

Math 13200 – Section 58 – Midterm 2

May 6, 2016

Information and Directions

- This exam will last 50 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Partial credit is given for showing your calculations and explaining your thoughts.

Name: _____ SOLUTIONS _____

Problem	Points	Score
1	16	
2	16	
3	16	
4	16	
5	20	
6	16	

Problem 1 (16 points)

In this problem, define the function $f(x) = \frac{1}{2}x^4 - \frac{2}{3}x^3 - 2x^2 + 2$.

Part 1 (8 points)

Find the critical points of f on the real line.

(Hint: The derivative of f exists everywhere, so there are no singular points, and we're considering the entire real line, so there are no end points. That leaves ...)

$$f'(x) = 2x^3 - 2x^2 - 4x$$

We're looking for stationary points, where $f'(x) = 0$

$$2x^3 - 2x^2 - 4x = 0$$

$$x^3 - x^2 - 2x = 0$$

$$x(x^2 - x - 2) = 0$$

$$(x-0)(x-2)(x+1) = 0$$

Thus the critical points
of f are $-1, 0, 2$

Part 2 (8 points)

For each point in your answer from Part 1, use the Second Derivative Test to determine whether it is a local maximum or a local minimum.

Second Derivative Test: if $f'(c) = 0$, then

$f''(c) > 0 \Rightarrow$ local minimum at c

$f''(c) < 0 \Rightarrow$ local Maximum at c

$f''(c) = 0 \Rightarrow$ No conclusion (could be ^{local}Min, ^{local}max, or neither)

