Math 13100 – Section 58 – Final Exam

March 15, 2016

Information and Directions

- This exam will last 120 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Partial credit is given for showing your calculations and explaining your thoughts.

Name: SOLUTIONS

Problem	Points	Score
1	8	
2	8	
3	4	
4	16	
5	16	
6	16	
7	16	
8	8	
9	8	

Problem 1 (8 points)

Consider the inequality

$$-5 < 2x - 1 < 7$$

Part 1 (4 points)

Express the set of solutions of this inequality as an interval.

$-5 < 2x - 1 < 7$

$-5 + 1 < 2x < 7 + 1$

$-4 < 2x < 8$

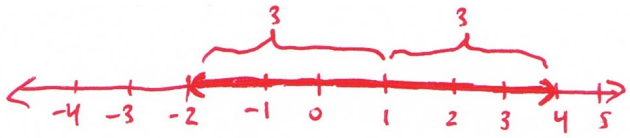
$-4/2 < 2x/2 < 8/2$

$-2 < x < 4$

Solutions are the real numbers x in the interval $(-2, 4)$

Part 2 (4 points)

For which numbers a and b does $|x - a| < b$ have the same solutions as the above inequality?



The center of the interval $(-2, 4)$ is the number 1, and the ends are a distance of 3 away from the center. Thus $-2 < x < 4$ is equivalent to $|x - 1| < 3$.

Alternatively, you can remember that

$$|x - a| < b \iff a - b < x < a + b$$

and look for the numbers a and b such that $\begin{matrix} a - b = -2 \\ a + b = 4 \end{matrix}$

You can solve by substitution, e.g., $a - b = -2$ implies $a = b - 2$, so $a + b = 4$ becomes $(b - 2) + b = 4$, hence $2b = 6$, hence $b = 3$, and $a = b - 2 = 1$.

Problem 2 (8 points)

Define the functions

$$f(x) = x^2 + 1 \qquad g(x) = \frac{\sqrt{x-2}}{x-5}$$

Part 1 (4 points)

What is the natural domain of g?

$\sqrt{x-2}$ is only defined when $x-2 \geq 0$,

and $\frac{1}{x-5}$ is only defined when $x-5 \neq 0$,

So $\frac{\sqrt{x-2}}{x-5}$ is only defined when $x \geq 2$, except $x = 5$

Written as a union of intervals, the domain is $[2, 5) \cup (5, \infty)$

Part 2 (4 points)

What is the value of $(f \circ g)(6)$?

