

Point-slope formula:

The equation of the line that goes through the point (x_1, y_1) and has slope m is

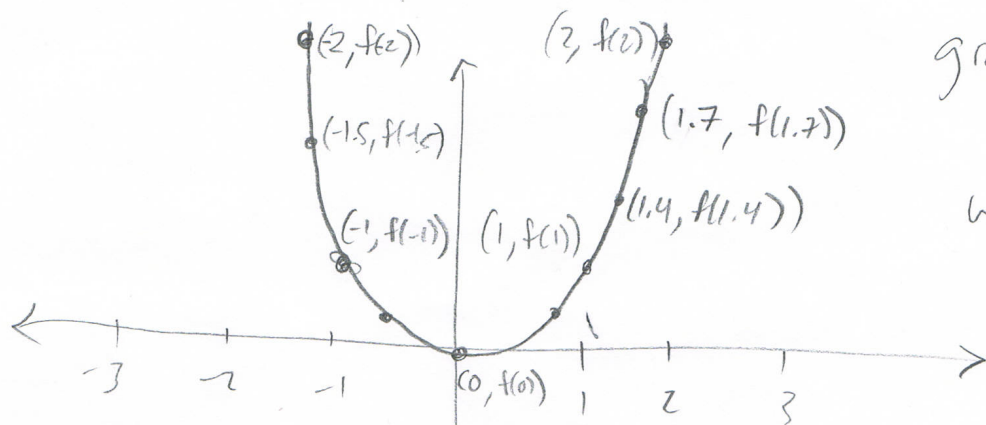
$$(y - y_1) = m(x - x_1)$$

Example: the line that goes through $(2, 1)$ and has slope 4 is $(y - 1) = 4(x - 2)$, i.e.,

$$y = 4x - 7$$

Remember, if f is a function, the graph of $y = f(x)$

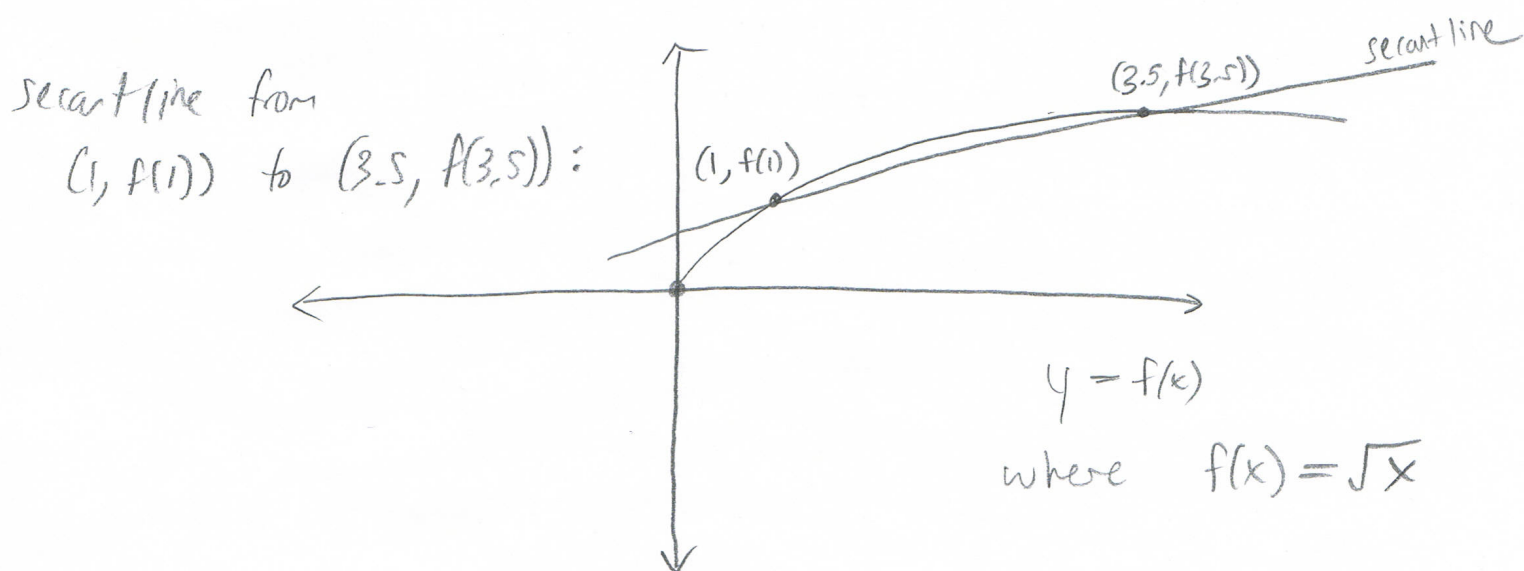
is the curve made by all the points $(x, f(x))$ in the plane