$$f''(x) = 6x^2 - 4x - 4$$

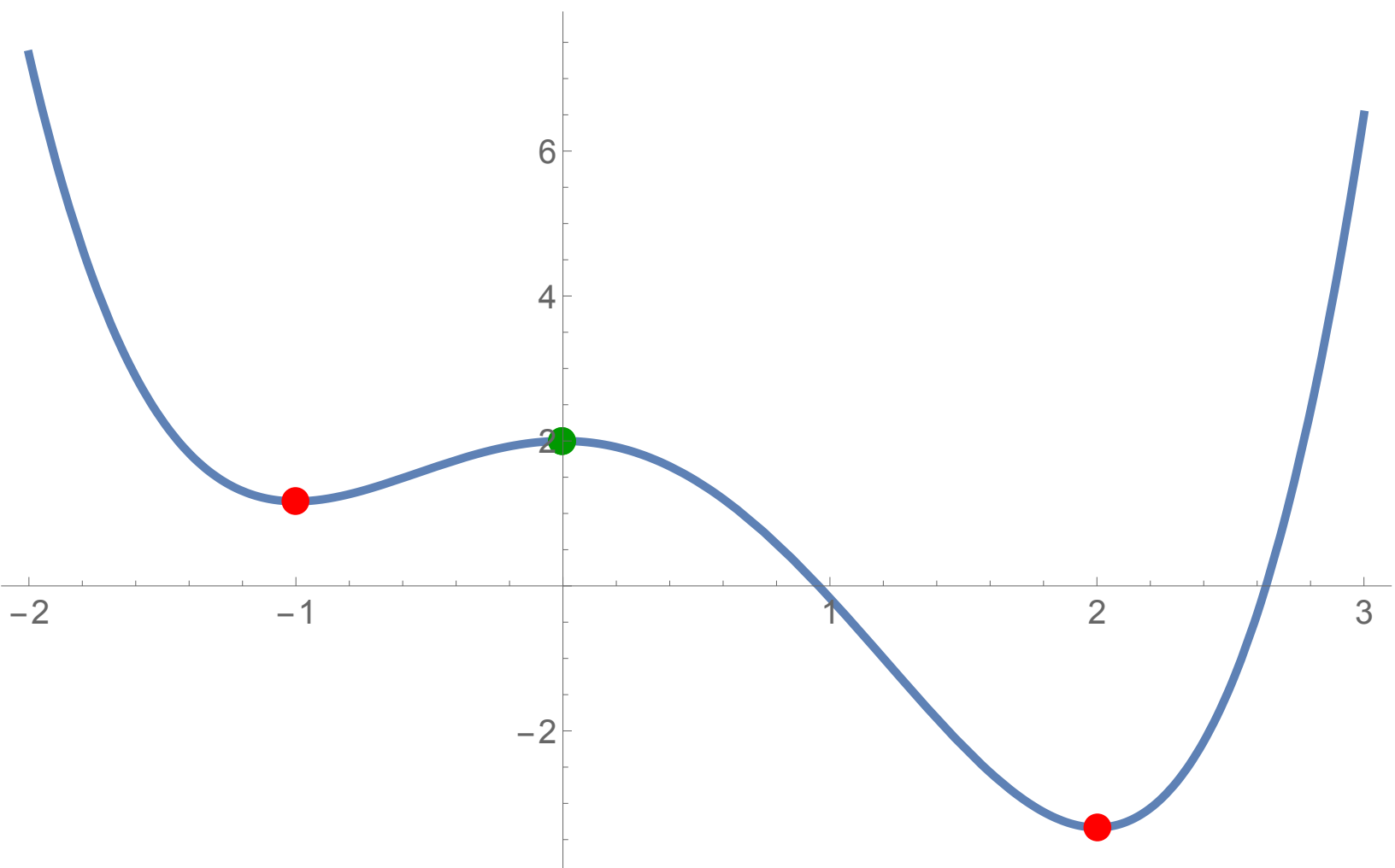
Therefore, -1 and 2 are local
Minima

$$f''(-1) = 6 + 4 - 4 = 6 > 0$$

$$f''(0) = 0 - 0 - 4 = -4 < 0$$

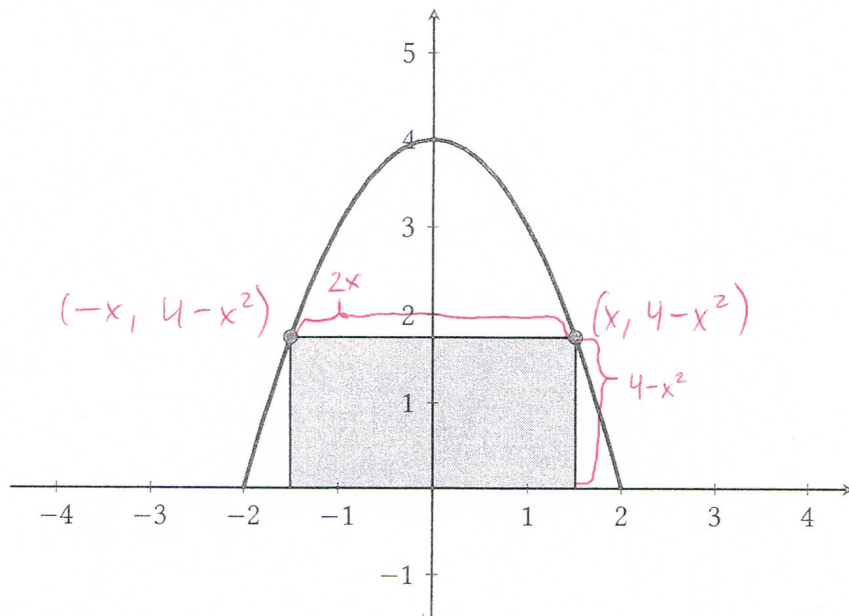
0 is a local maximum

$$f''(2) = 24 - 8 - 4 = 12 > 0$$



Problem 2 (16 points)

Suppose that I want to make a rectangle with the largest area I can, but there is a requirement that the rectangle has two corners on the x -axis, and the other two corners on the parabola $y = 4 - x^2$, where $y \geq 0$ (see the picture below).



What are the width and the height of the rectangle I should choose to get the maximum area?

The x -coordinate of the top-right corner is a number $0 \leq x \leq 2$

The height of the rectangle is $4 - x^2$

The width of the rectangle is $2x$

Thus the area is $(2x)(4 - x^2) = 8x - 2x^3$

We want to find the maximum of the function $A(x) = 8x - 2x^3$ on $[0, 2]$

Look at critical points. $A'(x) = 8 - 6x^2$

stationary points: $A'(x) = 8 - 6x^2 = 0$ when $x = \frac{2}{\sqrt{3}}$ or $x = -\frac{2}{\sqrt{3}}$, but only $\frac{2}{\sqrt{3}}$ is in $[0, 2]$

singular points: none, since $A'(x)$ exists everywhere

end points: 0 and 2

$$A(0) = 0 - 0 = 0$$

$$A(2) = 16 - 16 = 0$$

Since $A\left(\frac{2}{\sqrt{3}}\right) = \frac{32}{3\sqrt{3}}$ is biggest, that is where maximum occurs

$$\text{width} = 2\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$$

$$\text{height} = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{8}{3}$$

$$A\left(\frac{2}{\sqrt{3}}\right) = \frac{16}{\sqrt{3}} - \frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}}$$