$$(f \circ g)(6) = f(g(6))$$

$$= f\left(\frac{\sqrt{6-2}}{6-5}\right)$$

$$= f\left(\frac{\sqrt{4}}{1}\right)$$

$$= f(2)$$

$$= 2^2 + 1$$

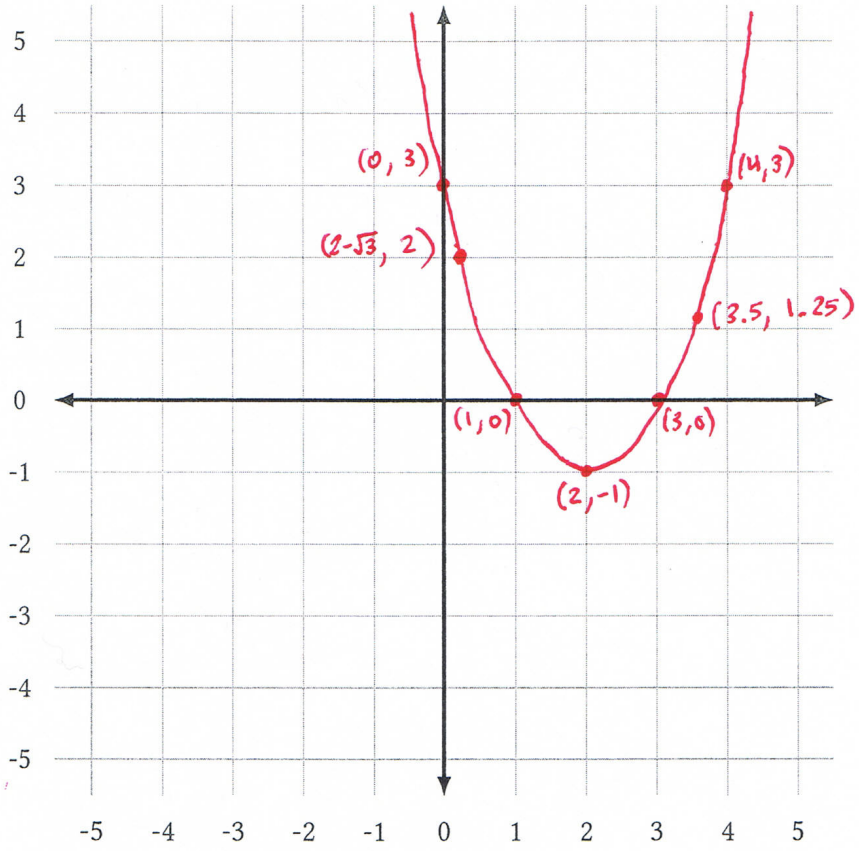
$$= 5$$

Problem 3 (4 points)

Graph the function $f(x) = (x-2)^2 - 1$.

For at least three different points on your graph, label the point with its x- and y-coordinates.

(Hint: take the graph of x^2 and translate it by the appropriate amounts horizontally and vertically.)



Problem 4 (16 points)

Part 1 (8 points)

Let $f(x)$ be a function. Give the precise, mathematical meaning of the statement

$$\text{“}\lim_{x\to 5} f(x) = 2\text{”}$$

For any $\varepsilon > 0$, there exists some $\delta > 0$

such that $0 < |x-5| < \delta \Rightarrow |f(x) - 2| < \varepsilon$

Part 2 (8 points)

Give a complete proof, without appealing to theorems about continuity, that

$$\lim_{x\to 5} x^2 - 3x - 8 = 2$$

We want to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-5| < \delta \Rightarrow |x^2 - 3x - 8 - 2| < \varepsilon$. Factoring the right side,

$$|x^2 - 3x - 10| < \varepsilon \iff |(x-5)(x+2)| < \varepsilon$$
$$\iff |x-5| \cdot |x+2| < \varepsilon$$

If we decide in advance to choose a $\delta < 1$, then by the triangle inequality, $|x+2| = |(x-5)+7| \leq |x-5| + |7| < \delta + 7 < 8$.

Thus, $|x-5| \cdot (\text{something} < 8) < \varepsilon$, hence $|x-5| < \frac{\varepsilon}{8}$.

Therefore, no matter what $\varepsilon > 0$ is, if we choose $\delta = \min\{1, \frac{\varepsilon}{8}\}$, we can guarantee that $0 < |x-5| < \delta \Rightarrow |x^2 - 3x - 10| < \varepsilon$.

(Keep writing on the back of the page if you need more space.)

Problem 5 (16 points)

Part 1 (8 points)

Let c be a real number, and let f be a function. Give the precise definition of what it means to say

"f is continuous at c"

(Hint: there are three conditions that must be met.)

- $f(c)$ is defined
- $\lim_{x \to c} f(x)$ exists
- $\lim_{x \to c} f(x) = f(c)$

Part 2 (8 points)

Are there any discontinuities of the function

$$f(x) = \frac{x^2 - 5x + 6}{x - 3} \; ?$$

(That is, can you find any c where f is not continuous at c?) If so, are they removable discontinuities? Explain your answer or thought process.

Theorem A from section 1.6 says that a rational function is continuous at any real number c where its denominator is not zero. Thus, f is continuous at any $c \neq 3$. At any $c \neq 3$, we have $f(c) = \frac{(c-2)(c-3)}{(c-3)} = c-2$.

f is not continuous at 3 because $f(3)$ is not defined.

However, if we define $f(3) = 1$, then $f(x) = x-2$ for all x, and since $x-2$ is continuous at 3, this demonstrates that the discontinuity that f had was removable, since we could turn f into a continuous function by choosing a new definition of $f(3)$.

Problem 6 (16 points)

In this problem, define the function $f(x) = (x^2 - 7)^5$.

Part 1 (8 points)

Using the chain rule, compute the derivative of f.

