

Find bases for both the kernel and image of

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 & 0 & 6 \\ 2 & 5 & -3 & 9 & 2 & 8 \end{bmatrix}$$

Explain how this relates to the rank-nullity theorem.

First, put A into RREF:

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 & 6 \\ 2 & 5 & -3 & 9 & 2 & 8 \end{bmatrix}$$

$(\text{row } 2) - 2 \cdot (\text{row } 1) \rightarrow$

$$\begin{bmatrix} 1 & 2 & 1 & 4 & 0 & 6 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix}$$

$(\text{row } 1) - 2 \cdot (\text{row } 2) \rightarrow$

$$\begin{bmatrix} 1 & 0 & 11 & 2 & -4 & 14 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix}$$

The original columns of A that correspond to the columns of $\text{RREF}(A)$ with pivots are a basis of $\text{im}(A)$.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\} \text{ is a basis of } \text{im}(A)$$

For each column of $\text{RREF}(A)$ without a pivot, there is a relation expressing it as a linear combination of the columns of $\text{RREF}(A)$ with pivots.

$$\begin{bmatrix} 11 \\ -5 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-5) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 2 \end{bmatrix} = (-4) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 14 \\ -4 \end{bmatrix} = 14 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rewriting these relations all on one side, and keeping track of the unused columns:

$$11 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-5) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 11 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 14 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 11 \\ -5 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 14 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-4) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 11 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -4 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 14 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$14 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 11 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 14 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words,

$$\begin{bmatrix} 1 & 0 & 11 & 2 & -4 & 14 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 2 & -4 & 14 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 2 & -4 & 14 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 11 & 2 & -4 & 14 \\ 0 & 1 & -5 & 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 14 \\ -4 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

These four vectors are a basis of $\ker(\text{RREF}(A))$. Since

$\ker(\text{RREF}(A)) = \ker(A)$, these four vectors are a basis of $\ker(A)$, too.

The rank-nullity theorem says that, for an $m \times n$ matrix A ,

$$\dim(\operatorname{im}(A)) + \dim(\operatorname{ker}(A)) = n$$

In this problem, A is 2×6 , and we found a basis of $\operatorname{im}(A)$ with 2 vectors, and a basis of $\operatorname{ker}(A)$ with 4 vectors, so

$$\begin{array}{rcl} \dim(\operatorname{im}(A)) & = & 2 \\ + \dim(\operatorname{ker}(A)) & = & 4 \\ \hline n & = & 6 \end{array}$$

as the rank-nullity theorem predicts.

Find the least-squares solutions to $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Are they exact solutions?

Least-squares solutions to $A\vec{x} = \vec{b}$ are the exact solutions to

$$A^T A \vec{x} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 6 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ -7 \end{bmatrix}$$

Find exact solutions to $\begin{bmatrix} 4 & 8 & 2 \\ 8 & 16 & 4 \\ 2 & 4 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 7 \\ 14 \\ -7 \end{bmatrix}$:

$$\left[\begin{array}{ccc|c} 4 & 8 & 2 & 7 \\ 8 & 16 & 4 & 14 \\ 2 & 4 & 6 & -7 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 7/4 \\ 8 & 16 & 4 & 14 \\ 2 & 4 & 6 & -7 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 7/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -21/2 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 7/4 \\ 0 & 0 & 5 & -21/2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1/2 & 7/4 \\ 0 & 0 & 1 & -21/10 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 14/5 \\ 0 & 0 & 1 & -21/10 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

So least-squares solutions to $A\vec{x} = \vec{b}$ are

$$\left\{ \begin{bmatrix} 14/5 - 2r \\ r \\ -21/10 \end{bmatrix} : r \in \mathbb{R} \right\}$$

For any least-squares solution $\vec{x} = \begin{bmatrix} 14/5 - 2r \\ r \\ -21/10 \end{bmatrix}$, we have

$$A\vec{x} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 14/5 - 2r \\ r \\ -21/10 \end{bmatrix} = \begin{bmatrix} (14/5 - 2r)1 + (r)2 + (-21/10)(-1) \\ (14/5 - 2r)1 + (r)2 + (-21/10)(0) \\ (14/5 - 2r)1 + (r)2 + (-21/10)(1) \\ (14/5 - 2r)1 + (r)2 + (-21/10)(2) \end{bmatrix} = \begin{bmatrix} 49/10 \\ 14/5 \\ 7/10 \\ -7/5 \end{bmatrix}$$

which is not equal to $\vec{b}$. So no, they are not exact solutions.

$$\begin{pmatrix} \begin{bmatrix} 49/10 \\ 14/5 \\ 7/10 \\ -7/5 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 2.8 \\ 0.7 \\ -1.4 \end{bmatrix} \text{ is as close as } A\vec{x} \text{ can ever be to } \vec{b} = \begin{bmatrix} 7 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{pmatrix}$$

COMMENTS

Can also check that there are no exact solutions by usual method

of forming augmented matrix $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array} \right]$, taking its

RREF, which is $\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{array} \right]$, and the fact that we get a row

$[0 \ 0 \ \dots \ 0 \ | \ 1]$ means the system is inconsistent.

