Math 13200 – Section 58 – Final Exam *Sample*

June 4, 2016

Information and Directions

- This exam will last 120 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- **Partial credit is given for showing your calculations and explaining your thoughts.**

Name: SOLUTIONS

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	

Problem 1 (10 points)

Part 1 (5 points)

What is the definition of the number e?

e is the unique real number with the property that $\ln(e) = 1$.

Part 2 (5 points)

What is $e^{\ln(2)+\ln(7)}$?

Because $e^{a+b} = e^a \cdot e^b$, and $e^{\ln(x)} = x$,

we have $e^{\ln(2)+\ln(7)} = e^{\ln(2)} \cdot e^{\ln(7)} = 2 \cdot 7 = 14$

Alternatively, since $\ln(ab) = \ln(a) + \ln(b)$,

we have $e^{\ln(2)+\ln(7)} = e^{\ln(14)} = 14$.

Problem 2 (10 points)

Part 1 (5 points)

What is the equation of the tangent line to $y = \cos(x)$ at $x = \frac{\pi}{4}$?

- Point: goes through $\left(\frac{\pi}{4}, \cos\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

- Slope: derivative is $\frac{d}{dx}\cos(x) = -\sin(x)$, so slope is $-\sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

By the point-slope formula, the equation of the tangent line is

$$\left(y - \frac{1}{\sqrt{2}}\right) = \left(-\frac{1}{\sqrt{2}}\right)\left(x - \frac{\pi}{4}\right)$$

$$y = \left(-\frac{1}{\sqrt{2}}\right)x + \left(\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

Part 2 (5 points)

What is the equation of the tangent line to $y = \ln(\sin(x))$ at $x = \frac{\pi}{2}$?

- Point: goes through $\left(\frac{\pi}{2}, \ln\left(\sin\left(\frac{\pi}{2}\right)\right)\right) = \left(\frac{\pi}{2}, \ln(1)\right) = \left(\frac{\pi}{2}, 0\right)$

- Slope: derivative is $\frac{d}{dx}\ln(\sin(x)) \overset{\text{chain rule}}{=} \frac{1}{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) = \frac{\cos(x)}{\sin(x)} = \cot(x)$

So slope is $\cot\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{0}{1} = 0$

By the point-slope formula, the equation of the tangent line is

$$(y - 0) = 0\left(x - \frac{\pi}{2}\right)$$

$$y = 0$$

Problem 3 (10 points)

Part 1 (5 points)

Find the derivative $\frac{d}{dx}\ \sin^3(x)\cos(x)$.

product rule

$$\frac{d}{dx}\left(\sin^3(x)\cos(x)\right) = \frac{d}{dx}\left(\sin^3(x)\right)\cos(x) + \sin^3(x)\,\frac{d}{dx}\left(\cos(x)\right)$$

Chain rule

$$= \left(3\sin^2(x)\cdot\frac{d}{dx}\left(\sin(x)\right)\right)\cos(x) + \sin^3(x)\,\frac{d}{dx}\left(\cos(x)\right)$$

$\frac{d}{dx}(\sin(x)) = \cos(x)$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

$$= \left(3\sin^2(x)\cos(x)\right)\cos(x) + \sin^3(x)\left(-\sin(x)\right)$$

Simplifying

$$= 3\sin^2(x)\cos^2(x) - \sin^4(x)$$

Part 2 (5 points)

Find the antiderivative $\int \sin^3(x)\cos(x)\,dx$.

u-substitution with $u = \sin(x)$

therefore, $du = \cos(x)\,dx$

$$\int \sin^3(x)\cos(x)\,dx = \int (\sin(x))^3\cos(x)\,dx = \int u^3\,du = \frac{u^4}{4} + C$$

$$= \frac{\sin^4(x)}{4} + C$$

Problem 4 (10 points)

Part 1 (5 points)

Find the derivative $\frac{d}{dx}\, e^x \sin(e^x)$.

product rule

$$\frac{d}{dx}\left(e^x \sin(e^x)\right) = \frac{d}{dx}(e^x)\sin(e^x) + e^x \frac{d}{dx}\left(\sin(e^x)\right)$$

chain rule

$$= \frac{d}{dx}(e^x)\sin(e^x) + e^x\left(\cos(e^x)\cdot \frac{d}{dx}(e^x)\right)$$

$\frac{d}{dx}(e^x) = e^x$

$$= e^x \sin(e^x) + e^x\left(\cos(e^x)\, e^x\right)$$

simplifying

$$= e^x \sin(e^x) + e^{2x}\cos(e^x)$$

Part 2 (5 points)

Find the antiderivative $\int e^x \sin(e^x)\, dx$.

u-substitution with $u = e^x$

therefore, $du = e^x dx$

$$\int e^x \sin(e^x)\, dx = \int \sin(e^x)\, e^x dx = \int \sin(u)\, du = -\cos(u) + C$$

$$= -\cos(e^x) + C$$

Problem 5 (10 points)

For this problem, define the function

$$G(x) = \int_{13}^{x} \ln(t^2 + 1)\, dt$$

Part 1 (5 points)

What is $G'(x)$?

