

Math 13200 – Section 58 – Final Exam

June 9, 2016

Information and Directions

- This exam will last 120 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- **Partial credit is given for showing your calculations and explaining your thoughts.**

Name: SOLUTIONS

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	

Problem 1 (10 points)

Part 1 (5 points)

What is the definition of $\ln(x)$? For which real numbers x is it defined?

$\ln(x)$ is defined for any real number $x > 0$

$$\ln(x) = \int_1^x \frac{1}{t} dt$$

Part 2 (5 points)

If $\ln(a) = 3$ and $\ln(b) = 5$, then what is $\ln(ab)$?

$$\ln(ab) = \ln(a) + \ln(b) = 3 + 5 = 8$$

Problem 2 (10 points)

Part 1 (5 points)

What is the equation of the tangent line to $y = \sin(x)$ at $x = \frac{\pi}{6}$?

• Point: goes through $(\frac{\pi}{6}, \sin(\frac{\pi}{6})) = (\frac{\pi}{6}, \frac{1}{2})$

• Slope: derivative is $\frac{d}{dx} \sin(x) = \cos(x)$, so slope of tangent line is $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

By point-slope formula, equation of tangent line is

$$(y - \frac{1}{2}) = (\frac{\sqrt{3}}{2})(x - \frac{\pi}{6})$$

$$y = (\frac{\sqrt{3}}{2})x + (\frac{1}{2} - \frac{\pi}{4\sqrt{3}})$$

Part 2 (5 points)

What is the equation of the tangent line to $y = \sin(\ln(x))$ at $x = 1$?

• Point: goes through $(1, \sin(\ln(1))) = (1, \sin(0)) = (1, 0)$

• Slope: derivative is $\frac{d}{dx} \sin(\ln(x)) = \cos(\ln(x)) \cdot \frac{d}{dx} \ln(x)$ (by chain rule)
 $= \cos(\ln(x)) \cdot \frac{1}{x}$

so slope of tangent line is $\cos(\ln(1)) \cdot \frac{1}{1} = \cos(0) \cdot 1 = 1$

By point-slope formula, equation of tangent line is

$$(y - 0) = 1 \cdot (x - 1)$$

$$y = x - 1$$

Problem 3 (10 points)

Part 1 (5 points)

Find the derivative $\frac{d}{dx} x \cos(x^2)$.

$$\begin{aligned}\frac{d}{dx} x \cos(x^2) &= \left(\frac{d}{dx} x \right) \cdot \cos(x^2) + x \cdot \left(\frac{d}{dx} \cos(x^2) \right) && \text{product rule} \\ &= (1) \cdot \cos(x^2) + x \cdot \left(\frac{d}{dx} \cos(x^2) \right) && \frac{d}{dx} x = 1 \\ &= \cos(x^2) + x \cdot (-\sin(x^2) \cdot \frac{d}{dx} x^2) && \text{chain rule} \\ &= \cos(x^2) + x \cdot (-\sin(x^2) \cdot (2x)) && \frac{d}{dx} x^2 = 2x \\ &= \cos(x^2) - 2x^2 \sin(x^2) && \text{simplifying}\end{aligned}$$

Part 2 (5 points)

Find the antiderivative $\int x \cos(x^2) dx$.

do u-substitution with $u = x^2$

therefore $du = 2x dx$

$$\begin{aligned}\int x \cos(x^2) dx &= \frac{1}{2} \int \cos(x^2) \cdot (2x) dx = \frac{1}{2} \int \cos(u) du \\ &= \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(x^2) + C\end{aligned}$$

Problem 4 (10 points)

Part 1 (5 points)

Find the derivative $\frac{d}{dx} e^{\sin(x)} \cos(x)$.

$$\begin{aligned}\frac{d}{dx} e^{\sin(x)} \cos(x) &= \left(\frac{d}{dx} e^{\sin(x)} \right) \cdot \cos(x) + e^{\sin(x)} \cdot \left(\frac{d}{dx} \cos(x) \right) && \text{product rule} \\ &= \left(e^{\sin(x)} \cdot \frac{d}{dx} \sin(x) \right) \cdot \cos(x) + e^{\sin(x)} \cdot \left(\frac{d}{dx} \cos(x) \right) && \begin{array}{l} \frac{d}{dx}(e^x) = e^x \\ \text{chain rule} \end{array} \\ &= \cos^2(x) e^{\sin(x)} - \sin(x) e^{\sin(x)} && \begin{array}{l} \frac{d}{dx} \sin(x) = \cos(x) \\ \frac{d}{dx} \cos(x) = -\sin(x) \end{array} \\ &= (\cos^2(x) - \sin(x)) e^{\sin(x)} && \text{simplifying/rewriting}\end{aligned}$$

Part 2 (5 points)

Find the antiderivative $\int e^{\sin(x)} \cos(x) dx$.

do u-substitution with $u = \sin(x)$

therefore $du = \cos(x) dx$

$$\int e^{\sin(x)} \cos(x) dx = \int e^u du = e^u + C = e^{\sin(x)} + C$$

Problem 5 (10 points)

For this problem, define the function

$$G(x) = \int_{17}^x e^{\cos(t)} dt$$

Part 1 (5 points)

What is $G'(x)$?