Since the columns of A are obviously linearly dependent, $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, that means $\text{Ker}(A)$ is more than just $\{\vec{0}\}$, so we could know immediately that there would be more than one least-squares solution.

Find the determinant of A via Gaussian elimination.

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 6 & 0 \\ 2 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 3 & 6 & 0 \\ 2 & 5 & 2 \end{bmatrix}$$

The determinant of this matrix is $\det(A)$.

$\left. \begin{array}{l} (\text{row } 2) - 3(\text{row } 1) \\ (\text{row } 3) - 2(\text{row } 1) \end{array} \right\}$ Both leave determinant unchanged.

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & -6 & -6 \\ 0 & -3 & -2 \end{bmatrix}$$

The determinant of this matrix is $\det(A)$.

$\left. \begin{array}{l} (\text{row } 2) \text{ times } (-\frac{1}{6}) \end{array} \right\}$ Multiplies determinant by $(-\frac{1}{6})$.

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{bmatrix}$$

The determinant of this matrix is $(-\frac{1}{6})\det(A)$.

$\left. \begin{array}{l} (\text{row } 1) - 4(\text{row } 2) \\ (\text{row } 3) + 3(\text{row } 2) \end{array} \right\}$ Both leave determinant unchanged.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of this matrix is $(-\frac{1}{6})\det(A)$.

$\left. \begin{array}{l} (\text{row } 1) + 2(\text{row } 3) \\ (\text{row } 2) - 1(\text{row } 3) \end{array} \right\}$ Both leave determinant unchanged.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of this matrix is $(-\frac{1}{6})\det(A)$.

But determinant of the identity matrix is 1. Thus

$$1 = (-\frac{1}{6})\det(A) \Rightarrow \det(A) = -6$$

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{x}$ are vectors in $\mathbb{R}^4$ such that

$$\det \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} = 12 \quad \det \begin{bmatrix} \vec{x} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} = -5$$

then what is the value of $\det \begin{bmatrix} 2\vec{a} + 2\vec{x} & \vec{d} & 3\vec{c} & \vec{b} \end{bmatrix}$?

$\det$ is multilinear as a function of the columns of a matrix, so

$$\begin{aligned} \det \begin{bmatrix} 2\vec{a} + 2\vec{x} & \vec{d} & 3\vec{c} & \vec{b} \end{bmatrix} &= 2 \cdot \det \begin{bmatrix} \vec{a} + \vec{x} & \vec{d} & 3\vec{c} & \vec{b} \end{bmatrix} \\ &= 6 \cdot \det \begin{bmatrix} \vec{a} + \vec{x} & \vec{d} & \vec{c} & \vec{b} \end{bmatrix} \\ &= 6 \cdot \det \begin{bmatrix} \vec{a} & \vec{d} & \vec{c} & \vec{b} \end{bmatrix} + 6 \cdot \det \begin{bmatrix} \vec{x} & \vec{d} & \vec{c} & \vec{b} \end{bmatrix} \end{aligned}$$

It takes one swap to get the columns in an order where we know the value of the determinant, and since the determinant is alternating,

$$\begin{aligned} &= (-6) \cdot \det \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} + (-6) \cdot \det \begin{bmatrix} \vec{x} & \vec{b} & \vec{c} & \vec{d} \end{bmatrix} \\ &= (-6) \cdot 12 + (-6) \cdot (-5) \\ &= \boxed{-42} \end{aligned}$$

Are these orthonormal bases of $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

No. The first basis consists of vectors that are all orthogonal to each other, but they are not unit vectors, so not orthonormal.

The second basis consists of unit vectors, but they are not orthogonal to each other, so not orthonormal.

