Math 19620 – Section 50 – Midterm 2

November 17, 2015

Information and Directions

- This exam will last 80 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Make sure to show all details of your work.

Name: Zev Chonoles

Problem	Points	Score
1	25	
2	25	
3	20	
4	20	
5	20	

Your score will be computed out of 100.

Problem 1 (25 points)

Consider this 3×5 matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 5 & 3 \\ 4 & 9 & 0 & 17 & 9 \\ 2 & 5 & -2 & 7 & 3 \end{bmatrix}$$

Part 1 (10 points)

Find a basis of the kernel of A.

First, find RREF(A)

$$\begin{bmatrix} 1 & 2 & 1 & 5 & 3 \\ 4 & 9 & 0 & 17 & 9 \\ 2 & 5 & -2 & 7 & 3 \end{bmatrix} \xrightarrow{\substack{(\text{row } 2) - 4(\text{row } 1) \\ (\text{row } 3) - 2(\text{row } 1)}} \begin{bmatrix} 1 & 2 & 1 & 5 & 3 \\ 0 & 1 & -4 & -3 & -3 \\ 0 & 1 & -4 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow{\substack{(\text{row } 1) - 2(\text{row } 2) \\ (\text{row } 3) - 1(\text{row } 2)}} \begin{bmatrix} 1 & 0 & 9 & 11 & 9 \\ 0 & 1 & -4 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ This is RREF(A)

For each column without a pivot, we get a relation:

$$9\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 9 \\ -4 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 11 \\ -3 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 9 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$11\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 9 \\ -4 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 11 \\ -3 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 9 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$9\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 9 \\ -4 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 11 \\ -3 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 9 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A basis of $\ker(A)$ is $\left\{ \begin{bmatrix} 9 \\ -4 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ -3 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

Part 2 (10 points)

Find a basis of the image of A.

For each column of RREF(A) with a pivot, take the corresponding column of A.

A basis of im(A) is $\left\{\begin{bmatrix}1\\4\\2\end{bmatrix}, \begin{bmatrix}2\\9\\5\end{bmatrix}\right\}$

Part 4 (5 points)

Complete the statement of the rank-nullity theorem: for any $m \times n$ matrix B, we have

dim(ker(B)) + dim(im(B)) = n

(fill in with mathematical expressions)

Use your work in this problem to find the specific values of each of these expressions for the matrix A. Make sure the integers you write below line up exactly with the corresponding expression above.

3 + 2 = 5

(fill in with integers)

Problem 2 (25 points)

Consider this 3×3 matrix and this basis of $\mathbb{R}^3$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 2 & 7 & -5 \end{bmatrix} \qquad \mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Part 1 (15 points)

Express the matrix A in the basis $\mathcal{B}$.

If B is the matrix A in the basis $\mathcal{B}$, then

$B = S^{-1}AS$ where $S = \left[\vec{v}_1 \middle| \vec{v}_2 \middle| \vec{v}_3\right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Need to find S^{-1}: take RREF of $[S \mid I_3]$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

Thus $S^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$.

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 2 & 7 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

This is A expressed in the basis $\mathcal{B}$.

Part 2 (10 points)

Let $\vec{w} = 5\vec{v}_1 + 7\vec{v}_2 + 3\vec{v}_3$, so that $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}$.

Express the vector $A\vec{w}$ in the basis $\mathcal{B}$, i.e. compute $[A\vec{w}]_{\mathcal{B}}$.

(You can do this directly, by computing $\vec{w}$, then computing $A\vec{w}$, then finding that vector's expression in the basis $\mathcal{B}$. There is also a much faster way using your answer from Part 1. Either is acceptable.)

Fast way: If B is the matrix A expressed in the basis $\mathcal{B}$, then $B[\vec{v}]_{\mathcal{B}} = [A\vec{v}]_{\mathcal{B}}$ for any vector $\vec{v} \in \mathbb{R}^3$

Thus $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$ is equal to $[A\vec{w}]_{\mathcal{B}}$.

Slow way: $\vec{w} = 5\vec{v}_1 + 7\vec{v}_2 + 3\vec{v}_3 = 5\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 7\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \\ 38 \end{bmatrix}$

$A\vec{w} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 2 & 7 & -5 \end{bmatrix} \begin{bmatrix} 15 \\ 28 \\ 38 \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \\ 36 \end{bmatrix}$

If $\vec{v}$ is any vector in $\mathbb{R}^3$, then $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v}$.

$[A\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 28 \\ 36 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix}$

Problem 3 (20 points)

Let V be the subspace of $\mathbb{R}^4$ with these vectors as a basis:

$$\left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Part 1 (10 points)

Find an orthonormal basis of V.

Gram-Schmidt:

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

$\vec{w}_2$ is formed by taking $\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1$ and dividing it by its length

$$\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - 0 \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ which has length } \sqrt{2}$$

so $\vec{w}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

$\vec{w}_3$ is formed by taking $\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2$ and dividing it by its length

$$\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} - 0\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ which has length } 1$$

so $\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Thus $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis of V

Part 2 (10 points)

Find the matrix of $\text{proj}_V : \mathbb{R}^4 \to \mathbb{R}^4$.

The matrix of proj_V is QQ^T, where $Q = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_k \\ | & & | \end{bmatrix}$

is a matrix whose columns are an orthonormal basis of V.