Problem 3 (16 points)

Part 1 (8 points)

The Mean Value Theorem for Derivatives says that if f is continuous on $[a, b]$ and differentiable on (a, b) , then there is at least one number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(Complete the formula by writing the correct expression in the box.)

Part 2 (8 points)

For the function $f(x) = x^3 - 12x$ and the interval $[-3, 3]$, find one of the numbers c that the Mean Value Theorem guarantees must exist.

We have $a = -3$, $b = 3$

$$\frac{f(b) - f(a)}{b - a} = \frac{(27 - 36) - (-27 - (-36))}{3 - (-3)} = \frac{(-9) - 9}{6} = -3$$

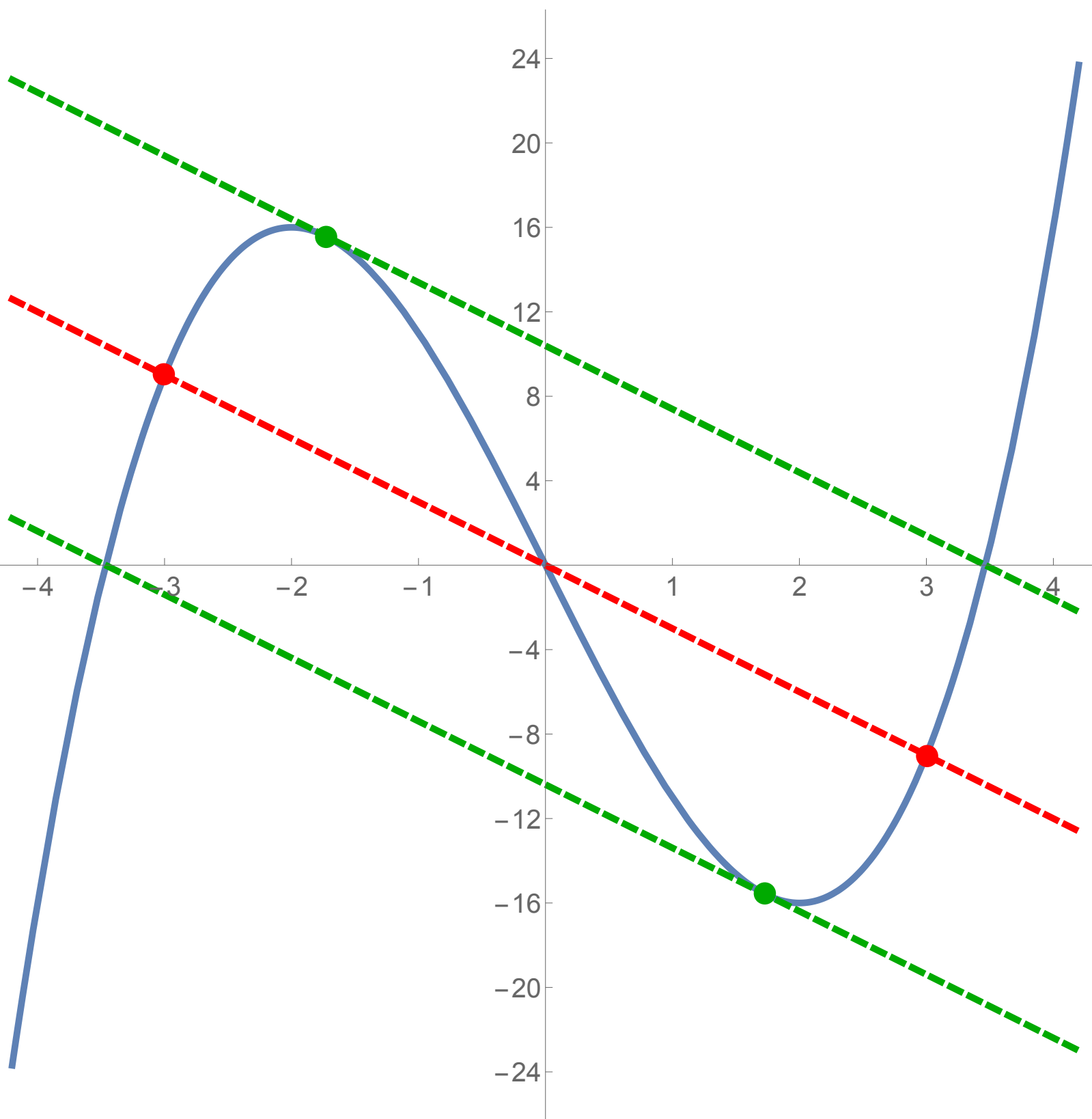
So we're looking for $c \in (-3, 3)$ with $f'(c) = -3$

