

$$\begin{cases} x + y - z = -2 \\ 3x - 5y + 13z = 18 \\ x - 2y + 5z = 7 \end{cases}$$

Find all solutions

{ augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 3 & -5 & 13 & 18 \\ 1 & -2 & 5 & 7 \end{array} \right]$$

{ (row 2) minus 3 · (row 1)
(row 3) minus 1 · (row 1)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -8 & 16 & 24 \\ 0 & -3 & 6 & 9 \end{array} \right]$$

{ (row 2) times $-\frac{1}{8}$
(row 3) times $-\frac{1}{3}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 1 & -2 & -3 \end{array} \right]$$

$$\text{set of solutions} = \left\{ \begin{bmatrix} 1-z \\ -3+2z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(The pivots are circled)

(row 1) minus
1 · (row 2)

(row 3) minus
1 · (row 2)

(x, y)

The pivots correspond
to non-free variables

They are determined by
the free variables,
(z)

$$\begin{cases} x + z = 1 \\ y - 2z = -3 \end{cases}$$

$$\begin{cases} x = 1 - z \\ y = -3 + 2z \end{cases}$$

Look at
p. 15

The following matrices are in reduced row echelon form

(make sure you understand why!)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 17 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[1 \quad 9 \quad -5 \quad 3 \quad 2]$$

$$[1]$$

$$\begin{bmatrix} 1 & 0 & 7 & 3 & 0 \\ 0 & 1 & 8 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivots are circled

Look at
p. 16

What value of k makes the vector $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

a multiple of the vector $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$?

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3k \\ 3 \end{bmatrix}$$

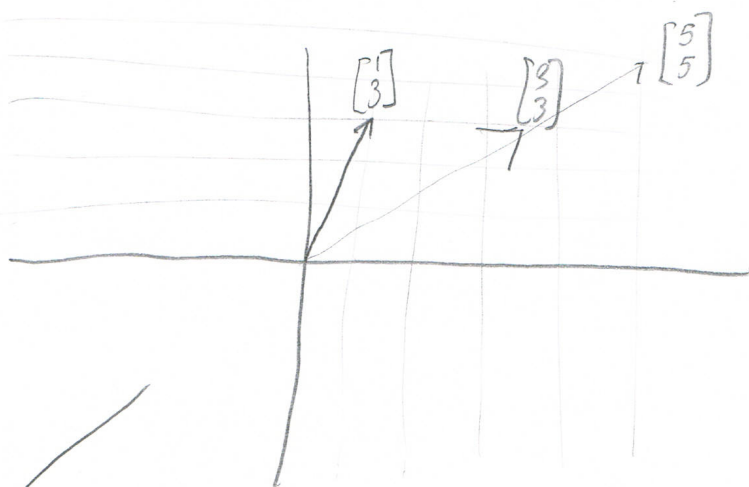
In order to have $\begin{bmatrix} 1+3k \\ 3 \end{bmatrix} = r \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ for some

real number r , must be $r = \frac{3}{5}$, hence

$$1 + 3k = \frac{3}{5} \cdot 5 = 3$$

$$3k = 2$$

$$k = \frac{2}{3}$$



$$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \checkmark$$

Are the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ linearly independent?

Approach 1: Clearly, no. ("by observation")

$$5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$9 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(-\frac{9}{2}\right) \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

etc. are all obvious non-trivial relations.

Approach 2: Is the rank of $\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ equal to 3?

$$\text{ref} \left(\begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Look at
p. 129

only one pivot $\Rightarrow$ rank = 1, not 3

Find $\ker \left(\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 7 & 9 & 5 \end{bmatrix} \right)$

So we want to find the set of $\vec{x} \in \mathbb{R}^4$ such that

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 7 & 9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is just solving a system of linear equations!

Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 5 & 0 \\ 2 & 7 & 9 & 5 & 0 \end{array} \right]$$



$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 5 & 0 \\ 0 & 1 & 1 & -5 & 0 \end{array} \right]$$



$$\left[\begin{array}{cccc|c} \textcircled{1} & 0 & 1 & 20 & 0 \\ 0 & \textcircled{1} & 1 & -5 & 0 \end{array} \right]$$

non-free variables

x_1, x_2

free variables

x_3, x_4

$$x_1 + x_3 + 20x_4 = 0$$

$$x_2 + x_3 + (-5)x_4 = 0$$

$$x_1 = -x_3 - 20x_4$$

$$x_2 = -x_3 + 5x_4$$

set of solutions =

$$\left\{ \begin{bmatrix} -r - 20s \\ -r + 5s \\ r \\ s \end{bmatrix} : r, s \in \mathbb{R} \right\}$$

= kernel of our matrix

What is the rank of $\begin{bmatrix} 17 & 24 \\ 9 & 5 \end{bmatrix}$?

Approach 1: direct computation (tedious)

need to
find
ref $\begin{bmatrix} 17 & 24 \\ 9 & 5 \end{bmatrix}$

$$\begin{bmatrix} 17 & 24 \\ 9 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{24}{17} \\ 9 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{24}{17} \\ 0 & -\frac{131}{17} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \frac{24}{17} \\ 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} \textcircled{1} & 0 \\ 0 & \textcircled{1} \end{bmatrix} \quad 2 \text{ pivots} \Rightarrow \text{rank} = 2$$

Approach 2: appeal to theorems (you must show around this level of explanation to get credit!)

We saw in the book that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if

$ad - bc \neq 0$. Since $85 - 216 = -131$ is not 0,

The matrix is invertible. The book also mentions that

an $n \times n$ matrix is invertible if and only if its rank is n .

Thus rank of $\begin{bmatrix} 17 & 24 \\ 9 & 5 \end{bmatrix}$ is 2.