Thus, with $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we have

$$QQ^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 4 (20 points)

Part 1 (10 points)

Compute the value of $\det(A)$, where

$$A = \begin{bmatrix} 3 & 6 & 3 \\ 4 & 10 & 0 \\ 2 & 0 & 8 \end{bmatrix}$$

We use Gauss-Jordan elimination.

$\begin{bmatrix} 3 & 6 & 3 \\ 4 & 10 & 0 \\ 2 & 0 & 8 \end{bmatrix}$ This matrix has determinant = $\det(A)$

$\downarrow$ (row 1) $\cdot \frac{1}{3}$ Multiplies determinant by $\frac{1}{3}$

$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 10 & 0 \\ 2 & 0 & 8 \end{bmatrix}$ This matrix has determinant = $\frac{1}{3}\det(A)$

$\downarrow$ (row 2) $-$ 4(row 1)
(row 3) $-$ 2(row 1) Both leave determinant unchanged

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -4 \\ 0 & -4 & 6 \end{bmatrix}$ This matrix has determinant = $\frac{1}{3}\det(A)$

$\downarrow$ (row 2) $\cdot \frac{1}{2}$ Multiplies determinant by $\frac{1}{2}$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & -4 & 6 \end{bmatrix}$ This matrix has determinant = $\frac{1}{6}\det(A)$

$\downarrow$ (row 1) $-$ 2(row 2)
(row 3) $+$ 4(row 2) Both leave determinant unchanged

$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$ This matrix has determinant = $\frac{1}{6}\det(A)$

$\downarrow$ (row 3) $\cdot -\frac{1}{2}$ Multiplies determinant by $-\frac{1}{2}$

$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ This matrix has determinant = $-\frac{1}{12}\det(A)$

$\downarrow$ (row 1) $-$ 5(row 3)
(row 2) $+$ 2(row 3) Both leave determinant unchanged

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This matrix has determinant = $-\frac{1}{12}\det(A)$

But det(identity matrix) = 1. Thus $1 = -\frac{1}{12}\det(A)$, so

$$\det(A) = -12$$

Part 2 (10 points)

Compute the value of $\det(B)$, where

$$B = \begin{bmatrix} 3 & 6 & 3 \\ 4 & 10 & 0 \\ 14 & 0 & 56 \end{bmatrix}$$

Note that B is the same as A, except that one of its rows has been multiplied by 7.

Since det is multilinear in the rows of a matrix,

$$\det\left(\begin{bmatrix} v_1 \\ v_2 \\ 7v_3 \end{bmatrix}\right) = 7 \det\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right)$$

Thus $\det(B) = 7 \cdot \det(A) = 7 \cdot (-12) = -84$

Problem 5 (20 points)

Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix}$$

Part 1 (15 points)

Find the least-squares solutions of $A\vec{x} = \vec{b}$.

The least-squares solutions are the exact solutions to

$$A^TA\vec{x} = A^T\vec{b}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 & -2 \\ 4 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So we want to find the solutions to $\begin{bmatrix} 10 & 4 & -2 \\ 4 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Take RREF of augmented matrix:

$$\left[\begin{array}{ccc|c} 10 & 4 & -2 & 0 \\ 4 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2/5 & -1/5 & 0 \\ 4 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2/5 & -1/5 & 0 \\ 0 & 2/5 & 4/5 & 0 \\ 0 & 4/5 & 8/5 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2/5 & -1/5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 4/5 & 8/5 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is RREF

There is one free variable, corresponding to the column without a pivot.

Least-squares solutions to $A\vec{x} = \vec{b}$ are $\left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

Part 2 (5 points)

Find the vector in $\text{im}(A)$ that is closest to $\vec{b}$, otherwise known as $\text{proj}_{\text{im}(A)}(\vec{b})$.

(For extra credit up to an additional 5 points, give an intuitive explanation of the answer to this part.)

For any least-squares solution $\vec{x}$, the value of $A\vec{x}$ is $\text{proj}_{\text{im}(A)}(\vec{b})$. Indeed, the idea of a least-squares solution to $A\vec{x}=\vec{b}$ is to be a choice of $\vec{x}$ that makes the distance $\|\vec{b}-A\vec{x}\|$ as small as possible.

Thus, we can take any least-squares solution $\vec{x}$, and multiply it by A to get $\text{proj}_{\text{im}(A)}(\vec{b})$.

$$A\vec{x} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \text{proj}_{\text{im}(A)}\left(\begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix}\right).$$

Since $\text{im}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} r \\ r \\ s \\ s \end{bmatrix} : r, s \in \mathbb{R}\right\}$,

the subspace of $\mathbb{R}^4$ comprised of the vectors whose 1st and 2nd coordinates are equal and whose 3rd and 4th coordinates are equal, the vector in $\text{im}(A)$ that is closest to $\vec{b} = \begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix}$, a vector whose 1st and 2nd, and whose 3rd and 4th, coordinates are opposites, must be $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

To choose any r or s other than $r=s=0$ would only make a vector $\begin{bmatrix} r \\ r \\ s \\ s \end{bmatrix}$ further from $\begin{bmatrix} -1 \\ 1 \\ -3 \\ 3 \end{bmatrix}$.