1st FTC says that $\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x)$

Therefore $G'(x) = \ln(x^2+1)$

Part 2 (5 points)

Is G concave up, concave down, or neither at the point $x = 2$?

2nd Derivative Test:

If $G''(x) > 0$, then G is concave up at x

If $G''(x) < 0$, then G is concave down at x

(No conclusion if $G''(x) = 0$)

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, so $G''(x) = \frac{d}{dx} G'(x) = \frac{d}{dx} \ln(x^2+1)$

Chain rule $= \frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}$

$G''(2) = \frac{4}{5} > 0$, so G is concave up at 2

Problem 6 (10 points)

Find the definite integral $\int_1^{e^3} \frac{\ln(x)}{x}\,dx$. For this problem, you must simplify fully.

u – substitution with $u = \ln(x)$

therefore, $du = \frac{1}{x}\,dx$

$1 \rightsquigarrow \ln(1) = 0$

$e^3 \rightsquigarrow \ln(e^3) = 3 \cdot \ln(e) = 3 \cdot 1 = 3$

$$\int_1^{e^3} \frac{\ln(x)}{x}\,dx = \int_1^{e^3} \ln(x) \cdot \frac{1}{x}\,dx = \int_0^3 u\,du = \left[\frac{u^2}{2}\right]_0^3$$

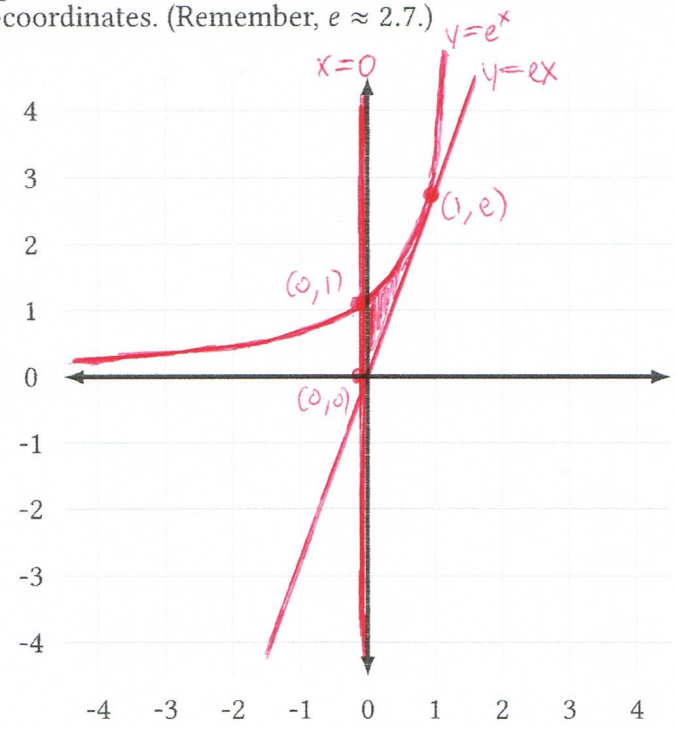
$$= \left(\frac{9}{2} - 0\right) = \frac{9}{2}$$

Problem 7 (10 points)

Consider the region bounded by the lines $x = 0$ and $y = ex$, and the curve $y = e^x$.

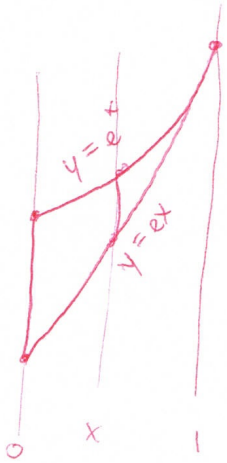
Part 1 (5 points)

Plot the region on this grid. For each intersection between the boundaries described above, label the point with its x- and y-coordinates. (Remember, $e \approx 2.7$.)



Part 2 (5 points)

Find the area of the region.



