

Math 19620 – Section 50 – Midterm 1

October 22, 2015

Information and Directions

- This exam will last 80 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Make sure to show all details of your work.

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Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	

Problem 1 (20 points)

Consider this system of linear equations:

$$1x + 4y + 15z = 12$$

$$2x - 2y + 20z = 4$$

$$5y + 5z = 10$$

Part 1 (4 points)

Write down the augmented matrix that represents this system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 4 & 15 & 12 \\ 2 & -2 & 20 & 4 \\ 0 & 5 & 5 & 10 \end{array} \right]$$

Part 2 (8 points)

Put the augmented matrix into reduced row echelon form. Circle the pivots.

$$\left[\begin{array}{ccc|c} 1 & 4 & 15 & 12 \\ 2 & -2 & 20 & 4 \\ 0 & 5 & 5 & 10 \end{array} \right] \xrightarrow{(\text{row } 2) - 2 \cdot (\text{row } 1)} \left[\begin{array}{ccc|c} 1 & 4 & 15 & 12 \\ 0 & -10 & -10 & -20 \\ 0 & 5 & 5 & 10 \end{array} \right]$$

$$\xrightarrow{(\text{row } 2) \text{ times } -\frac{1}{10}} \left[\begin{array}{ccc|c} 1 & 4 & 15 & 12 \\ 0 & 1 & 1 & 2 \\ 0 & 5 & 5 & 10 \end{array} \right] \xrightarrow{\substack{(\text{row } 1) - 4 \cdot (\text{row } 2) \\ (\text{row } 3) - 5 \cdot (\text{row } 2)}} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

pivots are circled

Part 3 (8 points)

Find all solutions of the system of linear equations.

pivots correspond to non-free variables

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightsquigarrow \begin{array}{l} x + 11z = 4 \\ y + z = 2 \end{array}$$

$$x = 4 - 11z$$

$$y = 2 - z$$

z is a free variable

$$\text{Set of solutions is } \left\{ \begin{bmatrix} 4-11r \\ 2-r \\ r \end{bmatrix} : r \in \mathbb{R} \right\}$$

Problem 2 (20 points)

Consider the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 \\ 9x_1 - x_2 \\ x_1 + 5x_2 \end{bmatrix}$$

Is this a linear transformation?

If so, find the matrix A such that $f(\vec{x}) = A\vec{x}$. If not, explain how you know it is not linear.

Approach 1: f is linear because the matrix $A = \begin{bmatrix} 3 & 2 \\ 9 & -1 \\ 1 & 5 \end{bmatrix}$ satisfies

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \text{ This also tells the matrix } A \text{ immediately.}$$

Approach 2: f is linear because, for any $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$,

$$\begin{aligned} \bullet f(\vec{x} + \vec{y}) &= f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 3(x_1 + y_1) + 2(x_2 + y_2) \\ 9(x_1 + y_1) - (x_2 + y_2) \\ (x_1 + y_1) + 5(x_2 + y_2) \end{bmatrix} \\ &= \begin{bmatrix} 3x_1 + 2x_2 + 3y_1 + 2y_2 \\ 9x_1 - x_2 + 9y_1 - y_2 \\ x_1 + 5x_2 + y_1 + 5y_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 \\ 9x_1 - x_2 \\ x_1 + 5x_2 \end{bmatrix} + \begin{bmatrix} 3y_1 + 2y_2 \\ 9y_1 - y_2 \\ y_1 + 5y_2 \end{bmatrix} \\ &= f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + f\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = f(\vec{x}) + f(\vec{y}) \end{aligned}$$

For any real number k ,

$$\bullet f(k\vec{x}) = f\left(k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix}\right) = \begin{bmatrix} 3kx_1 + 2kx_2 \\ 9kx_1 - kx_2 \\ kx_1 + 5kx_2 \end{bmatrix} = \begin{bmatrix} k(3x_1 + 2x_2) \\ k(9x_1 - x_2) \\ k(x_1 + 5x_2) \end{bmatrix} = k \begin{bmatrix} 3x_1 + 2x_2 \\ 9x_1 - x_2 \\ x_1 + 5x_2 \end{bmatrix} = k f(\vec{x})$$

By Theorem 2.1.2, the columns of A are

$$f(\vec{e}_1) = f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix} \quad f(\vec{e}_2) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \quad = k f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = k f(\vec{x})$$

$$\text{Thus, } A = \begin{bmatrix} 3 & 2 \\ 9 & -1 \\ 1 & 5 \end{bmatrix}.$$

Problem 3 (20 points)

Part 1 (5 points)

Write down the 2×2 matrix D_α that represents counterclockwise rotation by an angle α .

