant G$, *together with the maps in* (2.1.b) *form a finite cohomological Mackey functor on G.*

Proof. The axioms (M1), (M2), (M3), and (M5) are clearly satisfied for I_{L^H}, $H \leqslant G$. We only show that the Mackey axiom (M4), namely

$$\mathrm{res}^H_U(\mathrm{ind}^H_V(\mathfrak{P})) = \sum_{h \in U\backslash H/V} \mathrm{ind}^U_{U\cap {}^hV}(\mathrm{res}^{{}^hV}_{U\cap {}^hV}({}^h\mathfrak{P})), \tag{2.2.a}$$

holds for $U, V \leqslant H \leqslant G$ and all prime ideals $\mathfrak{P} \neq 0$ of $\mathcal{O}_{L^V}$.

Step 1. We show (2.2.a) in the case $V = U = 1$,

$$\mathrm{res}_1^H(\mathrm{ind}_1^H(\mathfrak{P})) = \mathrm{res}_1^H(\mathfrak{p}^f) = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^{ef} = \prod_{h \in H} h(\mathfrak{P}),$$

where $\mathfrak{p} = \mathcal{O}_{L^H} \cap \mathfrak{P}$, f and e are the residue degree and the ramification index of $\mathfrak{P}/\mathfrak{p}$, respectively, and $\mathfrak{p}\mathcal{O}_L = (\mathfrak{P}_1 \cdots \mathfrak{P}_r)^e$ with pairwise distinct prime ideals $\mathfrak{P}_1, ..., \mathfrak{P}_r$ of $\mathcal{O}_L$.

Step 2. We show (2.2.a) in the case $U = 1$. Using Step 1, (M3), and (M5) we have

$$\begin{aligned}
\mathrm{res}_1^H(\mathrm{ind}_V^H(\mathfrak{P}))^{|V|} &= (\mathrm{res}_1^H \circ \mathrm{ind}_V^H \circ \mathrm{ind}_1^V \circ \mathrm{res}_1^V)(\mathfrak{P}) \\
&= (\mathrm{res}_1^H \circ \mathrm{ind}_1^H)(\mathrm{res}_1^V(\mathfrak{P})) \\
&= \prod_{h \in H} {}^h(\mathrm{res}_1^V(\mathfrak{P})) \\
&= \prod_{h \in H/V} (\mathrm{res}_1^{{}^hV}(h(\mathfrak{P})))^{|V|}.
\end{aligned}$$

Since I_L is torsion free, (2.2.a) follows in this case.

Step 3. We show (2.2.a) in general. Since res_1^U is injective, it suffices to show that

$$(\mathrm{res}_1^H \circ \mathrm{ind}_V^H)(\mathfrak{P}) = \sum_{h \in U \backslash H/V} (\mathrm{res}_1^U \circ \mathrm{ind}_{U \cap {}^hV}^U)(\mathrm{res}_{U \cap {}^hV}^{{}^hV}(h(\mathfrak{P}))). \tag{2.2.b}$$

By Step 2 we have,

$$(\mathrm{res}_1^H \circ \mathrm{ind}_V^H)(\mathfrak{P}) = \sum_{h \in H/V} \mathrm{res}_1^{{}^hV}(h(\mathfrak{P})),$$

and for arbitrary $h \in H$ we have by Step 2,

$$\begin{aligned}
&(\mathrm{res}_1^U \circ \mathrm{ind}_{U \cap {}^hV}^U)(\mathrm{res}_{U \cap {}^hV}^{{}^hV}(h(\mathfrak{P}))) \\
&\quad = \sum_{u \in U/U \cap {}^hV} {}^u[\mathrm{res}_1^{U \cap {}^hV}(\mathrm{res}_{U \cap {}^hV}^{{}^hV}(h(\mathfrak{P})))] \\
&\quad = \sum_{u \in U/U \cap {}^hV} \mathrm{res}_1^{{}^{uh}V}(uh(\mathfrak{P})).
\end{aligned}$$

Now Eq. (2.2.b) follows, since if h runs through a set of representatives for $U\backslash H/V$ and for each such h, u_h runs through a set of representatives for $U/U \cap {}^hV$, then $u_h \cdot h$ runs through a set of representatives for H/V. This completes the proof. ∎

