

Math 13100 – Section 58 – Midterm 1

January 22, 2016

Information and Directions

- This exam will last 50 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Partial credit is given for showing your calculations and explaining your thoughts.

Name: SOLUTIONS

Problem	Points	Score
1	25	
2	25	
3	15	
4	35	

Problem 1 (25 points)

Consider the real numbers x that are solutions to this inequality:

$$11 < 3x + 8 < 23$$

Part 1 (10 points)

Express the set of solutions as an interval. Is it an open, closed, or half-open interval?

$$11 < 3x + 8 < 23$$

$$11 - 8 < 3x < 23 - 8$$

subtract 8 from each expression

$$3 < 3x < 15$$

simplify

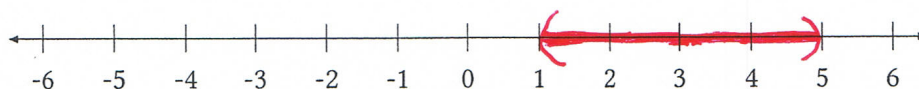
$$1 < x < 5$$

divide each expression by 3

The set of real numbers x that are solutions to $1 < x < 5$ is, by definition, the open interval $(1, 5)$.

Part 2 (5 points)

On this depiction of the real line, highlight / mark where the solutions of this inequality are.



Part 3 (10 points)

Find real numbers a and b such that this inequality has the same solutions as $|x - a| < b$.

Hint: Remember, $|x - a|$ is the distance of x away from a , so the solutions to the inequality $|x - a| < b$ are the real numbers that are at most b away from the "center" a .

We want to find values for a and b such that

$$x \text{ is a solution to } 1 < x < 5 \iff x \text{ is a solution to } |x - a| < b$$

(That is, we want $|x - a| < b$ to have the same solutions as the inequality from Part 1 of the problem.)

The inequality $|x - a| < b$ is equivalent to

$$-b < x - a < b$$

which is equivalent to

$$a - b < x < a + b$$

Thus, we want to find the values of a and b such that

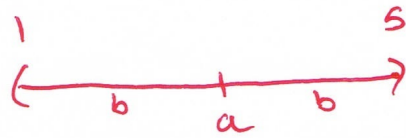
$$a - b = 1 \quad \text{and} \quad a + b = 5$$

Now just substitute: $a - b = 1 \iff a = b + 1$, so

$a + b = (b + 1) + b = 5$, so $2b = 4$, so $b = 2$, and thus

$$a = b + 1 = 2 + 1 = 3.$$

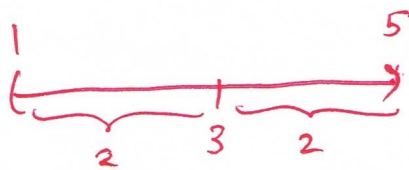
The more visual approach that the hint suggests is to observe that we're looking for an a and b such that



The number a that's in the center between 1 and 5 is their average, $\frac{1+5}{2} = 3$. The number b that represents "how far away from the center can we get" is then equal to the distance between the center and the ends:

$$5 - 3 = 2$$

$$3 - 1 = 2$$



Problem 2 (25 points)

Consider the functions

$$f(x) = 2 + \sqrt{x+4}$$

$$g(x) = \frac{1}{x-2}$$

Part 1 (10 points)

What is the natural domain of f ? What is the natural domain of g ?

(That is, for what inputs x are they each defined?)

$\sqrt{\text{blah}}$ is defined whenever $\text{blah} \geq 0$ ($\sqrt{0}$ is fine, it equals 0!)

$\frac{1}{\text{blah}}$ is defined whenever $\text{blah} \neq 0$

Thus f is defined whenever $x+4 \geq 0$, i.e., whenever x is in $[-4, \infty)$

g is defined whenever $x-2 \neq 0$, i.e., whenever x is in $(-\infty, 2) \cup (2, \infty)$

Part 2 (5 points)

Using your knowledge of what it means to add functions, calculate $(f+g)(12)$.

$(f+g)(x)$ is defined to be $f(x) + g(x)$

$$\text{thus } (f+g)(12) = f(12) + g(12)$$

$$= 2 + \sqrt{12+4} + \frac{1}{12-2}$$

$$= 2 + \sqrt{16} + \frac{1}{10}$$

$$= \frac{61}{10}$$

Part 3 (10 points)

Using your knowledge of what it means to compose functions, calculate $(g \circ f)(5)$.

