

Math 13200 – Section 58 – Midterm 3

May 27, 2016

Information and Directions

- This exam will last 50 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- **Partial credit is given for showing your calculations and explaining your thoughts.**

Name: _____

SOLUTIONS

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	20	
6	20	
7	20	

Problem 1 (10 points)

Find the antiderivative $\int x^2 \cos(x^3 + 5) dx$.

u -substitution with $u = x^3 + 5$

therefore $du = 3x^2 dx$

$$\begin{aligned}\int x^2 \cos(x^3 + 5) dx &= \frac{1}{3} \int \cos(x^3 + 5) \cdot (3x^2) dx = \frac{1}{3} \int \cos(u) du \\ &= \frac{1}{3} \sin(u) + C = \frac{1}{3} \sin(x^3 + 5) + C\end{aligned}$$

Problem 2 (10 points)

Find the definite integral $\int_0^{\pi/6} \sin^2(3x) \cos(3x) dx$.

u -substitution with $u = \sin(3x)$

therefore $du = 3\cos(3x) dx$ and $0 \rightsquigarrow \sin(3 \cdot 0) = 0$

$\frac{\pi}{6} \rightsquigarrow \sin(3 \cdot \frac{\pi}{6}) = 1$

$$\int_0^{\frac{\pi}{6}} \sin^2(3x) \cos(3x) dx = \frac{1}{3} \int_0^{\frac{\pi}{6}} (\sin(3x))^2 \cdot (3\cos(3x)) dx$$

$$= \frac{1}{3} \int_0^1 u^2 du = \frac{1}{3} \left[\frac{u^3}{3} \right]_0^1 = \frac{1}{3} \left(\frac{1}{3} - 0 \right) = \frac{1}{9}$$

Problem 3 (10 points)

Find the definite integral $\int_4^9 \frac{(\sqrt{x} + 3)^2}{\sqrt{x}} dx$.

u -substitution with $u = \sqrt{x} + 3$

therefore $du = \frac{1}{2\sqrt{x}} dx$ and $4 \rightsquigarrow \sqrt{4} + 3 = 5$

$9 \rightsquigarrow \sqrt{9} + 3 = 6$

$$\begin{aligned} \int_4^9 \frac{(\sqrt{x} + 3)^2}{\sqrt{x}} dx &= 2 \int_4^9 (\sqrt{x} + 3)^2 \cdot \left(\frac{1}{2\sqrt{x}}\right) dx = 2 \int_5^6 u^2 du \\ &= 2 \left[\frac{u^3}{3} \right]_5^6 = 2 \left(\frac{216}{3} - \frac{125}{3} \right) = \frac{182}{3} \end{aligned}$$

Problem 4 (10 points)

Find the definite integral $\int_{-18}^{18} \sin(x^{43}) dx$.

Hint: Because 43 is an odd number, for any number x it is true that

$$\sin((-x)^{43}) = \sin(-(x^{43})) = -\sin(x^{43})$$

By the Symmetry Theorem from section 4.5, if f is an odd function, then $\int_{-a}^a f(x) dx = 0$.

The hint points out that $\sin(x^{43})$ is an odd function.

$$\text{Therefore } \int_{-18}^{18} \sin(x^{43}) dx = 0.$$

Problem 5 (20 points)

Part 1 (10 points)

The Mean Value Theorem for Integrals says that if f is continuous on $[a, b]$, then there is at least one number c between a and b such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

(Complete the formula by writing the correct expression in the box.)

Hint: Remember, the MVT for Integrals is related to, but different than, the MVT for Derivatives.