$f'(x) = 3x^2 - 12$, so if $f'(c) = 3c^2 - 12 = -3$, then

$$3c^2 = 9$$

$$c^2 = 3$$

$$c = \sqrt{3} \text{ or } -\sqrt{3}$$



Problem 4 (16 points)

Part 1 (8 points)

Find the indefinite integral (i.e., the antiderivative)

$$\int 4x^3 + \sec^2(x) dx$$

$$\frac{d}{dx}(x^4 + \tan(x)) = \frac{d}{dx}(x^4) + \frac{d}{dx}(\tan(x)) = 4x^3 + \sec^2(x)$$

$$\text{Therefore } \int 4x^3 + \sec^2(x) dx = x^4 + \tan(x) + C$$

Part 2 (8 points)

Find the indefinite integral (i.e., the antiderivative)

$$\int 6 \sin^5(x) \cos(x) dx$$

$$\frac{d}{dx}(\sin^6(x)) = 6 \sin^5(x) \cdot \frac{d}{dx}(\sin(x)) = 6 \sin^5(x) \cos(x)$$

(Chain Rule)

$$\text{Therefore } \int 6 \sin^5(x) \cos(x) = \sin^6(x) + C$$

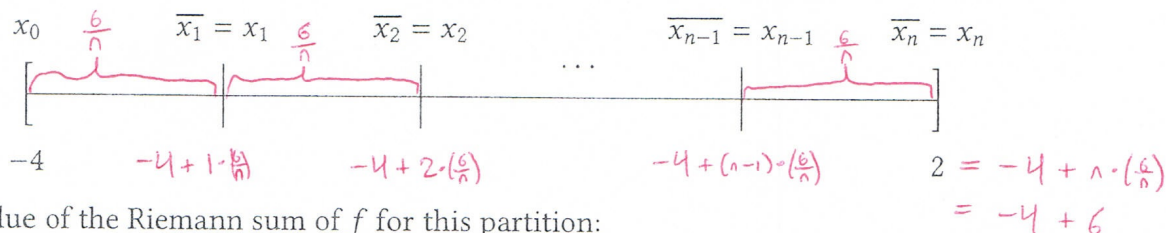
Problem 5 (20 points)

In this problem, define the function $f(x) = x^2 + x - 2$.

Part 1 (10 points)

Let n be some number.

Partition the interval $[-4, 2]$ into n pieces of equal length, and choose the sample points $\bar{x}_i = x_i$.



Find the value of the Riemann sum of f for this partition:

$$\sum_{i=1}^n f(\bar{x}_i) \Delta x_i$$

Your answer should be an explicit formula depending only on n (no occurrences of $\sum$, f , i , etc.)

Hints:

- To find Δx_i , first determine the length of the interval $[-4, 2]$. If the interval is divided into n equal pieces, what is the length of each piece?
- To find a formula for $\bar{x}_i$, note that $\bar{x}_i$ is ahead of the number -4 by $i \times$ the width of one piece.
- Here are some of the special sums-of-powers formulas, for your convenience:

$$\sum_{i=1}^n 1 = n \quad \sum_{i=1}^n i = \frac{1}{2}n^2 + \frac{1}{2}n \quad \sum_{i=1}^n i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Each $\Delta x_i = \frac{6}{n}$ (the length of the interval $[-4, 2]$ is 6, and it's broken into n equal pieces of length $\frac{6}{n}$)

