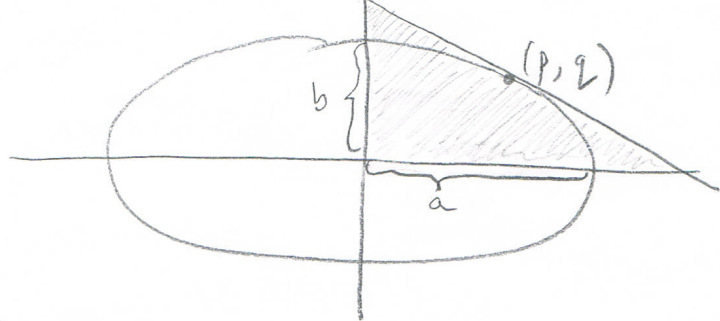




equation of ellipse:

$$b^2 x^2 + a^2 y^2 = a^2 b^2$$

(equivalently)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(Note that a and b are positive)

If (p, q) is a point on the ellipse in the first quadrant, then because

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1, \text{ we have}$$

$$\frac{q^2}{b^2} = 1 - \frac{p^2}{a^2}$$

$$q^2 = b^2 \left(1 - \frac{p^2}{a^2}\right)$$

$$q^2 = \frac{b^2}{a^2} (a^2 - p^2)$$

$$q = \frac{b}{a} \sqrt{a^2 - p^2}$$

Thus, points on the ellipse in the first quadrant are $(p, \frac{b}{a} \sqrt{a^2 - p^2})$ for p ranging between 0 and a .

Our goal is to make the tangent line create a triangle of least area with the x -axis and y -axis.

Slope of tangent line to ellipse at $(p, \frac{b}{a}\sqrt{a^2-p^2})$:

(implicit differentiation)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \implies \frac{d}{dx} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = \frac{d}{dx} (1)$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{x}{a^2}}{\frac{y}{b^2}}$$

$$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$$

Thus the slope of the tangent line to the ellipse at the point is

$$\frac{-b^2 p}{a^2 \cdot \frac{b}{a}\sqrt{a^2-p^2}} = \frac{-bp}{a\sqrt{a^2-p^2}}$$

The tangent line goes through $(p, \frac{b}{a}\sqrt{a^2-p^2})$ so the equation is

$$(y - \frac{b}{a}\sqrt{a^2-p^2}) = \left(\frac{-bp}{a\sqrt{a^2-p^2}} \right) (x - p)$$

This comes to

$$y = \left(-\frac{bp}{a\sqrt{a^2-p^2}} \right) x + \frac{bp^2}{a\sqrt{a^2-p^2}} + \frac{b}{a}\sqrt{a^2-p^2}$$

$$y = \frac{b}{a} \left(\frac{-p}{\sqrt{a^2-p^2}} \right) x + \frac{b}{a} \left(\frac{p^2}{\sqrt{a^2-p^2}} + \sqrt{a^2-p^2} \right)$$

$$y = \frac{b}{a} \left(\frac{-p}{\sqrt{a^2-p^2}} \right) x + \frac{b}{a} \left(\frac{p^2 + (a^2-p^2)}{\sqrt{a^2-p^2}} \right)$$

$$y = \frac{b}{a} \left(\frac{-p}{\sqrt{a^2-p^2}} \right) x + \frac{b}{a} \left(\frac{a^2}{\sqrt{a^2-p^2}} \right)$$

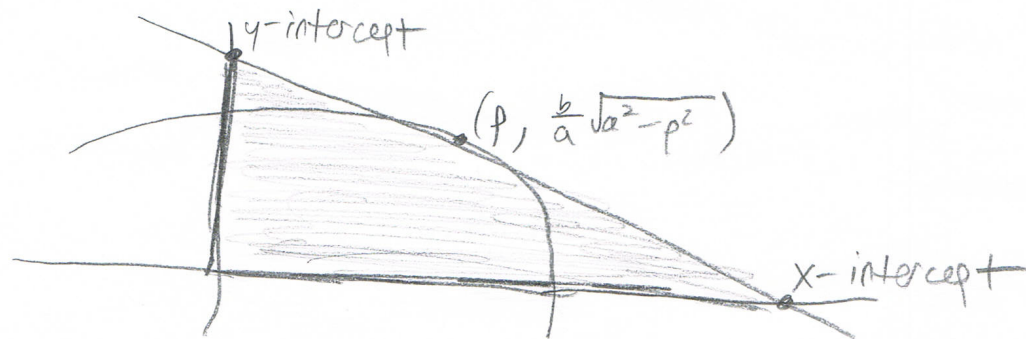
y-intercept is value of y when $x=0$:

$$y\text{-intercept of tangent line} = \frac{b}{a} \left(\frac{a^2}{\sqrt{a^2-p^2}} \right)$$

x-intercept is value of x when $y=0$:

$$x\text{-intercept of tangent line} = \frac{-\frac{b}{a} \left(\frac{a^2}{\sqrt{a^2-p^2}} \right)}{\frac{b}{a} \left(\frac{-p}{\sqrt{a^2-p^2}} \right)} = \frac{a^2}{p}$$