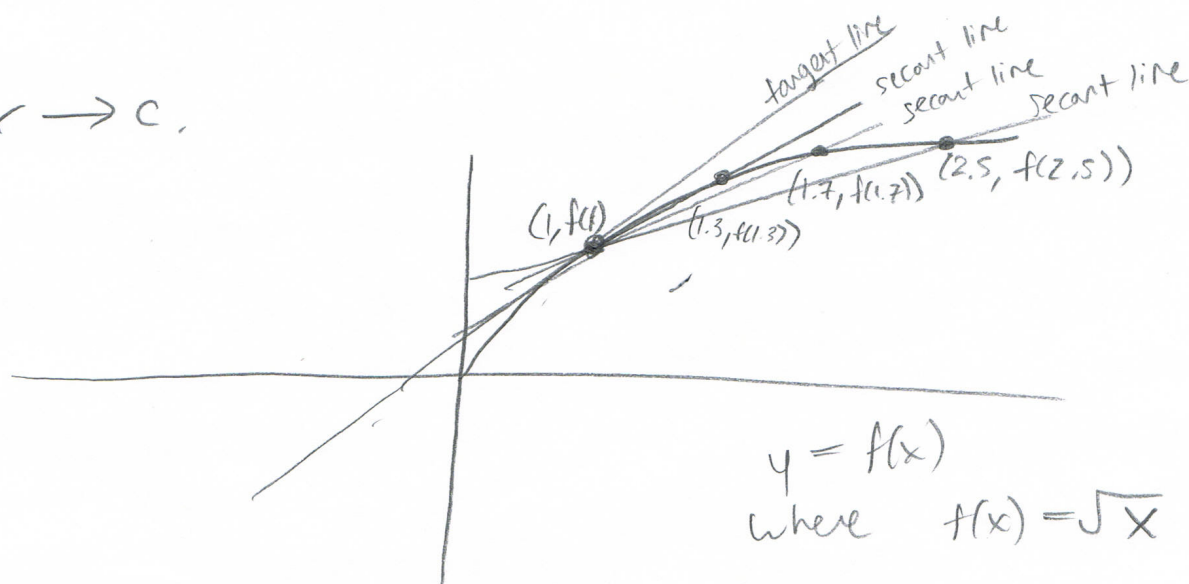
graph of
 $y = f(x)$
where $f(x) = x^2$

Sometimes not enough to draw
a few and connect them -
really think about all points!

If we graph $y = f(x)$ for some function f , and draw the line connecting two points on the graph, that line is called a "secant line"



The "tangent line to the graph $y = f(x)$ at c ", or alternatively, "... at the point $(c, f(c))$ ", is the limit of the secant lines from $(c, f(c))$ to $(x, f(x))$ as $x \rightarrow c$.



The slope of the tangent line to $y=f(x)$ at c is the limit of the slopes of the secant lines

$$\text{slope of tangent line at } c = \lim_{x \rightarrow c} \left[\text{slope of secant line between } (c, f(c)) \text{ and } (x, f(x)) \right]$$

$$= \lim_{x \rightarrow c} \frac{\text{"rise"}}{\text{"run"}}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

If we use the letter h to mean the quantity $x - c$, then taking a limit as $x \rightarrow c$ is the same as taking the limit as $h \rightarrow 0$. Also, if $h = x - c$, then $x = c + h$.

Thus,

$$\begin{aligned} \text{slope of tangent line at } c &= \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{(c+h) - c} \\ &= \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \end{aligned}$$

We can use this to define an entirely new function:

Given a function f , the derivative of f is

the function defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

For example, if $f(x) = x^2 - 5x$, then

$$\begin{aligned} f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(2+h)^2 - 5(2+h)] - [2^2 - 5(2)]}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 4h + 4 - 10 - 5h - 4 + 10}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 - h}{h} = \lim_{h \rightarrow 0} (h - 1) = 0 - 1 = -1$$

To find the equation for a tangent line, use the point-slope formula.

The tangent line to the graph $y=f(x)$ at $(c, f(c))$ by definition goes through the point $(c, f(c))$ and has slope equal to the derivative $f'(c)$.