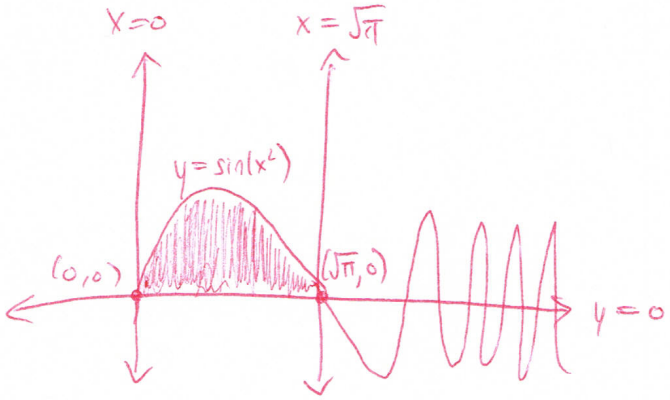
$$\text{Area} = \int_{\text{one side}}^{\text{other side}} (\text{length of slice at } x)\, dx$$

$$= \int_0^1 (e^x - ex)\, dx = \left[e^x - e\frac{x^2}{2}\right]_0^1$$

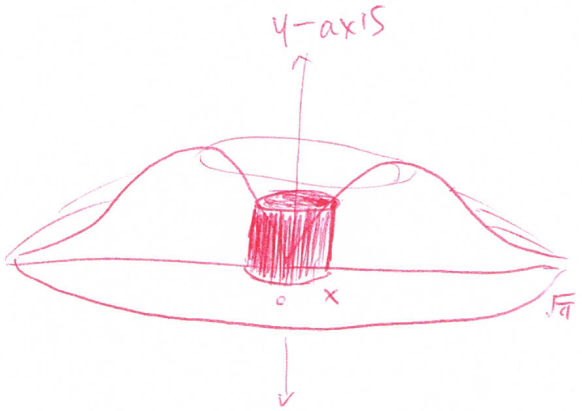
$$= \left(e - \frac{e}{2}\right) - (1 - 0) = \frac{e}{2} - 1$$

Problem 8 (10 points)

Consider the region bounded by the lines $y = 0$, $x = 0$, and $x = \sqrt{\pi}$, and the curve $y = \sin(x^2)$. If we rotate the region around the y-axis, what is the volume of the resulting shape?



($\sin(x^2)$ between 0 and $\sqrt{\pi}$ looks approximately like $\sin(x)$ does between 0 and π)



Shell at radius x has height $\sin(x^2)$

area of shell at radius x $= 2\pi rh = 2\pi x \sin(x^2)$

$$\text{Volume} = \int_{\text{inner radius}}^{\text{outer radius}} (\text{area of shell at radius } x)\, dx = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx$$

$$\int_0^{\sqrt{\pi}} 2\pi \times \sin(x^2)\, dx = \pi \int_0^{\sqrt{\pi}} \sin(x^2) \cdot 2x\, dx$$

u - substitution with $u = x^2$

therefore, $du = 2x\, dx$

$0 \rightsquigarrow 0^2 = 0$

$\sqrt{\pi} \rightsquigarrow (\sqrt{\pi})^2 = \pi$

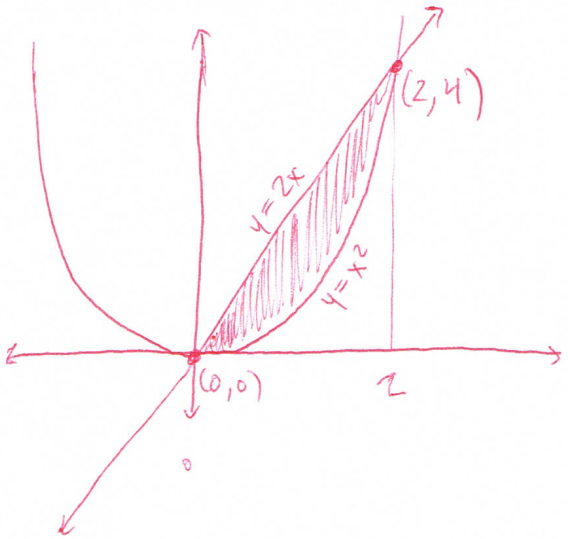
$$= \pi \int_0^{\pi} \sin(u)\, du = \pi \left[-\cos(u) \right]_0^{\pi}$$

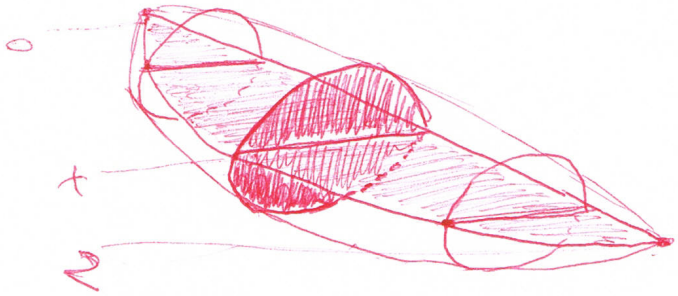
$$= \pi \left(-(-1) - (-1) \right) = 2\pi$$

Problem 9 (10 points)

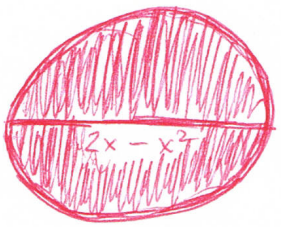
Consider the region bounded by the line $y = 2x$ and the curve $y = x^2$.