Part 2 (10 points)

For the function $f(x) = x^2 - 1$ and the interval $[-3, 6]$, find the number c that the Mean Value Theorem for Integrals guarantees must exist.

$$a = -3, \quad b = 6$$

$$\frac{1}{6 - (-3)} \int_{-3}^6 x^2 - 1 \, dx = \frac{1}{9} \int_{-3}^6 x^2 - 1 \, dx = \frac{1}{9} \left[\frac{x^3}{3} - x \right]_{-3}^6$$

$$= \frac{1}{9} \left((72 - 6) - ((-9) - (-3)) \right) = \frac{1}{9} (66 - (-6)) = 8$$

So we're looking for numbers c such that $\overset{-3}{\parallel} a < c < \overset{6}{\parallel} b$

$$\text{and } f(c) = c^2 - 1 = 8.$$

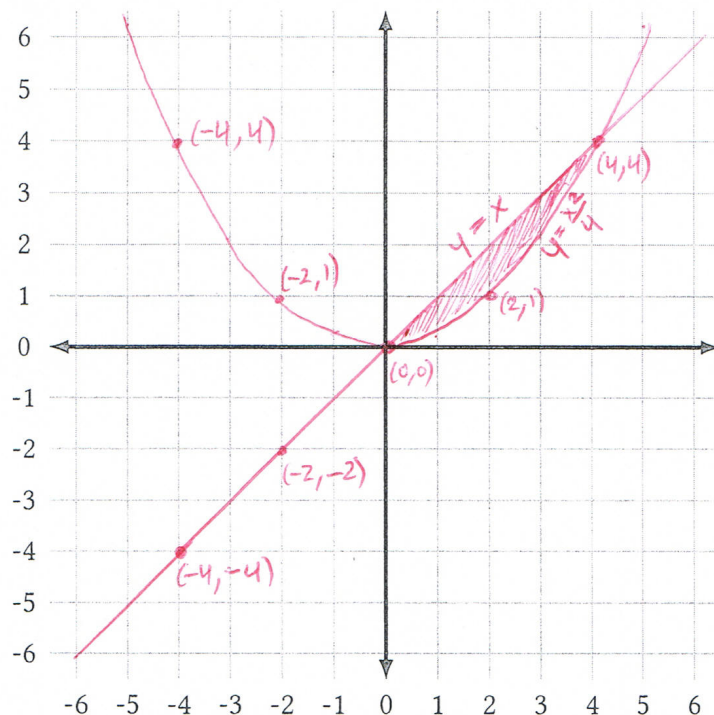
$$c^2 - 1 = 8 \Rightarrow c^2 = 9 \Rightarrow c = \pm 3$$

Since we want $-3 < c < 6$, the only option is $c = 3$.

Problem 6 (20 points)

Consider the region bounded by the line $y = x$ and the curve $y = \frac{x^2}{4}$.

Here is a grid for your convenience if you would like to plot this region (optional – no points).



Part 1 (10 points)

Find the area of the region R .

Slicing vertically, the length of the slice at x is $x - \frac{x^2}{4}$

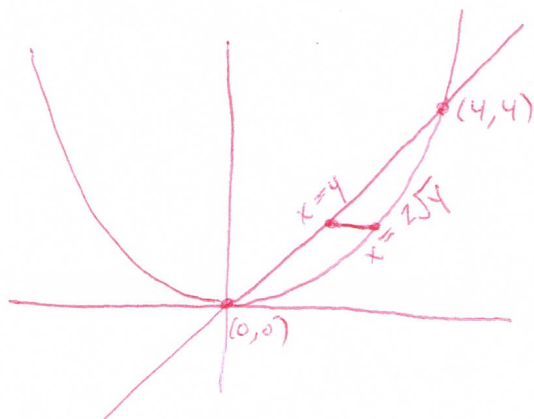
(the line $y = x$ is on top, the curve $y = \frac{x^2}{4}$ is on the bottom)

$$\text{Area} = \int_{\text{one side}}^{\text{other side}} (\text{length of slice at } x) \, dx = \int_0^4 x - \frac{x^2}{4} \, dx = \left[\frac{x^2}{2} - \frac{x^3}{12} \right]_0^4$$

$$= \left(8 - \frac{16}{3} \right) - (0 - 0) = \frac{8}{3}$$

Alternative solution for Problem 6, Part 1:

Slicing horizontally, the length of the slice at y is $2\sqrt{y} - y$



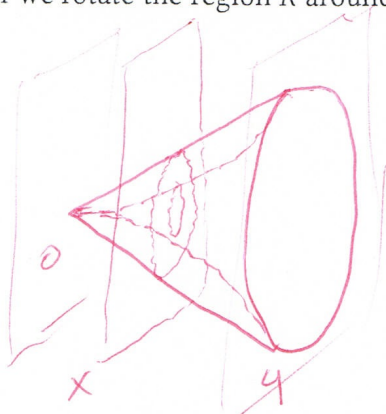
(The curve $y = \frac{x^2}{4}$ becomes $x = \sqrt{4y} = 2\sqrt{y}$)

$$\text{Area} = \int_{\text{one side}}^{\text{other side}} (\text{length of slice at } y) dy = \int_0^4 2\sqrt{y} - y dy$$

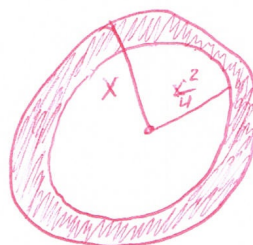
$$\begin{aligned} &= \left[2 \frac{y^{3/2}}{3/2} - \frac{y^2}{2} \right]_0^4 = \left[\frac{4y^{3/2}}{3} - \frac{y^2}{2} \right]_0^4 = \left(\frac{32}{3} - 8 \right) - (0 - 0) \\ &= \frac{8}{3} \end{aligned}$$

Part 2 (10 points)

If we rotate the region R around the x -axis, what is the volume of the resulting shape?



Slice at x looks like



Area of "washer" is $\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$

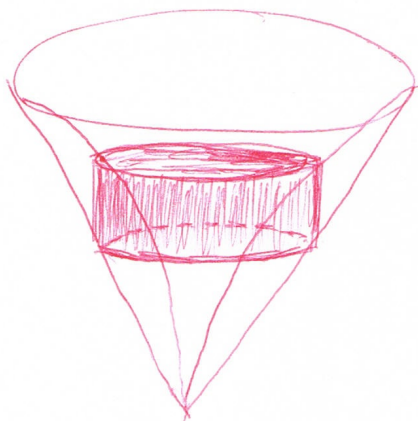
Area of slice at x is $\pi x^2 - \pi\left(\frac{x^2}{4}\right)^2$

$$\text{Volume} = \int_{\text{one side}}^{\text{other side}} (\text{area of slice at } x) \, dx = \int_0^4 \pi x^2 - \pi \frac{x^4}{16} \, dx$$

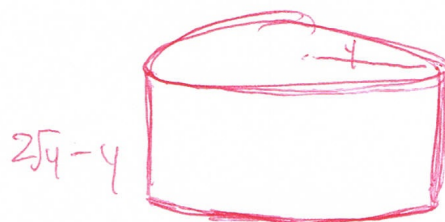
$$= \pi \int_0^4 x^2 - \frac{x^4}{16} \, dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{80} \right]_0^4 = \pi \left(\left(\frac{64}{3} - \frac{64}{5} \right) - (0 - 0) \right)$$

$$= \frac{128\pi}{15}$$

Alternative solution for Problem 6, Part 2:



Slice at y looks like:



Area of "shell" is $2\pi(\text{radius})(\text{height})$

Area of slice at radius y is $2\pi y (2\sqrt{y} - y)$

$$\text{Volume} = \int_{\text{inner radius}}^{\text{outer radius}} (\text{area of slice at radius } y) dy = \int_0^4 2\pi y (2\sqrt{y} - y) dy$$

