

Math 13100 – Section 58 – Midterm 2

February 10, 2016

Information and Directions

- This exam will last 50 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Partial credit is given for showing your calculations and explaining your thoughts.

Name: _____

SOLUTIONS

Problem	Points	Score
1	30	
2	20	
3	25	
4	25	

Problem 1 (30 points)

Part 1 (10 points)

Let c be a real number, and let f be a function. Give the precise definition of what it means to say

" f is continuous at c "

(Hint: there are three conditions that must be met.)

- $f(c)$ is defined
- $\lim_{x \rightarrow c} f(x)$ exists
- $\lim_{x \rightarrow c} f(x) = f(c)$

Part 2 (5 points)

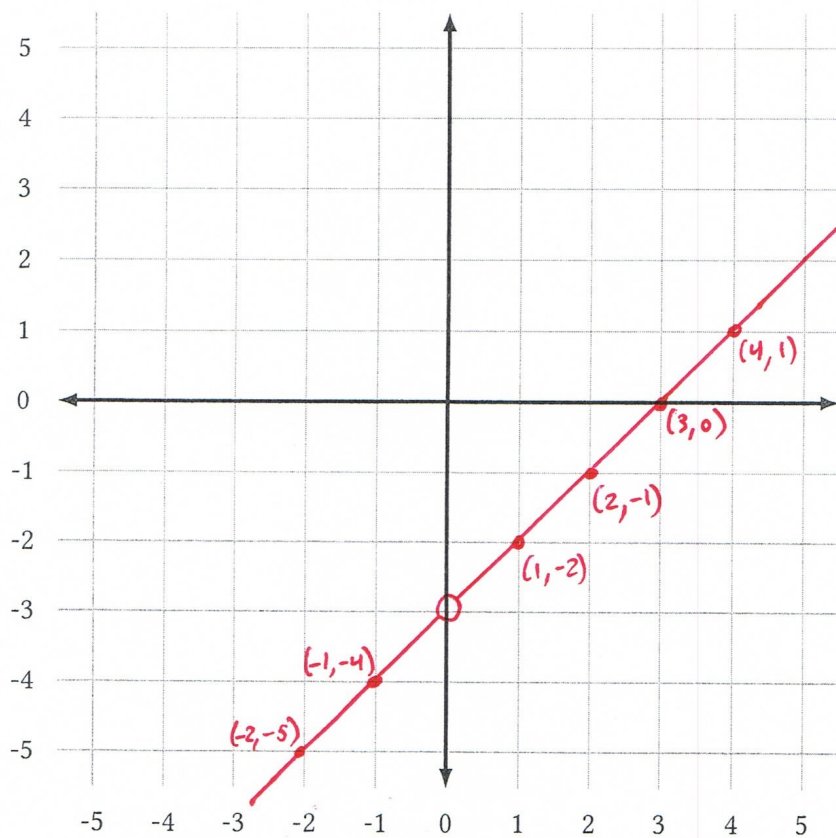
Let $f(x) = \frac{x^2 - 3x}{x}$. On this depiction of the plane, draw the graph of the equation $y = f(x)$.

For at least three different points on your graph, label the point with its x - and y -coordinates.

Division by zero
is undefined, leaves
a "hole" in the graph

For any $x \neq 0$,
we can cancel in
the fraction $\frac{x^2 - 3x}{x}$
to get $x - 3$

Thus the graph is
the graph of $y = x - 3$,
but with a hole at the point $(0, -3)$.



Part 3 (10 points)

Let $f(x) = \frac{x^2 - 3x}{x}$, as in Part 2.

Are there any discontinuities of f ? (That is, can you find any c where f is **not** continuous at c ?) Explain your answer.

By Theorem A in section 1.6, a rational function (such as $\frac{x^2 - 3x}{x}$) is continuous at any input where it is defined, i.e., wherever its denominator is not zero.

The only discontinuity of $\frac{x^2 - 3x}{x}$ is at the real number 0.

Part 4 (5 points)

If f does have any discontinuities, then for each discontinuity, say whether it is removable or not. Explain your answer.