1st Fundamental Theorem of Calculus says $\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$

$$\text{Therefore } G'(x) = \frac{d}{dx} \left(\int_{17}^x e^{\cos(t)} dt \right) = e^{\cos(x)}$$

Part 2 (5 points)

Is G concave up, concave down, or neither at the point $x = \frac{\pi}{2}$?

2nd Derivative Test:

$$G''(x) > 0 \Rightarrow G \text{ concave up at } x$$

$$G''(x) < 0 \Rightarrow G \text{ concave down at } x$$

$$G''(x) = 0 \Rightarrow (\text{no conclusion, could be anything})$$

$$\begin{aligned} G''(x) &= \frac{d}{dx} G'(x) = \frac{d}{dx} e^{\cos(x)} = e^{\cos(x)} \cdot \frac{d}{dx} (\cos(x)) \quad (\text{by chain rule}) \\ &= -\sin(x) e^{\cos(x)} \end{aligned}$$

$$G''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) e^{\cos\left(\frac{\pi}{2}\right)} = -1 \cdot e^0 = -1 < 0, \text{ so } G \text{ is concave down at } \frac{\pi}{2}$$

Problem 6 (10 points)

Find the definite integral $\int_{\ln(3)}^{\ln(8)} e^x \sqrt{e^x + 1} dx$. For this problem, you must simplify fully.

do u-substitution with $u = e^x + 1$

therefore $du = e^x dx$

$$\ln(8) \rightsquigarrow e^{\ln(8)} + 1 = 8 + 1 = 9$$

$$\ln(3) \rightsquigarrow e^{\ln(3)} + 1 = 3 + 1 = 4$$

$$\int_{\ln(3)}^{\ln(8)} e^x \sqrt{e^x + 1} dx = \int_{\ln(3)}^{\ln(8)} \sqrt{e^x + 1} \cdot e^x dx = \int_4^9 \sqrt{u} du$$

$$= \left[\frac{u^{3/2}}{3/2} \right]_4^9 = \left(\frac{2}{3} \cdot 9^{3/2} - \frac{2}{3} \cdot 4^{3/2} \right)$$

$$= \left(\frac{2}{3} \cdot 27 - \frac{2}{3} \cdot 8 \right)$$

$$= \left(\frac{54}{3} - \frac{16}{3} \right)$$

$$= \frac{38}{3}$$

Problem 7 (10 points)

Consider the region bounded by the curves $y = 2 \sin(x)$ and $y = 2 \cos(x)$, between their intersection at $x = \frac{\pi}{4}$ and their intersection at $x = \frac{5\pi}{4}$.

Part 1 (5 points)

Plot the region on this grid. For each of the two intersections described above, label the point with its x - and y -coordinates.



Part 2 (5 points)

Find the area of the region.

$$\text{Area} = \int_{\text{one side}}^{\text{otherside}} (\text{length of slice at } x) \, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (2 \sin(x) - 2 \cos(x)) \, dx$$

$$= \left[-2 \cos(x) - 2 \sin(x) \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