Example: we already computed that if $f(x)=x^2-5x$,

then $f'(2) = -1$. Since the tangent line to

$y=f(x)$ at 2 passes through the point $(2, f(2))$
 $= (2, 2^2 - 5 \cdot 2)$
 $= (2, -1)$

The equation of the tangent line at 2 is

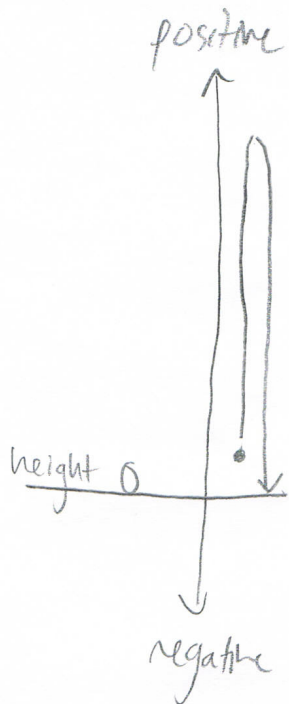
$$\underbrace{(y - (-1))}_{\substack{\text{y-coordinate} \\ \text{of point}}} = \underbrace{(-1)}_{\text{slope}} (\underbrace{x - 2}_{\substack{\text{x-coordinate} \\ \text{of point}}})$$

$$y = -x + 1$$

The derivative $f'(c)$ says how quickly f is changing at c .

Example: if $f(x) =$ the height of a ball at time x ,
(IMPORTANT)

then $f'(x) =$ the velocity of the ball at time x



If $f'(2) > 0$, then that means the ball is traveling in the positive direction at time 2, i.e. it's going up

If $f'(2) = 0$, then it's not moving at time 2

If $f'(2) < 0$, then it's traveling down at time 2

Acceleration = the derivative of velocity
= the derivative of the derivative of f
= how quickly is the velocity changing

If a ball's velocity is constant, then the velocity isn't changing at all - the acceleration is 0

Acceleration at time $x > 0 \Rightarrow$ velocity is increasing at time x

Acceleration at time $x = 0 \Rightarrow$ velocity isn't changing at time x

Acceleration at time $x < 0 \Rightarrow$ velocity is decreasing at time x

If $f(x)$ = position of object at time x ,

$v(x)$ = velocity of object at time $x = f'(x)$

$a(x)$ = acceleration of object at time $x = v'(x) = f''(x)$

DERIVATIVE RULES (THE FUNDAMENTAL ONES)

If $f(x) = 1$ for all x ,
then $f'(x) = 0$ for all x

$$\frac{d}{dx}(1) = 0$$

$$D_x(1) = 0$$

If $f(x) = x$ for all x ,
then $f'(x) = 1$ for all x

$$\frac{d}{dx}(x) = 1$$

$$D_x(x) = 1$$

derivative $(f+g)'(x)$ is
equal to $f'(x) + g'(x)$

$$\frac{d}{dx}(f+g) = \frac{d}{dx}(f) + \frac{d}{dx}(g)$$

$$D_x(f+g) = D_x(f) + D_x(g)$$

PRODUCT RULE:

derivative of $(fg)(x)$ is
equal to $f'(x)g(x) + f(x)g'(x)$

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f \frac{d}{dx}(g)$$

$$D_x(fg) = D_x(f)g + f D_x(g)$$

CHAIN RULE

derivative of $(f \circ g)(x)$ is
equal to $f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}(f \circ g) = \left(\frac{d}{dx}(f) \circ g \right) \cdot \frac{d}{dx}(g)$$

$$D_x(f \circ g) = \left(D_x(f) \circ g \right) \cdot D_x(g)$$

Example of the product rule:

$$f(x) = x^2 - 2x + 3$$

$$f'(x) = 2x - 2$$

$$g(x) = 2x^2 + 5x - 4$$

$$g'(x) = 4x + 5$$

Remember, the definition of the product of two functions

$$(fg)(x) \stackrel{\text{definition}}{=} f(x) \cdot g(x)$$

$$\text{So } (fg)(x) = (x^2 - 2x + 3)(2x^2 + 5x - 4)$$

$$= 2x^4 + 5x^3 - 4x^2$$

$$- 4x^3 - 10x^2 + 8x$$

$$+ 6x^2 + 15x - 12$$

$$= 2x^4 + x^3 - 8x^2 + 23x - 12$$

Product Rule:

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$= (2x - 2)(2x^2 + 5x - 4) + (x^2 - 2x + 3)(4x + 5)$$

$$= 8x^3 + 3x^2 - 16x + 23$$

DERIVATIVE RULES THAT ARE CONSEQUENCES OF THE FUNDAMENTAL ONES

POWER RULE:

$$\frac{d}{dx}(x^a) = ax^{a-1} \quad \text{for any real number } a$$

$$\text{e.g. } \frac{d}{dx}(\sqrt[3]{x}) = \frac{d}{dx}(x^{1/3}) = (1/3)x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\frac{d}{dx}(x^5) = 5x^4$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