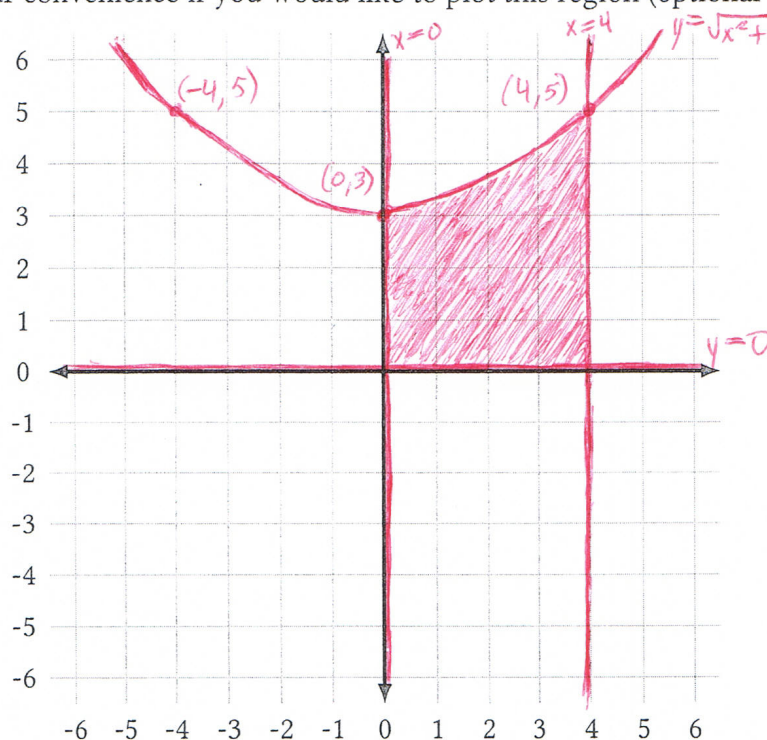
$$= 4\pi \int_0^4 y^{3/2} dy - 2\pi \int_0^4 y^2 dy = 4\pi \left[\frac{y^{5/2}}{5/2} \right]_0^4 - 2\pi \left[\frac{y^3}{3} \right]_0^4$$

$$= 4\pi \left(\frac{64}{5} - 0 \right) - 2\pi \left(\frac{64}{3} - 0 \right) = \frac{256\pi}{5} - \frac{128\pi}{3} = \frac{128\pi}{15}$$

Problem 7 (20 points)

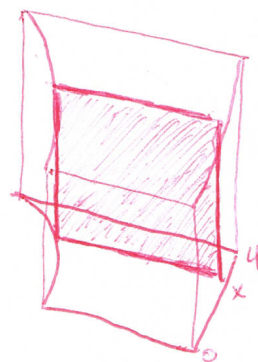
Consider the region R bounded by the lines $y = 0$, $x = 0$, and $x = 4$, and the curve $y = \sqrt{x^2 + 9}$.

Here is a grid for your convenience if you would like to plot this region (optional - no points).

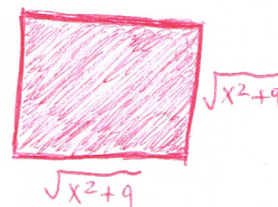


Part 1 (10 points)

If a shape has the region R as its base, and every cross-section of the shape which is perpendicular to the x -axis is a square, what is the volume of the shape?



Slice at x looks like



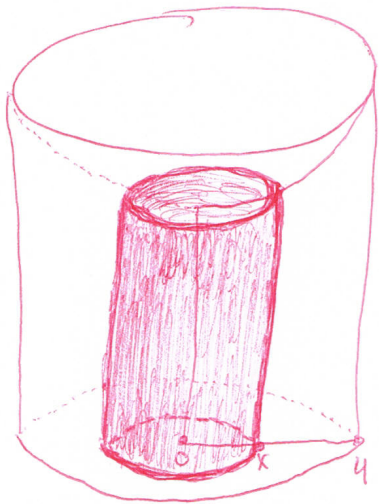
Area of slice at x is $(\sqrt{x^2+9})^2 = x^2 + 9$

$$\text{Volume} = \int_0^4 (\text{area of slice at } x) dx = \int_0^4 (x^2 + 9) dx$$

$$= \left[\frac{x^3}{3} + 9x \right]_0^4 = \left(\frac{64}{3} + 36 \right) - (0 + 0) = \frac{172}{3}$$

Part 2 (10 points)

If we rotate the region R around the y -axis, what is the volume of the resulting shape?