The discontinuity of f at 0 is removable, since the limit $\lim_{x \rightarrow 0} f(x)$ exists. By changing $f(0)$ from being undefined to being the value of this limit (namely, -3), we now have a function which meets the definition of being continuous at 0:

- $f(0)$ is defined ✓ (we have just defined it to be -3)
- $\lim_{x \rightarrow 0} f(x)$ exists ✓ (this remains true no matter what we do to $f(0)$)
- $\lim_{x \rightarrow 0} f(x) = f(0)$ ✓ (we chose $f(0)$ to be equal to this limit)

Problem 2 (20 points)

Part 1 (10 points)

Let c be a real number, and let f be a function.

Give the precise, mathematical meaning of the statement

$$\lim_{x \rightarrow c} f(x) = \infty$$

(Hint: your answer shouldn't contain the letter ϵ .)

For any $N > 0$, there exists some $\delta > 0$ such that

$$0 < |x - c| < \delta \implies f(x) > N$$

Part 2 (10 points)

Give an example of a function f such that $\lim_{x \rightarrow 2} f(x) = \infty$.

(Hint: remember, "equals infinity" is not the same thing as "does not exist".)

$f(x) = \frac{1}{(x-2)^2}$, i.e., $f(x) = \frac{1}{x^2 - 4x + 4}$, is one such function.

The problem doesn't ask for a proof, but here it is:

For any $N > 0$, there does in fact exist a $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies \frac{1}{(x-2)^2} > N,$$

namely $\delta = \frac{1}{\sqrt{N}}$. That's because

$$0 < |x - 2| < \frac{1}{\sqrt{N}} \implies (x-2)^2 < \frac{1}{N} \implies N(x-2)^2 < 1 \implies N < \frac{1}{(x-2)^2}$$

Problem 3 (25 points)

Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x - 1 & \text{if } x < 0 \end{cases}$$

Thus, for example,

$$f(0) = 0^2 = 0$$

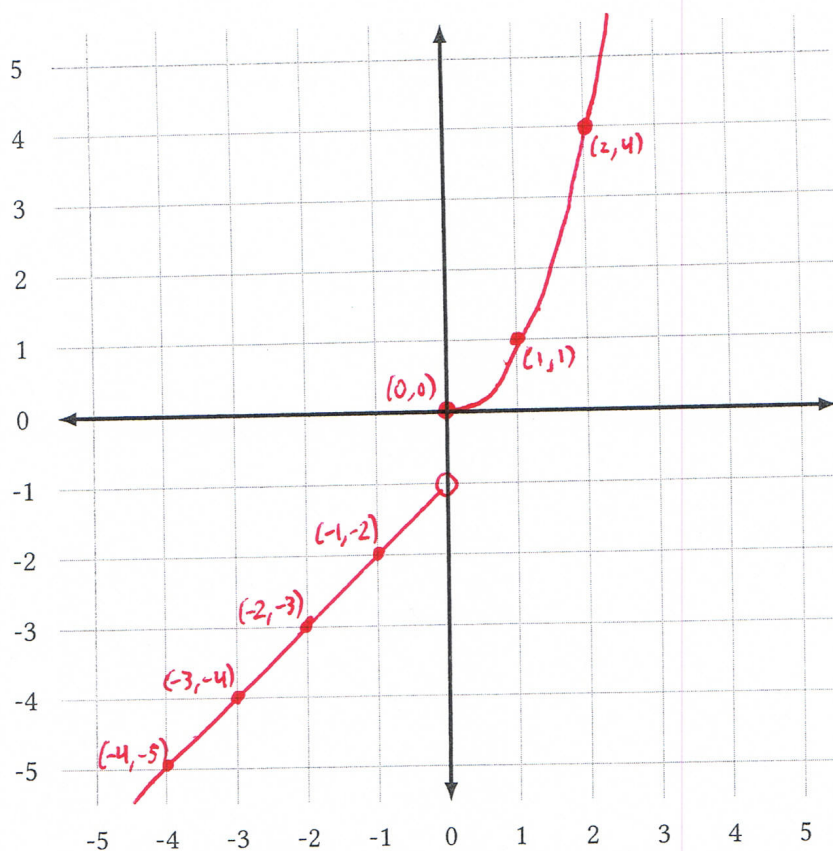
$$f(2) = 2^2 = 4$$

$$f(-2) = (-2) - 1 = -3$$

Part 1 (5 points)

On this depiction of the plane, draw the graph of the equation $y = f(x)$.