2.3. THEOREM. (i) *If $H \leqslant G$ is not p-hypo-elementary for a prime p, then*

$$\bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ even}}} \mathrm{cl}(L^{H_0})_p^{|H_0|} \cong \bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ odd}}} \mathrm{cl}(L^{H_0})_p^{|H_0|}.$$

(ii) *If $H \leqslant G$ is not hypo-elementary, then*

$$\bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ even}}} \mathrm{cl}(L^{H_0})^{|H_0|} \cong \bigoplus_{\substack{H_0 < \cdots < H_n = H \\ n \text{ odd}}} \mathrm{cl}(L^{H_0})^{|H_0|}.$$

Proof. This follows immediately from Theorem 1.8 and Proposition 2.2. ∎

2.4. COROLLARY. (i) *If p is a prime and $H \leqslant G$ is not p-hypo-elementary, then*

$$\sum_{U \leqslant H} |U|\, \mu(U, H)[\mathrm{cl}(L^U)_p] = 0$$

in $K_0(\mathrm{ab})$. In particular, $\mathrm{cl}(L^H)_p$ is uniquely determined by the groups $\mathrm{cl}(L^U)_p$, $U < H$.

(ii) *If $H \leqslant G$ is not hypo-elementary, then*

$$\sum_{U \leqslant H} |U|\, \mu(U, H)[\mathrm{cl}(L^U)] = 0$$

in $K_0(\mathrm{ab})$. In particular, $\mathrm{cl}(L^H)$ is uniquely determined by the groups $\mathrm{cl}(L^U)$, $U < H$.

Proof. This follows from Theorem 2.3 by taking the associated elements in the Grothendieck group $K_0(\mathrm{ab})$ and using Eq. (1.6.b). ∎

3. REMARKS AND EXAMPLES

We first compare Theorem 2.3 with the following two classical results. For a number field K we denote by $h_K = |\mathrm{cl}(K)|$ its class number.

3.1. THEOREM (Dirichlet, [8]). *Let $1 < d \in \mathbb{N}$ be square-free, and let $L = \mathbb{Q}(\sqrt{d}, \sqrt{-d})$. Then*

$$2h_L = Q \cdot h_{\mathbb{Q}(\sqrt{d})} \cdot h_{\mathbb{Q}(\sqrt{-d})},$$

where $Q \in \{1, 2\}$ depends on d.

3.2. Theorem (Brauer, [4]). *Let G be a finite group and let*

$$\sum_{H \leqslant G} a_H \cdot \operatorname{ind}_H^G(1_H) = 0$$

be a relation in the character ring $R(G)$ of G, with integers a_H, $H \leqslant G$. Then the product

$$\prod_{H \leqslant G} h_{L^H}^{a_H}$$

assumes only finitely many values, as L runs over all Galois extensions of $K = \mathbb{Q}$ with Galois group G.

3.3. *Remark.* (i) Theorem 3.2 can be considered as a generalization of Theorem 3.1, since there is a relation

$$\operatorname{ind}_1^G(1) - \operatorname{ind}_{H_1}^G(1) - \operatorname{ind}_{H_2}^G(1) - \operatorname{ind}_{H_3}^G(1) + 2 \cdot \operatorname{ind}_G^G(1) = 0 \quad (3.3.a)$$

for the group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where H_1, H_2, H_3 denote the three subgroups of order 2. Both theorems make use of number theoretic results. Brauer, for instance, uses the theory of Artin L-functions and the analytic class number formula. In contrast to this we used very little knowledge from number theory to verify that class groups carry the structure of a cohomological Mackey functor. The rest followed from the general theory of cohomological Mackey functors. Note that the character relations used in Theorem 3.2 come from the idempotents $\varepsilon_H^{(G)} \in \mathbb{Q} \otimes \Omega(G)$ with noncyclic $H \leqslant G$; cf. [17, Lemma 3.7]. More precisely, the kernel of the natural map $i_G: \mathbb{Q} \otimes \Omega(G) \to \mathbb{Q} \otimes R(G)$, $[G/H] \mapsto \operatorname{ind}_H^G(1)$, has as a $\mathbb{Q}$-basis the set of idempotents $\varepsilon_H^{(G)}$, where H runs through a set of representatives for the conjugacy classes of noncyclic subgroups of G (cf. Proposition 1.7). Therefore, Brauer obtains more relations than we do in Theorem 2.3. But his relations are only exact up to a factor which may assume finitely many values. For the relations coming from idempotents $\varepsilon_H^{(G)}$ with H noncyclic but hypo-elementary, this finiteness result can not be obtained by our general approach.