Each $\bar{x}_i = -4 + i \cdot \left(\frac{6}{n}\right)$ (the number $\bar{x}_i$ is i piece-lengths more than the number -4)

$$\text{Thus } f(\bar{x}_i) = \bar{x}_i^2 + \bar{x}_i - 2$$

$$= \left(-4 + i\left(\frac{6}{n}\right)\right)^2 + \left(-4 + i\left(\frac{6}{n}\right)\right) - 2$$

$$= \left(16 - 8i\left(\frac{6}{n}\right) + i^2\left(\frac{36}{n^2}\right)\right) + \left(-4 + i\left(\frac{6}{n}\right)\right) - 2$$

$$= 10 - \left(\frac{42}{n}\right)i + \left(\frac{36}{n^2}\right)i^2$$

$$\begin{aligned}\text{Therefore } f(\bar{x}_i) \Delta x_i &= \left(10 - \left(\frac{42}{n} \right) i + \left(\frac{36}{n^2} \right) i^2 \right) \left(\frac{6}{n} \right) \\ &= \frac{60}{n} - \left(\frac{252}{n^2} \right) i + \left(\frac{216}{n^3} \right) i^2\end{aligned}$$

Therefore the Riemann sum for this partition is

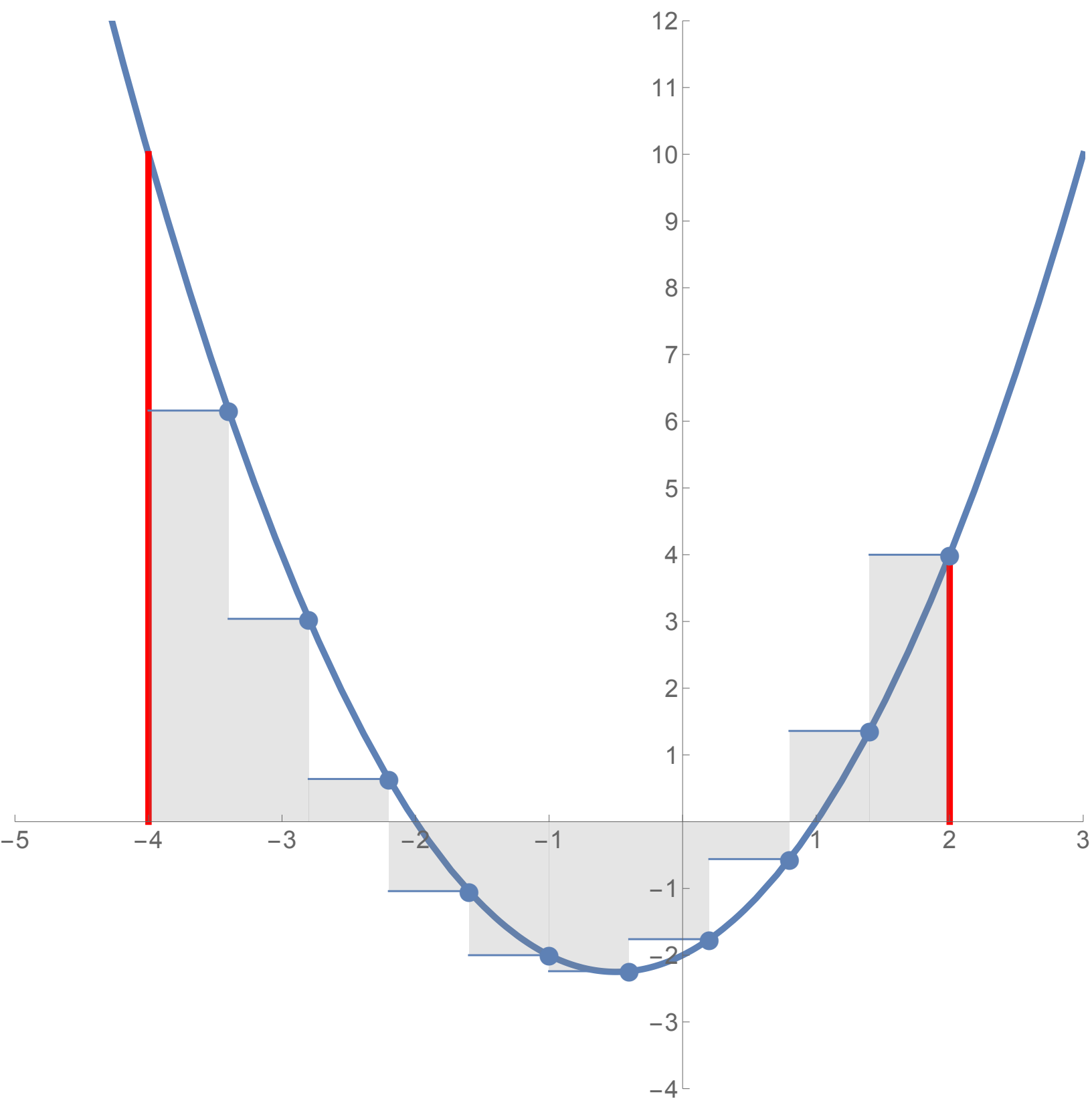
$$\begin{aligned}\sum_{i=1}^n f(\bar{x}_i) \Delta x_i &= \sum_{i=1}^n \left(\frac{60}{n} - \left(\frac{252}{n^2} \right) i + \left(\frac{216}{n^3} \right) i^2 \right) \\ &= \sum_{i=1}^n \left(\frac{60}{n} \right) - \sum_{i=1}^n \left(\frac{252}{n^2} \right) i + \sum_{i=1}^n \left(\frac{216}{n^3} \right) i^2\end{aligned}$$