If a shape has this region as its base, and every cross-section of the shape which is perpendicular to the x-axis is a circle, what is the volume of the shape?





Slice at x looks like



Circle of diameter d has area $\pi r^2 = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$

$$\text{Volume} = \int_{\text{one side}}^{\text{other side}} (\text{area of slice at } x)\,dx = \int_0^2 \frac{\pi}{4}(2x - x^2)^2\,dx =$$

$$\frac{\pi}{4}\int_0^2 4x^2 - 4x^3 + x^4\,dx$$

$$\frac{\pi}{4}\int_0^2 4x^2 - 4x^3 + x^4\,dx = \frac{\pi}{4}\left[\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right]_0^2$$

$$= \frac{\pi}{4}\left(\left(\frac{32}{3} - 16 + \frac{32}{5}\right) - (0)\right)$$

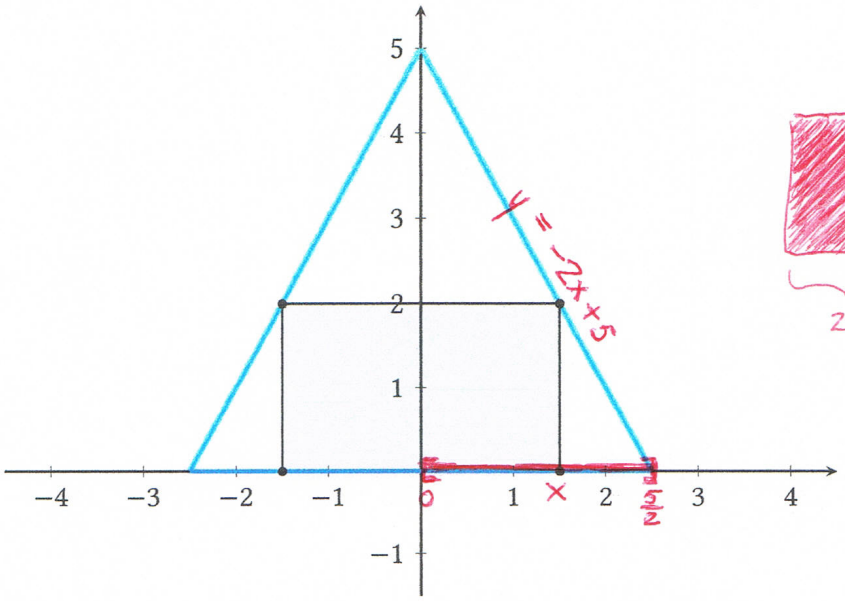
$$= \frac{\pi}{4}\left(\frac{160}{15} - \frac{240}{15} + \frac{96}{15}\right)$$

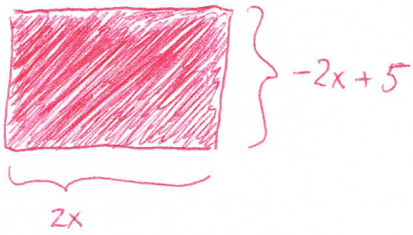
$$= \frac{\pi}{4}\left(\frac{16}{15}\right)$$

$$= \frac{4\pi}{15}$$

Problem 10 (10 points)

Suppose that I want to make a rectangle with the largest area I can, but it is required that the rectangle sits inside the triangle formed by $y = 0$, $y = -2x + 5$, and $y = 2x + 5$ (see the picture below).





What are the width and the height of the rectangle I should choose to get the maximum area?

Area of the rectangle is (base) · (height) = $(2x)(-2x+5)$

$= -4x^2 + 10x$

So we are looking for the maximum of the function $A(x) = -4x^2 + 10x$ on the interval $[0, \frac{5}{2}]$

Look at critical points:

Stationary points: $A'(x) = -8x + 10$, so $A'(x) = 0$ when $x = \frac{5}{4}$

Singular points: none, $A'(x)$ always exists

end points: $0, \frac{5}{2}$

$A(0) = 0$

$A(\frac{5}{4}) = \frac{25}{4}$

$A(\frac{5}{2}) = 0$

Since $A(\frac{5}{4}) = \frac{25}{4}$ is biggest, maximum area occurs when $x = \frac{5}{4}$

width $= 2 \cdot \frac{5}{4} = \frac{5}{2}$ height $= -2(\frac{5}{4}) + 5 = \frac{5}{2}$