For at least three different points on your graph, label the point with its x - and y -coordinates.



Part 2 (10 points)

Remember that $f(x)$ was defined in Part 1 of this problem.

What are the values of these one-sided limits?

$$\lim_{x \rightarrow 0^+} f(x) = \underline{0}$$

$$\lim_{x \rightarrow 0^-} f(x) = \underline{-1}$$

You aren't required to give any formal proof for these answers, though any manner of explanation for your answers that you provide below can be helpful in case you end up needing partial credit.

For $x > 0$, the function $f(x)$ is defined to be x^2 .

$$\text{Thus } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2 = 0.$$

For $x < 0$, the function $f(x)$ is defined to be $x - 1$.

$$\text{Thus } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x - 1 = 0 - 1 = -1.$$

Part 3 (10 points)

Remember that $f(x)$ was defined in Part 1 of this problem.

Does the two-sided limit $\lim_{x \rightarrow 0} f(x)$ exist? Justify your answer.

(Hint: for this question, a great strategy would be to cite your answers in Part 2 and appeal to a theorem from the book that relates one-sided limits to the existence of a two-sided limit.)

By Theorem A from section 1.1, a two-sided limit

$\lim_{x \rightarrow c} f(x)$ exists if, and only if, both one-sided limits

$$\lim_{x \rightarrow c^+} f(x)$$

$$\lim_{x \rightarrow c^-} f(x)$$

exist, and moreover, the one-sided limits are equal to each other.

Since $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$, the two-sided limit $\lim_{x \rightarrow 0} f(x)$

does not exist.

Problem 4 (25 points)

Part 1 (10 points)

Let L be a real number, and let f be a function.

Give the precise, mathematical meaning of the statement

$$\lim_{x \rightarrow \infty} f(x) = L$$

(Hint: your answer shouldn't contain the letter δ .)

For any $\varepsilon > 0$, there exists some $M > 0$ such that

$$x > M \implies |f(x) - L| < \varepsilon$$

Part 2 (5 points)

What is the value of $\lim_{x \rightarrow \infty} \frac{3x^2}{9x^2 - 2x + 5}$? You don't have to justify your answer here.

The limit equals $\frac{1}{3}$.

(Proof in the next part of the problem.)

Part 3 (10 points)

Give a proof that $\lim_{x \rightarrow \infty} \frac{3x^2}{9x^2 - 2x + 5}$ is equal to your answer in Part 2.

Of course you are always welcome to create formal proof from scratch, but it would probably be easier to appeal to certain theorem(s) from the book about how limits and operations can be manipulated. Please show individual steps and justify why you are allowed to do what you're doing.

Since we're interested in what happens to the quantity $\frac{3x^2}{9x^2 - 2x + 5}$ as x gets bigger and bigger, we can change what happens at $x = 0$ (or any particular x , for that matter) without affecting the value of the limit.

$$\frac{3x^2}{9x^2 - 2x + 5} = \frac{3x^2}{9x^2 - 2x + 5} \cdot \frac{1/x^2}{1/x^2} = \frac{3}{9 - 2/x + 5/x^2}$$

The above manipulation is valid for any $x \neq 0$. When $x = 0$, the expression $\frac{1/x^2}{1/x^2}$ involves division by 0 and therefore is undefined.

So as explained, we still have that

$$\lim_{x \rightarrow \infty} \frac{3x^2}{9x^2 - 2x + 5} = \lim_{x \rightarrow \infty} \frac{3}{9 - 2/x + 5/x^2}$$

The limits $\lim_{x \rightarrow \infty} 3$ and $\lim_{x \rightarrow \infty} (9 - 2/x + 5/x^2)$ both exist; in fact,

$$\lim_{x \rightarrow \infty} 3 = 3 \quad \lim_{x \rightarrow \infty} (9 - 2/x + 5/x^2) = 9 - 0 + 0 = 9$$

Having checked that the requirements of Theorem A from section 1.3 are met (the separate limits exist), we are justified in doing this:

$$\lim_{x \rightarrow \infty} \frac{3}{9 - 2/x + 5/x^2} = \frac{\lim_{x \rightarrow \infty} 3}{\lim_{x \rightarrow \infty} (9 - 2/x + 5/x^2)} = \frac{3}{9} = \frac{1}{3}$$