For the group $G = Z/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in Dirichlet's Theorem 3.1 we obtain only one relation, coming from the idempotent

$$\varepsilon_G^{(G)} = \tfrac{1}{2}(2[G/G] - [G/H_1] - [G/H_2] - [G/H_3] + [G/1]), \quad (3.3.b)$$

since G is not p-hypo-elementary for odd primes p. In particular we obtain

$$\operatorname{cl}(\mathbb{Q}(\sqrt{d}))_p \oplus \operatorname{cl}(\mathbb{Q}(\sqrt{-d}))_p \cong \operatorname{cl}(\mathbb{Q}(\sqrt{d}, \sqrt{-d}))_p$$

for odd primes (since $\mathrm{cl}(\mathbb{Q})$ and $\mathrm{cl}(\mathbb{Q}(i))$ are trivial), showing that the factor Q in Theorem 3.1 has to be a power of 2 or $\frac{1}{2}$.

But note that for all primes which do not divide the order of the Galois group G we obtain the exact relation, even for the class groups, not only for the class numbers. Moreover, we do not need the restriction that the base field is $\mathbb{Q}$, as in Theorems 3.1 and 3.2.

(ii) There have been improvements of Brauer's Theorem 3.2. Walter proved in [15, Theorem 2.1] that each relation

$$\sum_{H \leqslant G} a_H \cdot \mathrm{ind}_H^G(1) = \sum_{H \leqslant G} b_H \cdot \mathrm{ind}_H^G(1)$$

in the character ring $R(G)$ with $a_H, b_H \in \mathbb{N}_0$, induces a relation

$$\bigoplus_{H \leqslant G} \mathrm{cl}(L^H)_\pi^{a_H} \cong \bigoplus_{H \leqslant G} \mathrm{cl}(L^H)_\pi^{b_H}$$

in the category ab, where π is the set of all primes not dividing $|G|$. This is covered by our results, since each relation in the character ring comes from idempotents $\varepsilon_H^{(G)} \in \mathbb{Q} \otimes \Omega(G)$, $H \leqslant G$ noncyclic, and since each noncyclic $H \leqslant G$ is non-p-hypo-elementary if p does not divide $|G|$.

(iii) Roggenkamp and Scott have an approach in [10] which is very similar to the one we outlined here. What they call "Hecke actions" is basically a functor in k-Funct(kG-per, k-Mod). They showed that Picard groups (in particular, class groups) give rise to a Hecke action. In particular, they have a result analogous to Corollary 1.4. But they did not determine all possible relations in $\mathbb{Z}_p G$-per explicitly.

If L/K is a Galois extension of number fields with Galois group G, for practical uses one would rather like to determine the class group of L from the class groups of all smaller intermediate fields, than the class group of K by all the other ones, as Corollary 2.4 does.

3.4. Proposition. *If L/K is a Galois extension of number fields with Galois group G and p a prime such that there is a non-p-hypo-elementary subgroup $H \leqslant G$ with $\mu(1, H) \neq 0$, then $\mathrm{cl}(L)_p$ is uniquely determined by the p-parts of the class groups of all smaller intermediate fields, and this in an explicit way which is independent of the specific Galois extension L/K.*

Proof. This follows immediately from Corollary 2.4 (a) and the fact that the isomorphisms type of an abelian group is uniquely determined by its class in $K_0(\mathrm{ab})$. ∎

3.5. *Remarks.* Unfortunately not too much is known about the number $\mu(1, G)$ for a finite group G. The following results were brought to my attention by the referee:

S. Bouc showed in [1, Corollary to Proposition 6] that $\mu(1, G) = m(G)\,|G'|$ for an integer $m(G)$, where G' is the derived subgroup of G.

