

Let me go over the details of finding all of the 2×2 orthogonal matrices.

- Theorem 5.3.3 in the book (which I stated in class) says that an $n \times n$ matrix A is orthogonal if, and only if, its columns are an orthonormal basis of $\mathbb{R}^n$.
- So a 2×2 matrix A is orthogonal if and only if its columns $\vec{v}, \vec{w}$ are an orthonormal basis of $\mathbb{R}^2$. That means the columns must have length 1, and they must be orthogonal to each other:

$$\|\vec{v}\| = 1, \quad \|\vec{w}\| = 1, \quad \vec{v} \cdot \vec{w} = 0$$

- The vectors in $\mathbb{R}^2$ that have length 1 are precisely the vectors on the unit circle, so if $\vec{v}$ and $\vec{w}$ have length 1, then there are some angles θ and ψ such that

$$\vec{v} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \vec{w} = \begin{bmatrix} \cos(\psi) \\ \sin(\psi) \end{bmatrix}$$

- Since $\vec{v} \cdot \vec{w} = 0$, we have that

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\psi) \\ \sin(\psi) \end{bmatrix} = \cos(\theta) \cos(\psi) + \sin(\theta) \sin(\psi) = \cos(\theta - \psi) = 0$$

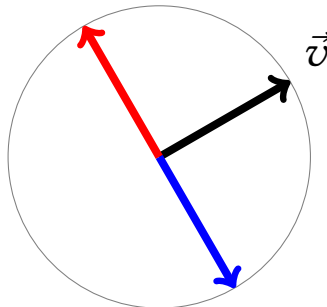
Therefore $\psi = \theta \pm \frac{\pi}{2}$.

Another way of seeing that result is that for any vector $\vec{v} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ on the unit circle, there are exactly two vectors on the unit circle that are perpendicular to $\vec{v}$, namely

$$\vec{v} \text{ rotated counterclockwise by } \frac{\pi}{2} = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix}$$

and

$$\vec{v} \text{ rotated clockwise by } \frac{\pi}{2} = \begin{bmatrix} \cos(\theta - \frac{\pi}{2}) \\ \sin(\theta - \frac{\pi}{2}) \end{bmatrix}$$



- So there are two possibilities:

$$\psi = \theta + \frac{\pi}{2} \quad \implies \quad \vec{w} = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = \vec{v} \text{ rotated counterclockwise by } \frac{\pi}{2}$$

and

$$\psi = \theta - \frac{\pi}{2} \quad \implies \quad \vec{w} = \begin{bmatrix} \cos(\theta - \frac{\pi}{2}) \\ \sin(\theta - \frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \end{bmatrix} = \vec{v} \text{ rotated clockwise by } \frac{\pi}{2}$$

- Thus, an orthogonal 2×2 matrix A is either of the form

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \underbrace{D_\theta}_{\substack{\text{counterclock} \\ \text{rotation by } \theta}}$$

for some angle θ , or it is of the form

$$A = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} = \underbrace{\underbrace{D_\theta}_{\substack{\text{counterclock} \\ \text{rotation by } \theta}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\substack{\text{reflection} \\ \text{about } x\text{-axis}}}}_{\substack{\text{reflection about the} \\ \text{line through angle } \frac{\theta}{2}}}$$

for some angle θ .

- We can summarize this by saying that the orthogonal linear transformations $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (that is, the ones that preserve length and perpendicularity) are just rotations and reflections. Note that rotations have determinant 1, and reflections have determinant -1 .