Express the matrix $A = \begin{bmatrix} 4 & 3 & -2 \\ -2 & 11 & -4 \\ -6 & 15 & -4 \end{bmatrix}$ in the basis

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}. \text{ Interpret the result.}$$

The matrix A in the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is $S^{-1}AS$, where

$$S = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3]. \text{ So we compute } S^{-1} \text{ first:}$$

$$[S | I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array} \right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array} \right]$$

$$\text{Thus } S^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix}. \text{ Now,}$$

$$S^{-1}AS = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -2 \\ -2 & 11 & -4 \\ -6 & 15 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ -2 & 3 & -1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 4 & 2 \\ 5 & 8 & 4 \\ 5 & 12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(Here, we multiplied A and S first, then multiplied the result by S^{-1} , but we could have done S^{-1} and A first, then S , since matrix multiplication is associative: $X \text{ times } (Y \text{ times } Z) = (X \text{ times } Y) \text{ times } Z$

Just be careful not to change the order, since matrix multiplication is not commutative:

$$X \text{ times } Y \neq Y \text{ times } X$$

The matrix $B = S^{-1}AS$ is the matrix with the property that

$$B[\vec{v}]_{\mathcal{B}} = [A\vec{v}]_{\mathcal{B}} \quad \text{for any vector } \vec{v}.$$

Since

$$B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$B \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = 4 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$B \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$S \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [\vec{v}_1 | \vec{v}_2 | \vec{v}_3] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we must have that

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

(As we'll see after the midterm,
the numbers 5, 4, 2 are called eigenvalues of A ,
and the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ are called eigenvectors of A .)

Let V be the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}$.

Find an orthonormal basis of V , and an orthonormal basis of $V^\perp$.

First, we need a basis of V .

We're told that $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix}\right)$, or in other words,

$$V = \text{im}(A) \text{ where } A = \begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 1 & 4 & 2 \\ 1 & 0 & -1 \end{bmatrix}.$$

To find a basis of $\text{im}(A)$, we compute $\text{RREF}(A)$ and take the columns of A that end up with pivots.

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 1 & 4 & 2 \\ 1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \\ 0 & -4 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \\ 0 & -4 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \textcircled{1} & 0 & -1 \\ 0 & \textcircled{1} & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, we've found that $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}\right\}$ is a basis of V .

Now use Gram-Schmidt to make an orthonormal basis of V .

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} \quad \vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$\vec{w}_2$ is made by taking $\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1) \vec{w}_1$, and then dividing it by its length (to make a unit vector)

$$\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1) \vec{w}_1 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \text{ has length } \sqrt{2^2 + (-2)^2 + 2^2 + (-2)^2} = 4$$

so $\vec{w}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$. Thus, $\left\{\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}\right\}$ is an orthonormal basis of V .

Approach 1 to getting an orthonormal basis of $V^\perp$: use $\text{im}(A)^\perp = \text{ker}(A^T)$

We now can write V in many ways:

$$V = \text{im} \left(\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & -1 \\ 1 & 4 & 2 \\ 1 & 0 & -1 \end{bmatrix} \right)$$

(original)

$$V = \text{im} \left(\begin{bmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 4 \\ 1 & 0 \end{bmatrix} \right)$$

(basis)

$$V = \text{im} \left(\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \right)$$

(orthonormal basis)

(I'll use the second option, but mathematically any are valid and will lead to a correct answer.)

Since $V = \text{im} \left(\begin{bmatrix} 1 & 4 \\ 1 & 0 \\ 1 & 4 \\ 1 & 0 \end{bmatrix} \right)$, we have that $V^\perp = \text{ker} \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 4 & 0 \end{bmatrix} \right)$.

We'll find a basis for this kernel, then make an orthonormal basis from it.

First, we put into RREF:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 4 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 0 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The relations are very simple:

(third column) (first column) (second column)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(fourth column) (first column) (second column)

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\} \text{ is a basis of } V^\perp$$

Now use Gram-Schmidt to make an orthonormal basis:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

to get $\vec{w}_2$, take $\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1) \vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - 0 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ and divide it by its length.

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}. \quad \text{Thus, } \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\} \text{ is an orthonormal basis of } V^\perp.$$

Approach 2 to getting an orthonormal basis of $V^\perp$: use $V^\perp = \ker(\text{proj}_V)$.

Since $V^\perp = \ker(\text{proj}_V)$, we can find the matrix for $\text{proj}_V: \mathbb{R}^4 \rightarrow \mathbb{R}^4$,

then get a basis for its kernel, then use Gram-Schmidt.

The matrix for proj_V is QQ^T where Q is a matrix whose columns are an orthonormal basis of V . (p.232)

$$\text{matrix of } \text{proj}_V = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

To find a basis of $\ker(\text{proj}_V)$, put it into RREF:

$$\begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The relations are the same as in Approach 1, so the process from here is essentially identical.