In [2, Proposition 3] it is shown that, if G has a normal subgroup N with $1 \neq N \neq G$ which has no complement in G (i.e., there is no subgroup $H \leqslant G$ with $HN = G$ and $H \cap N = 1$), then $m(G) = 0$. Moreover, in [2, Proposition 5] there are stated several equivalent conditions for a finite *solvable* group G to satisfy $m(G) = 0$. One of them is the existence of a normal subgroup N as above. The same proposition also gives a nice formula for $\mu(1, G)$ in the solvable case.

If G_1 and G_2 are two finite groups without common factor group except 1, then $\mu(1, G_1 \times G_2) = \mu(1, G_1)\,\mu(1, G_2)$, cf. [1, Proposition 7(ii)] or [3, Lemma 22].

The methods in [1] also allow to compute $m(G)$ for nonsolvable groups, as for example, for the series $G = \mathrm{SL}_2(p)$, p an odd prime:

$$m(G) = \begin{cases} 2, & \text{if } p^2 \equiv 1 \bmod 5;\ p^2 \equiv 1 \bmod 16, \\ 1, & \text{if } p^2 \equiv 1 \bmod 5;\ p^2 \equiv 9 \bmod 16, \\ 0, & \text{if } p^2 \equiv 4 \bmod 5;\ p^2 \equiv 1 \bmod 16, \\ -1, & \text{if } p^2 \equiv 4 \bmod 5;\ p^2 \equiv 9 \bmod 16. \end{cases}$$

If G is a p-group, then $\mu(1, G) \neq 0$, if and only if G is elementary abelian (in the other case, the Frattini subgroup $\Phi(G)$, i.e., the intersection of all maximal subgroups of G, is a nontrivial proper normal subgroup of G without complement in G), and in the elementary abelian case one has $m(G) = (-1)^r p^{\binom{r}{2}}$, if $|G| = p^r$ (cf. [11, 5, Example 2]).

A check with the computer program package GAP for computational group theory yields that among the 168 nonhypo-elementary groups G of order between 1 and 95 there are exactly 50 with $\mu(1, G) \neq 0$.

EXAMPLE 3.6. Let L/K be a Galois extension of number fields with Galois group G:

(i) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Then G is not p-hypo-elementary for all odd primes p and we obtain from the idempotent $\varepsilon_G^{(G)}$ in (3.3.b) the relation

$$\mathrm{cl}(K)_{2'} \oplus \mathrm{cl}(K)_{2'} \oplus \mathrm{cl}(L)_{2'} \cong \mathrm{cl}(K_1)_{2'} \oplus \mathrm{cl}(K_2)_{2'} \oplus \mathrm{cl}(K_3)_{2'},$$

if K_1, K_2, K_3 are the proper intermediate fields of L/K.

(ii) Let $G = S_3$, the symmetric group on three letters. Then G is not p-hypo-elementary for all primes p except $p = 3$, and we obtain from the idempotent

$$\varepsilon_G^{(G)} = \tfrac{1}{2}([G/1] - 2[G/C_2] - [G/C_3] + 2[G/G]) \in \mathbb{Q} \otimes \Omega(G),$$

with C_2 and C_3 the Sylow subgroups for the primes 2 and 3, respectively, the relation

$$\mathrm{cl}(K)_{3'} \oplus \mathrm{cl}(K)_{3'} \oplus \mathrm{cl}(L)_{3'} \cong \mathrm{cl}(E)_{3'} \oplus \mathrm{cl}(E)_{3'} \oplus \mathrm{cl}(F)_{3'},$$

with $[E:K] = 3$ and $[F:K] = 2$.

(iii) The smallest nonhypo-elementary group is the dihedral group D_{12} of order 12. Since $\mu(1, D_{12}) = -6 \neq 0$, Proposition 3.4 applies to $G = D_{12}$ and all primes p. Also $G = S_4$, the symmetric group on 4 letters, is not hypo-elementary with $\mu(1, S_4) = -12$. Unfortunately there are, for fixed p, minimal non-p-hypo-elementary groups G with $\mu(1, G) = 0$, as for example the group $G = \langle a, b \mid a^6 = 1, b^4 = 1, bab^{-1} = a^5, b^2 = a^3 \rangle$ of order 12 for the prime 2. For this G we obtain by our approach no information about the 2-part of $\mathrm{cl}(L)$ from the 2-parts of the class groups of all proper subfields containing K.

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