QUOTIENT RULE:

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

"low dee high minus high dee low over low squared"

$$\text{e.g. if } f(x) = x^2 + 1, \quad g(x) = 5x - 3,$$

$$\left(\frac{f}{g}\right)'(x) = \frac{(5x-3)(2x) - (x^2+1)(5)}{(5x-3)^2} = \frac{5x^2 - 6x - 5}{25x^2 - 30x + 9}$$

$$\frac{d}{dx}(c \cdot f) = c \cdot \frac{d}{dx}(f) \quad \text{if } c \text{ is any number}$$

e.g.

$$\frac{d}{dx} \left(5 \cdot \frac{(x^2+1)}{(x^3-17x)} \right) = 5 \cdot \frac{d}{dx} \left(\frac{(x^2+1)}{(x^3-17x)} \right)$$

any crazy fraction

take the factor of 5 out

Example:

splitting derivative over sum

$$\frac{d}{dx}(x^3 - 7x^2 + 5x - 2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(-7x^2) + \frac{d}{dx}(5x) + \frac{d}{dx}(-2)$$

pulling out
constant factors

$$= \frac{d}{dx}(x^3) + (-7) \frac{d}{dx}(x^2) + (5) \frac{d}{dx}(x) + (-2) \frac{d}{dx}(1)$$

power rule

$$= 3x^2 + (-7)(2x) + (5)(1) + (-2)(0)$$

$$= 3x^2 - 14x + 5$$

Example:

$$\frac{d}{dx}(7x^{19}) = 7 \frac{d}{dx}(x^{19}) = 7 \cdot 19 \cdot x^{18} = 133x^{18}$$

Example of the chain rule:

If $f(x) = x^2 + 5$, and $g(x) = 2x^2 - 7x + 1$,

then

definition of composition of functions, " $\circ$ "

$$(f \circ g)(x) = f(g(x))$$

$$= f(2x^2 - 7x + 1)$$

tedious!

$$= (2x^2 - 7x + 1)^2 + 5$$

$$= 4x^4 - 28x^3 + 53x^2 - 14x + 6$$

$$\text{so } (f \circ g)'(x) = 16x^3 - 84x^2 + 106x - 14$$

Chain rule gets the same answer

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = 2x, \quad g'(x) = 4x - 7 \quad \text{so}$$

$$f'(g(x)) \cdot g'(x) = f'(2x^2 - 7x + 1) \cdot g'(x)$$

$$= 2(2x^2 - 7x + 1) \cdot (4x - 7)$$

$$= (4x^2 - 14x + 2) \cdot (4x - 7)$$

$$= 16x^3 - 84x^2 + 106x - 14 \quad \checkmark$$

Another example of the chain rule:

$$\frac{d}{dx}((2x+1)^5) = ?$$

$$\text{let } f(x) = x^5, \quad g(x) = 2x+1$$

$$\text{then } (f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^5$$

so we want to use the chain rule:

$$\begin{aligned} \frac{d}{dx}((2x+1)^5) &= f'(g(x)) \cdot g'(x) \\ &= f'(2x+1) \cdot g'(x) \end{aligned}$$

Since $f'(x) = 5x^4$, $g'(x) = 2$, we get

$$\frac{d}{dx}((2x+1)^5) = 5(2x+1)^4 \cdot 2 = 10(2x+1)^4$$

Key skill: know how to write complicated functions as compositions of simpler ones

If $f(x)$ = height of ball at time x

$v(x)$ = velocity of ball at time x

$a(x)$ = acceleration of ball at time x

then $v(x) = f'(x)$ and $a(x) = v'(x) = f''(x)$

Usually, we start counting at time $x=0$

So for example,

"When does the ball hit the ground?" $\leadsto$ For what $x > 0$ is it true that $f(x) = 0$?
"height=0" means "hit ground"

"What is the ball's velocity when it hits the ground?" $\leadsto$ Find the specific number c that is the answer to the above question (i.e., what time does the ball hit the ground) then compute $v(c)$

"When does the ball reach its maximum height?" $\leadsto$

$v(x) > 0$ $v(x) < 0$

The ball travels in a straight line up and down. Its maximum height is when it stops moving up and starts moving down, i.e. the exact moment when $v(x) = 0$.

So solve $v(x) = 0$ for x .