$g(x) = x^5$ $\quad h(x) = x^2 - 7$

$g'(x) = 5x^4$ $\quad h'(x) = 2x$

$f(x) = (x^2-7)^5 = (g \circ h)(x)$

so $f'(x) = g'(h(x)) \cdot h'(x)$

$= 5(x^2-7)^4 \cdot (2x)$

$= 10x(x^2-7)^4$

Part 2 (8 points)

Find the equation of the tangent line to the graph of $y = f(x)$ at 3, i.e., the tangent line that goes through the point

$$(3, f(3)) = (3, (3^2 - 7)^5) = (3, 2^5) = (3, 32)$$

The tangent line to the graph of $y = f(x)$ at 3 is characterized by these two properties:

- It goes through $(3, f(3)) = (3, 32)$.
- Its slope is $f'(3) = 10 \cdot 3 \cdot (3^2 - 7)^4 = 30 \cdot 2^4 = 480$

By the point-slope formula, the equation of the line is

$$y - 32 = 480(x - 3)$$

$$y = 480x - 1408$$

Problem 7 (16 points)

Part 1 (8 points)

Using implicit differentiation, find $\frac{dy}{dx}$ where $x^2y^2 + 4xy = 12y$.

$\frac{d}{dx}(x^2y^2 + 4xy) = \frac{d}{dx}(12y)$

$\frac{d}{dx}(x^2y^2) + \frac{d}{dx}(4xy) = \frac{d}{dx}(12y)$

product rule + chain rule

$\left(2xy^2 + x^2 \cdot 2y\frac{dy}{dx}\right) + 4\left(y + x\frac{dy}{dx}\right) = 12\frac{dy}{dx}$

$\left(2xy^2 + 4y\right) + \left(2x^2y + 4x - 12\right)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2xy^2 - 4y}{2x^2y + 4x - 12}$

Part 2 (8 points)

Find the equation of the tangent line to the graph of

$$x^2y^2 + 4xy = 12y$$

that goes through the point $(2, 1)$.

The tangent line to the graph at $(2,1)$ is characterized by:

- It goes through $(2,1)$
- Its slope is $\frac{dy}{dx}$ when $x=2$, $y=1$:

$$\frac{-2(2)(1)^2 - 4(1)}{2(2)^2(1) + 4(2) - 12} = \frac{-8}{4} = -2$$

By the point-slope formula, the equation of the line is

$$y - 1 = (-2)(x - 2)$$

$$y = -2x + 5$$

Problem 8 (8 points)

Use derivatives to approximate the value of $\sqrt{16.1}$.

(Hint: you are being asked to approximate $f(16 + 0.1)$, where $f(x) = \sqrt{x}$. You should start by finding $f(16)$ and $f'(16)$.)

To approximate the value of $f(x+h)$, we use

$$f(x+h) \underset{(\text{approx})}{\approx} f(x) + f'(x)\,h$$

In this problem, $f(x) = \sqrt{x} = x^{1/2}$, so $f'(x) = \frac{1}{2\sqrt{x}} = (1/2)\,x^{(-1/2)}$.

Thus, $f(16) = \sqrt{16} = 4$ and $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$,

so $f(16 + 0.1) \approx 4 + \frac{1}{8} \cdot 0.1 = 4.0125$

(The actual value is 4.0124805... So this is pretty close.)

Problem 9 (8 points)

Each edge of a cube is growing at a rate of 3 cm/sec. How quickly is the volume of the cube increasing when the edges of the cube are exactly 12 cm long?

(Hint: The volume of a cube whose edges have length x is equal to x^3.)

We want to find the rate of change of volume, $\frac{dV}{dt}$

We know the relationship between volume and edge length: $V = x^3$

We know the rate of change of edge length: $\frac{dx}{dt} = 3$ cm/sec

We know we're asking about the time at which edges $x = 12$ cm

Taking the derivative $\frac{d}{dt}$ of the relationship $V = x^3$,

we get $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$, so

$$\frac{dV}{dt} = 3(12\text{ cm})^2 (3\text{cm/sec}) = 1296\text{ cm}^3/\text{sec}$$