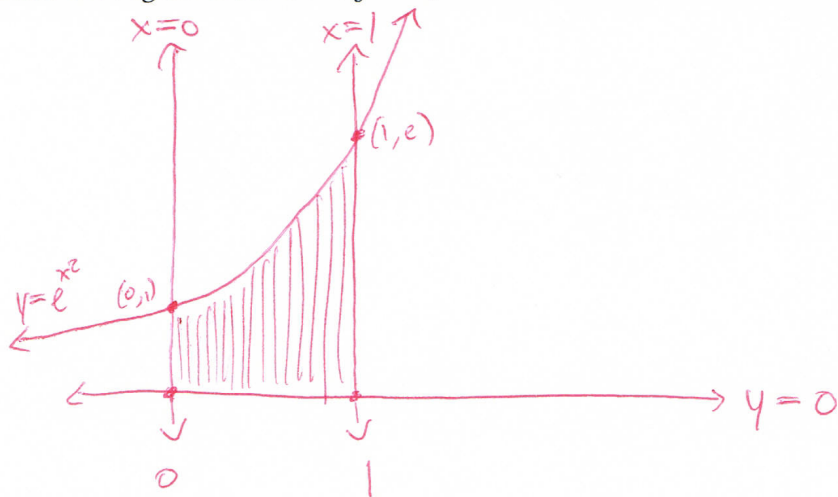
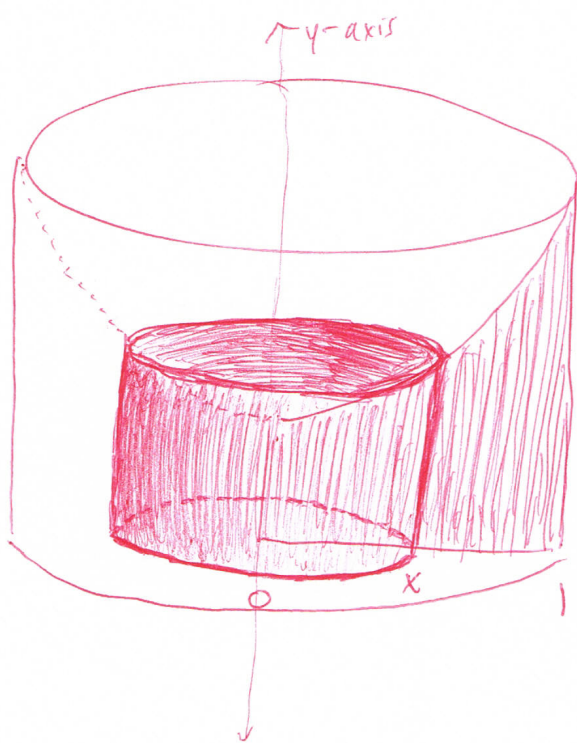
$$= \left(-2 \cos\left(\frac{5\pi}{4}\right) - 2 \sin\left(\frac{5\pi}{4}\right) \right) - \left(-2 \cos\left(\frac{\pi}{4}\right) - 2 \sin\left(\frac{\pi}{4}\right) \right)$$

$$= (\sqrt{2} + \sqrt{2}) - (-\sqrt{2} - \sqrt{2}) = 4\sqrt{2}$$

Problem 8 (10 points)

Consider the region bounded by the lines $y = 0$, $x = 0$, and $x = 1$, and the curve $y = e^{x^2}$.

If we rotate the region around the y -axis, what is the volume of the resulting shape?



Shell at radius x has height e^{x^2}

Area of shell at radius x is

$$2\pi r h$$

$$= 2\pi x e^{x^2}$$

$$\text{Volume} = \int_{\text{inner radius}}^{\text{outer radius}} (\text{area of shell at radius } x) dx = \int_0^1 2\pi x e^{x^2} dx = \pi \int_0^1 e^{x^2} 2x dx$$

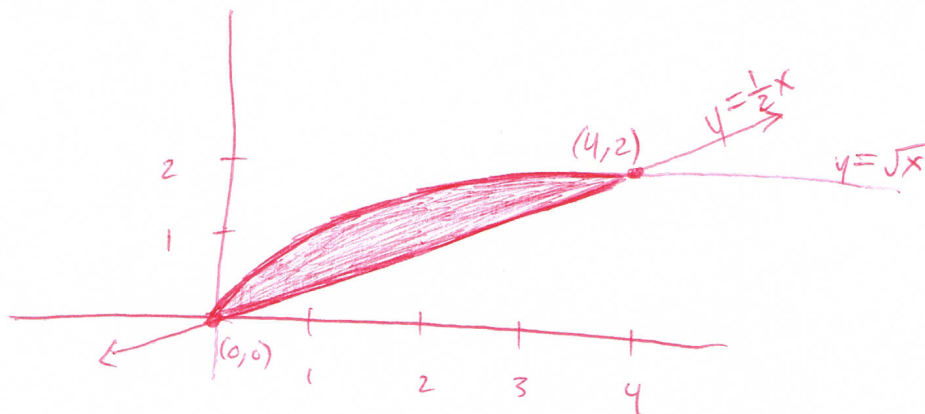
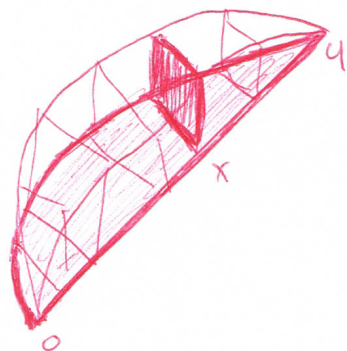
(do u -substitution with $u = x^2$, so $du = 2x dx$)
 $0 \rightsquigarrow 0^2, 1 \rightsquigarrow 1^2$

$$= \pi \int_0^1 e^u du = \pi [e^u]_0^1 = \pi (e^1 - e^0) = \pi (e - 1)$$

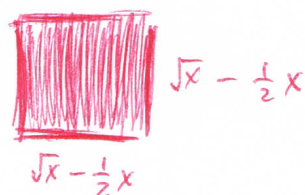
Problem 9 (10 points)

Consider the region bounded by the line $y = \frac{1}{2}x$ and the curve $y = \sqrt{x}$.