Thus area of triangle formed by tangent line is



$$\begin{aligned} \frac{1}{2} \text{ base} \cdot \text{height} &= \frac{1}{2} \cdot \left(\frac{a^2}{p} \right) \cdot \left(\frac{b}{a} \cdot \frac{a^2}{\sqrt{a^2 - p^2}} \right) \\ &= \underbrace{\frac{a^3 b}{2}}_{\text{constant}} \cdot \frac{1}{p \sqrt{a^2 - p^2}} \end{aligned}$$

This is a function of p :

$$A(p) = \frac{a^3 b}{2} \cdot \frac{1}{p \sqrt{a^2 - p^2}}$$

We want to find value of p that makes area as small as possible:

We know we need to look for values of p that have $A'(p) = 0$

$$A'(p) = \frac{a^3 b}{2} \cdot \left(\frac{p\sqrt{a^2-p^2} \cdot \frac{d}{dp}(1) - 1 \cdot \frac{d}{dp}(p\sqrt{a^2-p^2})}{(p\sqrt{a^2-p^2})^2} \right) \quad \text{Quotient Rule}$$

$$= \frac{a^3 b}{2} \cdot \left(\frac{0 - \frac{d}{dp}(p\sqrt{a^2-p^2})}{p^2(a^2-p^2)} \right)$$

$$= -\frac{a^3 b}{2} \cdot \frac{\frac{d}{dp}(p\sqrt{a^2-p^2})}{p^2(a^2-p^2)}$$

Product Rule

$$\frac{d}{dp}(p\sqrt{a^2-p^2}) = \sqrt{a^2-p^2} + p \cdot \frac{d}{dp}(\sqrt{a^2-p^2})$$

$$= \sqrt{a^2-p^2} + p \cdot \overset{\text{chain rule}}{\frac{1}{2\sqrt{a^2-p^2}}} \cdot \frac{d}{dp}(a^2-p^2)$$

$$= \sqrt{a^2-p^2} + p \cdot \frac{1}{2\sqrt{a^2-p^2}} \cdot (-2p)$$

$$= \sqrt{a^2-p^2} - \frac{p^2}{\sqrt{a^2-p^2}}$$

$$= \frac{(a^2-p^2)}{\sqrt{a^2-p^2}} - \frac{p^2}{\sqrt{a^2-p^2}}$$

$$= \frac{a^2-2p^2}{\sqrt{a^2-p^2}}$$

Thus $A'(p)$ is

$$A'(p) = -\frac{a^3 b}{2} \cdot \frac{\left(\frac{a^2 - 2p^2}{\sqrt{a^2 - p^2}} \right)}{p^2 (a^2 - p^2)}$$

$$= -\frac{a^3 b}{2} \cdot \frac{(a^2 - 2p^2)}{p^2 (a^2 - p^2)^{3/2}}$$

(Remember, $0 \leq p \leq a$, and a and b are positive numbers)

Thus the only way we can have $A'(p) = 0$ is if

the numerator $a^2 - 2p^2 = 0$, i.e., if $p = \frac{1}{\sqrt{2}} a$.

The equation of the tangent line to the ellipse at $(p, \frac{b}{a}\sqrt{a^2-p^2})$

when $p = \frac{a}{\sqrt{2}}$ is $\left(\frac{1}{\sqrt{2}}a, \frac{1}{\sqrt{2}}b\right)$

$$y = \frac{b}{a} \left(\frac{-\left(\frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - \frac{a^2}{2}}} \right) x + \frac{b}{a} \left(\frac{a^2}{\sqrt{a^2 - \frac{a^2}{2}}} \right)$$

$$y = \frac{b}{a} \left(\frac{-\left(\frac{a}{\sqrt{2}}\right)}{\left(\frac{a}{\sqrt{2}}\right)} \right) x + \frac{b}{a} \left(\frac{a^2}{\left(\frac{a}{\sqrt{2}}\right)} \right)$$

$$y = \left(-\frac{b}{a}\right)x + b\sqrt{2}$$

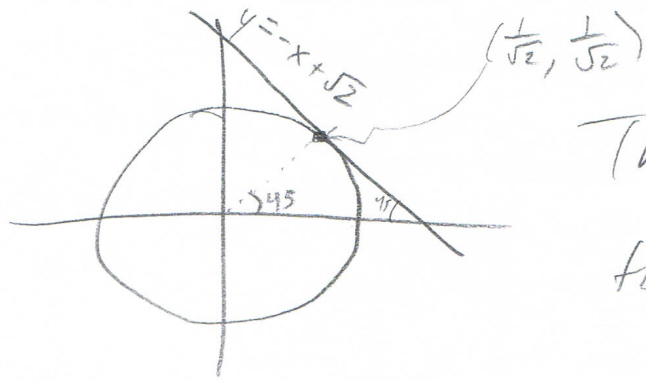
This is the equation of the line that is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant

which makes a triangle with the axes whose area is smallest, out of all such tangent lines

We can also check that this answer makes sense

in the case that the ellipse is just the unit circle

$$(a=1, b=1)$$



The answer we got suggests
the equation of the line is

$$y = -x + \sqrt{2}$$

which does indeed go through the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$,

i.e. the triangle is just a 45-45-90 right triangle

The tangent line is all "smushed up" against the corner
as far as it can go, making a triangle of least area