$$= \left(\frac{60}{n} \right) \sum_{i=1}^n 1 - \left(\frac{252}{n^2} \right) \sum_{i=1}^n i + \left(\frac{216}{n^3} \right) \sum_{i=1}^n i^2$$

$$= \left(\frac{60}{n} \right) (n) - \left(\frac{252}{n^2} \right) \left(\frac{1}{2} n^2 + \frac{1}{2} n \right) + \left(\frac{216}{n^3} \right) \left(\frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n \right)$$

$$= 60 - \left(126 + \frac{126}{n} \right) + \left(72 + \frac{108}{n} + \frac{36}{n^2} \right)$$

$$= 6 - \frac{18}{n} + \frac{36}{n^2}$$



Part 2 (10 points)

Find the value of the definite integral

$$\int_{-4}^2 f(x) dx$$

by using your answer from Part 1 and the **definition** of the definite integral as a limit of Riemann sums whose pieces are getting smaller and smaller.

(You may not use the 2nd Fundamental Theorem of Calculus, other than to check your answer obtained from the definition.)

From Part 1 = the Riemann sum $\sum_{i=1}^n f(\bar{x}_i) \Delta x_i$ for the partition of $[-4, 2]$ into n equal pieces of length $\frac{6}{n}$ is

$$6 - \frac{18}{n} + \frac{36}{n^2}$$

$$\int_{-4}^2 f(x) dx = \text{limit of (Riemann sum of partition } P \text{ of } [-4, 2])$$

as (length of largest piece of P) $\rightarrow 0$

$$= \lim_{n \rightarrow \infty} \left(\text{Riemann sum of partition of } [-4, 2] \text{ into pieces of length } \frac{6}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left(6 - \frac{18}{n} + \frac{36}{n^2} \right)$$

$$= 6$$

$$\text{(Double-check: } \int_{-4}^2 x^2 + x - 2 dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x \right]_{-4}^2 = \left(\frac{8}{3} + 2 - 4 \right) - \left(-\frac{64}{3} + 8 + 8 \right) = 6)$$

Problem 6 (16 points)

For this problem, define the function

$$G(x) = \int_{\sqrt{\pi}}^x \cos(t^2) dt$$

Part 1 (8 points)

What is $G(\sqrt{\pi})$?

(Hint: Use basic properties of integrals.)

For any function f and any number a , $\int_a^a f(x) dx = 0$

$$\text{Thus } G(\sqrt{\pi}) = \int_{\sqrt{\pi}}^{\sqrt{\pi}} \cos(t^2) dt = 0$$

Part 2 (8 points)

What is $G'(\sqrt{\pi})$?

(Hint: Use the 1st Fundamental Theorem of Calculus.)

$$\text{1st FTC: } \frac{d}{dx} \left(\int_{\sqrt{\pi}}^x \cos(t^2) dt \right) = \cos(x^2)$$

$$\text{Therefore } G'(\sqrt{\pi}) = \cos((\sqrt{\pi})^2) = \cos(\pi) = -1$$