If a shape has this region as its base, and every cross-section of the shape which is perpendicular to the x -axis is a square, what is the volume of the shape?



Slice at x looks like



Area of slice at x is $(\sqrt{x} - \frac{1}{2}x)^2$

$$= x - x^{3/2} + \frac{1}{4}x^2$$

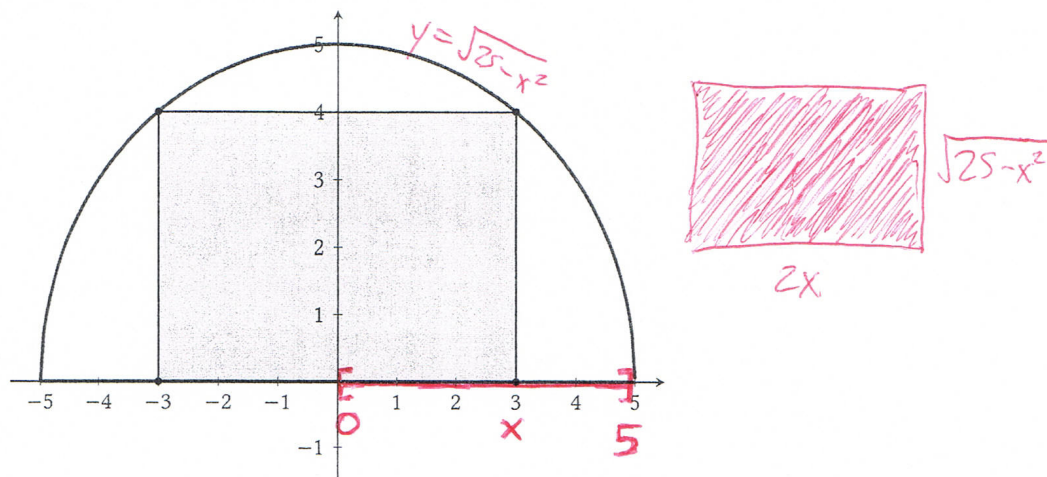
Volume = $\int_0^4 (\text{area of slice at } x) dx = \int_0^4 x - x^{3/2} + \frac{1}{4}x^2 dx$

other side
one side

$$= \left[\frac{1}{2}x^2 - \frac{x^{5/2}}{5/2} + \frac{1}{12}x^3 \right]_0^4 = \left(8 - \frac{64}{5} + \frac{16}{3} \right) - (0) = \frac{8}{15}$$

Problem 10 (10 points)

Suppose that I want to make a rectangle with the largest area I can, but it is required that the rectangle sits inside the semicircle $y = \sqrt{25 - x^2}$ (see the picture below).



What are the width and the height of the rectangle I should choose to get the maximum area?

$$\text{Area of rectangle} = (\text{width}) \cdot (\text{height}) = 2x\sqrt{25-x^2}$$

We want to find the Maximum of $A(x) = 2x\sqrt{25-x^2}$ on the interval $[0, 5]$

$$\begin{aligned} A'(x) &= \frac{d}{dx} 2x\sqrt{25-x^2} = \left(\frac{d}{dx} 2x\right)\sqrt{25-x^2} + 2x\left(\frac{d}{dx} \sqrt{25-x^2}\right) \\ &= 2\sqrt{25-x^2} + 2x\left(\frac{1}{2\sqrt{25-x^2}} \cdot \frac{d}{dx}(25-x^2)\right) \\ &= 2\sqrt{25-x^2} + 2x\left(\frac{-x}{\sqrt{25-x^2}}\right) = \frac{50-4x^2}{\sqrt{25-x^2}} \end{aligned}$$

Look at critical points of $A(x)$ on $[0, 5]$:

Stationary points: $A'(x) = 0 \Leftrightarrow 50 - 4x^2 = 0 \Leftrightarrow x = \pm \frac{5}{\sqrt{2}}$, but only one in $[0, 5]$ is $\frac{5}{\sqrt{2}}$

Singular points: $A'(x)$ doesn't exist $\Leftrightarrow 25 - x^2 = 0 \Leftrightarrow x = \pm 5$, but only one in $[0, 5]$ is 5

End points: 0 and 5

Maximum area occurs when $x = \frac{5}{\sqrt{2}}$, so

$$\begin{aligned} A(0) &= 0 \\ A\left(\frac{5}{\sqrt{2}}\right) &= 25 \\ A(5) &= 0 \end{aligned}$$

$$\text{width} = 2\left(\frac{5}{\sqrt{2}}\right) = 5\sqrt{2} \quad \text{height} = \sqrt{25 - \left(\frac{5}{\sqrt{2}}\right)^2} = \frac{5}{\sqrt{2}}$$