(Be sure to notice what order g and f are in!)

$(g \circ f)(x)$ is defined to be $g(f(x))$

$$\begin{aligned}\text{thus } (g \circ f)(5) &= g(f(5)) \\ &= g(2 + \sqrt{5+4}) \\ &= g(2 + \sqrt{9}) \\ &= g(5) \\ &= \frac{1}{5-2} \\ &= \frac{1}{3}\end{aligned}$$

Problem 3 (15 points)

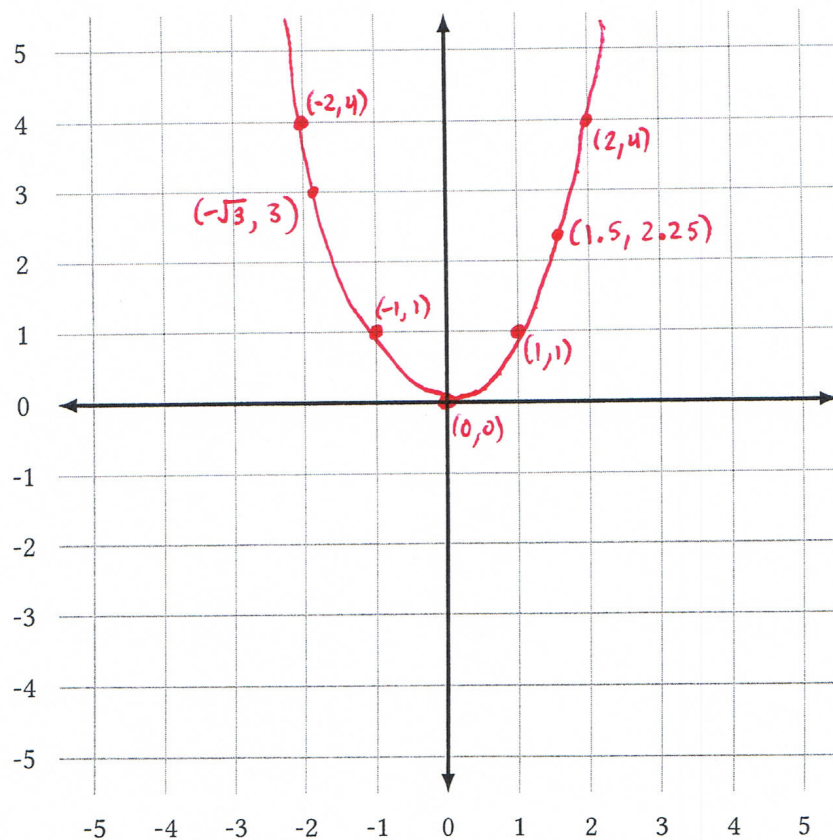
Consider the function

$$f(x) = x^2$$

Part 1 (5 points)

On this depiction of the plane, draw the graph of the equation $y = f(x)$.

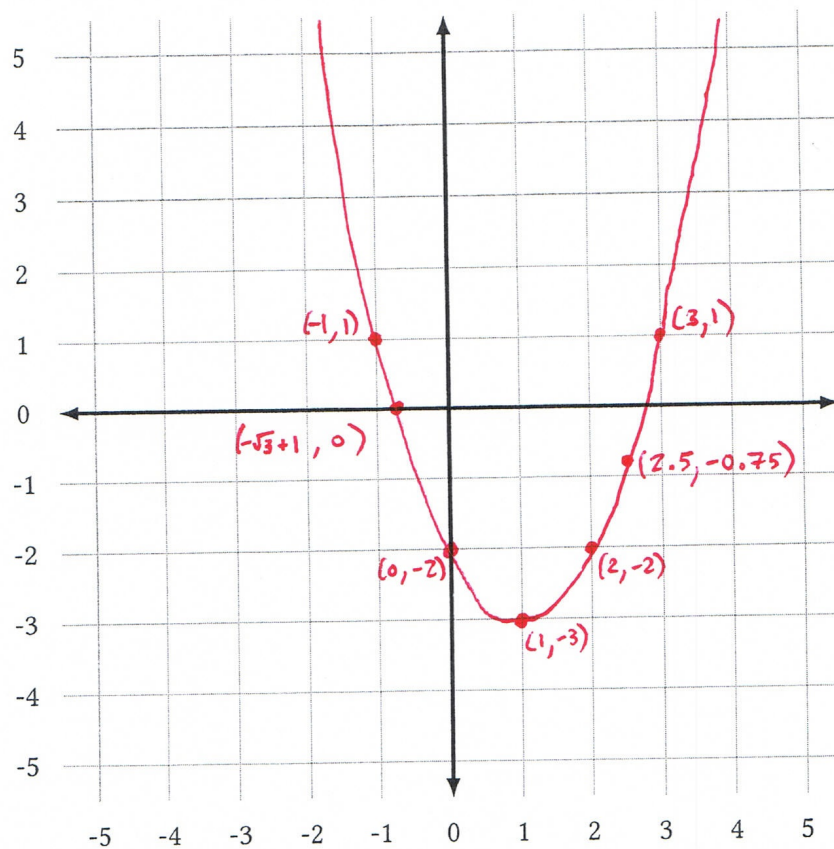
For at least three different points on your graph, label the point with its x - and y -coordinates.