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Part 2 (5 points)

Two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ are perpendicular if and only if their dot product $\vec{v} \cdot \vec{w}$ equals 0.

Part 3 (10 points)

Since D_α is a 2×2 matrix, and $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ is a 2×1 vector, their product $D_\alpha \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ is a 2×1 vector.

Find a value of α such that the vector $D_\alpha \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ is perpendicular to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}\cos(\alpha) - \sin(\alpha) \\ \sqrt{3}\sin(\alpha) + \cos(\alpha) \end{bmatrix}$$

$$\begin{aligned} \text{dot product of } \begin{bmatrix} \sqrt{3}\cos(\alpha) - \sin(\alpha) \\ \sqrt{3}\sin(\alpha) + \cos(\alpha) \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{ is } (\sqrt{3}\cos(\alpha) - \sin(\alpha)) \cdot (1) + (\sqrt{3}\sin(\alpha) + \cos(\alpha)) \cdot (0) \\ & = \sqrt{3}\cos(\alpha) - \sin(\alpha) \end{aligned}$$

Thus, $D_\alpha \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ when $\sqrt{3}\cos(\alpha) - \sin(\alpha) = 0$,

i.e. $\tan(\alpha) = \sqrt{3}$, which is true for $\alpha = \frac{\pi}{3}$ or $\alpha = \frac{4\pi}{3}$, or $\alpha = \frac{7\pi}{3}, \dots$
or $\alpha = -\frac{2\pi}{3}$, or $\alpha = -\frac{5\pi}{3}, \dots$

Problem 4 (20 points)

Consider this 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Part 1 (10 points)

Does the matrix A have an inverse?

If so, find it. If not, explain how you know there is no inverse.

We want to find RREF of $[A | I_3]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\substack{(\text{row } 3) - (\text{row } 1) \\ (\text{row } 2) - 2(\text{row } 1)}} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -2 & 3 & -1 & 0 & 1 \end{array} \right]$$

$(\text{row } 1) - 2(\text{row } 2)$
 $(\text{row } 3) + 2(\text{row } 2)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right] \xrightarrow{(\text{row } 3) \text{ times } (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$(\text{row } 1) - 5(\text{row } 3)$
 $(\text{row } 2) + 2(\text{row } 3)$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -20 & 8 & 5 \\ 0 & 1 & 0 & 8 & -3 & -2 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

Since this is of the form $[I_3 | B]$ for some matrix B , we conclude from Theorem 2.4.5 that A is invertible and that

$$A^{-1} = \begin{bmatrix} -20 & 8 & 5 \\ 8 & -3 & -2 \\ 5 & -2 & -1 \end{bmatrix}$$

Part 2 (10 points)

What is the rank of the matrix A ?

Show your work, or explain how you know. You can appeal to Part 1 and theorems from the book.

Approach 1: By Part 1, we know A is invertible.

By Theorem 2.4.3, an $n \times n$ matrix is invertible $\Leftrightarrow$ its rank $= n$.

Thus, the invertible 3×3 matrix A has $\text{rank}(A) = 3$.

Approach 2: The left half of our computation in Part 1 is just RREF of A .

Since $\text{RREF}(A)$ has 3 pivots, we conclude that $\text{rank}(A) = 3$.

Approach 3: (Do hard work from Part 1 again.)

Problem 5 (20 points)

Let the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, and consider the linear transformation $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Part 1 (10 points)

Find the kernel of f , also known as the kernel of A .

$$\ker(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Solving

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ means solving the system of linear equations } \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \end{cases}$$

So we form the augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array} \right]$, find RREF of it:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{array} \right] \xrightarrow{\text{row 2} - 2(\text{row 1})} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \text{ This is RREF.}$$

Free variables are x_2, x_3 ; and x_1 is non-free since there is a pivot in its column.

$$x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 = -2x_2 - 3x_3$$

x_2, x_3 can be any real numbers

Kernel of $A =$

$$\text{Set of solutions to system} = \left\{ \begin{bmatrix} -2r - 3s \\ r \\ s \end{bmatrix} : r, s \in \mathbb{R} \right\}$$

Part 2 (5 points)

Find the image of f , also known as the image of A .

Your answer should be expressed as the span of some set of vectors.

The image of A is the span of its columns.

$$\text{im}(A) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right)$$

Because there are redundant columns, it is also true that

$$\text{im}(A) = \text{span} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right)$$

However, I was not requiring you to observe this.

Part 3 (5 points)

Are the columns of the matrix A linearly independent?

If so, explain how you know. If not, write a non-trivial linear relation between the columns.

No, the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are not linearly independent.

Examples of non-trivial linear relations:

$$(2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-6) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\frac{3}{2}\right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$