Area of "shell" is $2\pi(\text{radius})(\text{height})$

Area of slice at radius x is

$$2\pi x \sqrt{x^2 + 9}$$

$$\text{Volume} = \int_{\text{inner radius}}^{\text{outer radius}} (\text{area of slice at radius } x) dx = \int_0^4 2\pi x \sqrt{x^2 + 9} dx$$

do u -substitution with $u = x^2 + 9$

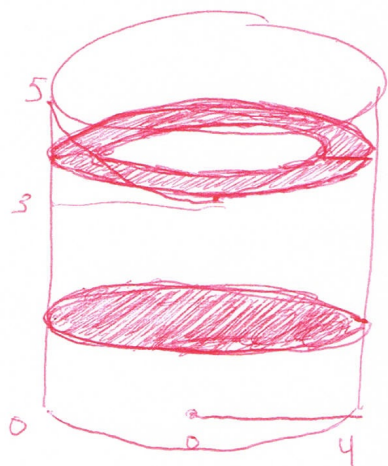
therefore $du = 2x dx$, $0 \rightsquigarrow 0^2 + 9 = 9$

$4 \rightsquigarrow 4^2 + 9 = 25$

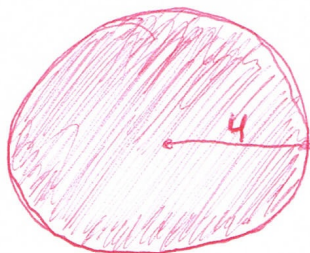
$$\int_0^4 2\pi x \sqrt{x^2 + 9} dx = \pi \int_0^4 \sqrt{x^2 + 9} \cdot (2x) dx = \pi \int_9^{25} \sqrt{u} du = \pi \left[\frac{u^{3/2}}{3/2} \right]_9^{25}$$

$$= \frac{2}{3} \pi (25^{3/2} - 9^{3/2}) = \frac{2}{3} \pi (125 - 27) = \frac{196\pi}{3}$$

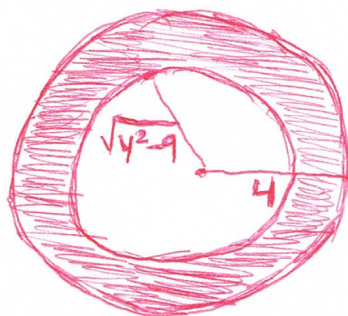
Alternative solution for Problem 7, Part 2



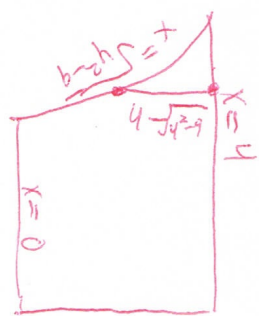
Slice at y (when $y \leq 3$) looks like



Slice at y (when $y > 3$) looks like



(The curve $y = \sqrt{x^2 + 9}$ becomes
 $x = \sqrt{y^2 - 9}$)



Area of slice at y (when $y \leq 3$) is 16π

Area of slice at y (when $y > 3$) is $16\pi - \pi(y^2 - 9)$

$$\text{Volume} = \int_0^5 (\text{area of slice at } y) dy = \int_0^5 \begin{pmatrix} 16\pi & \text{if } y \leq 3 \\ 16\pi - \pi(y^2 - 9) & \text{if } y > 3 \end{pmatrix} dy =$$

$$\int_0^3 16\pi dy + \int_3^5 (16\pi - \pi(y^2 - 9)) dy = 48\pi + \left[16\pi y - \frac{\pi y^3}{3} + 9\pi y \right]_3^5 = 98\pi - \left(\frac{125\pi}{3} - \frac{27\pi}{3} \right) = \frac{196\pi}{3}$$