Part 2 (10 points)

On this depiction of the plane, draw the graph of the equation $y = f(x - 1) - 3$.

For at least three different points on your graph, label the point with its x - and y -coordinates.



$y = f(x)$ $\rightsquigarrow$ $y = f(x) + c$ shift graph vertically by c
($c > 0$ means shift up, $c < 0$ means shift down)

$y = f(x)$ $\rightsquigarrow$ $y = f(x - c)$ shift graph horizontally by c
($c > 0$ means shift right, $c < 0$ means shift left)

Problem 4 (35 points)

Part 1 (20 points)

Let c and L be real numbers, and let f be a function. Give the precise, mathematical meaning of the statement

$$\lim_{x \rightarrow c} f(x) = L$$

For any $\varepsilon > 0$, there exists some $\delta > 0$

such that $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$

Part 2 (15 points)

Give a formal proof that $\lim_{x \rightarrow 1} f(x) = 3$, where $f(x) = x^2 - 3x + 5$.

Our goal is to prove that

"For any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |x - 1| < \delta \implies |f(x) - 3| < \varepsilon"$$

(We do this by describing what δ would work given an ε .
At the end, we'll be able to say "if ε is this, then δ is that.")

We work backwards: if $|f(x) - 3| < \varepsilon$, that means

$$|f(x) - 3| < \varepsilon \iff |x^2 - 3x + 5 - 3| < \varepsilon$$

$$\iff |x^2 - 3x + 2| < \varepsilon$$

$$\iff |(x-2)(x-1)| < \varepsilon$$

$$\iff |x-2| |x-1| < \varepsilon$$

What should we choose δ to be so that requiring $0 < |x-1| < \delta$ guarantees $|x-2| |x-1| < \varepsilon$?

First, we need to put $|x-2|$ in terms of $|x-1|$. That's the only quantity we have direct "control" of via our choice of δ , so if we can get $|x-2|$ under the control of δ also, we've made progress.

We use the triangle inequality. We want to write $|x-2|$ as $|x-1+a|$ for some a , so that we can use the triangle inequality to know this is less than $|x-1| + |a|$.

$$\begin{aligned}|x-2| &= |x-1+(-1)| \leq |x-1| + |-1| \\ &= |x-1| + 1\end{aligned}$$

Thus, whenever $|x-1| < \delta$, we know that $|x-2| < \delta + 1$.

So, our earlier question becomes: what should we choose δ to be

so that requiring $0 < |x-1| < \delta$ guarantees $\underbrace{|x-2|}_{\text{something that is } < \delta+1} \cdot \underbrace{|x-1|}_{\text{something that is } < \delta} < \epsilon$

If we decide in advance to never choose a δ bigger than 17, for instance, then we know that $\delta+1$ will never be bigger than 18.

Then our question becomes: what number less than 17 should we choose δ to be

so that requiring $0 < |x-1| < \delta$ guarantees $\underbrace{|x-2|}_{\text{something that is } < 18} \cdot \underbrace{|x-1|}_{\text{something that is } < \delta} < \epsilon$

Now we can divide both sides by 18 to conclude that choosing $\delta = \frac{\epsilon}{18}$

works (as long as it's less than 17). So for something that works for any ϵ ,

we can declare our choice of δ to be $\min \{17, \frac{\epsilon}{18}\}$.