

Math 19620 – Section 50 – Final Exam

December 10, 2015

Information and Directions

- This exam will last 120 minutes.
- This is a closed-book exam.
- No electronic devices are allowed to be used during this exam.
- Make sure to show all details of your work.

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Problem	Points	Score
1	20	
2	20	
3	20	
4	15	
5	10	
6	10	
7	5	

Problem 1 (20 points)

Consider the linear transformation $f : \mathbb{R}^5 \rightarrow \mathbb{R}^3$ defined by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_3 - 2x_4 + 8x_5 \\ x_1 + 2x_3 + 7x_5 \\ 3x_1 - 3x_4 + 9x_5 \end{bmatrix}$$

Part 1 (5 points)

Find the solutions to $f(\vec{x}) = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$.

$$f(\vec{x}) = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} 2 & 0 & 2 & -2 & 8 \\ 1 & 0 & 2 & 0 & 7 \\ 3 & 0 & 0 & -3 & 9 \end{bmatrix}$$

Form augmented matrix, take RREF:

$$\left[\begin{array}{ccccc|c} 2 & 0 & 2 & -2 & 8 & 4 \\ 1 & 0 & 2 & 0 & 7 & 6 \\ 3 & 0 & 0 & -3 & 9 & 3 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 4 & 2 \\ 1 & 0 & 2 & 0 & 7 & 6 \\ 3 & 0 & 0 & -3 & 9 & 3 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 4 & 2 \\ 0 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & -3 & 0 & -3 & -3 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 3 & 6 & 9 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 5 & 4 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array} \right]$$

pivots are circled

x_2, x_5 are free variables

$$x_1 + 5x_5 = 4 \quad \rightarrow \quad x_1 = 4 - 5x_5$$

$$x_3 + 1x_5 = 1 \quad \rightarrow \quad x_3 = 1 - x_5$$

$$x_4 + 2x_5 = 3 \quad \rightarrow \quad x_4 = 3 - 2x_5$$

$$\text{Solutions are } \left\{ \begin{bmatrix} 4-5r \\ s \\ 1-r \\ 3-2r \\ r \end{bmatrix} : r, s \in \mathbb{R} \right\}$$

Part 2 (10 points)

Find a basis of the kernel of f .

In part 1, we found that RREF of A is $\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

For each column without a pivot, relation gives an element of a basis of the kernel.

$$0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ is a basis of $\ker(f)$.

Part 3 (5 points)

Find a basis of the image of f .

In part 1, we found that RREF of A is $\begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

Columns of A that ended up with pivots form a basis of image

Thus $\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \right\}$ is a basis of $\text{im}(f)$.

Problem 2 (20 points)

Consider this 3×3 matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Part 1 (10 points)

Is A invertible? If so, find its inverse. If not, explain how you know it is not invertible.

$$\text{RREF}[A | I_3] = \text{RREF} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & -1/2 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1/2 & -1/2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1/2 & -1/2 \end{array} \right] \quad \text{This is RREF}$$

But the left side is not I_3 , so A is not invertible

Part 2 (10 points)

Is A diagonalizable? If so, find a basis of $\mathbb{R}^3$ in which A becomes diagonal, and find what diagonal matrix A becomes in this basis. If not, explain how you know it is not diagonalizable.

$$\text{char poly } f_A(x) = \det(A - xI) = \det \begin{bmatrix} 1-x & -1 & 1 \\ 2 & -1-x & 0 \\ 0 & -1 & 2-x \end{bmatrix}$$

Laplace cofactor expansion:

$$f_A(x) = (1-x) \det \begin{bmatrix} -1-x & 0 \\ -1 & 2-x \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & 0 \\ 0 & 2-x \end{bmatrix} + 1 \det \begin{bmatrix} 2 & -1-x \\ 0 & -1 \end{bmatrix}$$

$$= (1-x)(x^2 - x - 2) - (-1)(-2x + 4) + 1(-2)$$

$$= x^2 - x - 2 - x^3 + x^2 + 2x - 2x + 4 - 2$$

$$= -x^3 + 2x^2 - x$$

$$= (-1)(x)(x^2 - 2x + 1)$$

$$= (-1)(x-0)(x-1)^2$$

Thus eigenvalues of A are 0 and 1

$$\text{alg mult}(0) = 1 \quad \text{alg mult}(1) = 2$$

Eigenspace associated to the eigenvalue 0 is

$$E_0 = \ker(A - 0I) = \ker\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \ker\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

Take RREF to find basis of eigenspace

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{This is RREF}$$

Relation for column without pivot:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of E_0 , so $\text{geomult}(0) = \dim(E_0) = 1$

Eigenspace associated to the eigenvalue 1 is

$$E_1 = \ker(A - 1I) = \ker\left(\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \ker\begin{bmatrix} 0 & -1 & 1 \\ 2 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Take RREF to find basis of eigenspace

$$\begin{bmatrix} 0 & -1 & 1 \\ 2 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is RREF

Relation for column without pivot:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Thus } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis of } E_1, \text{ so}$$

$$\text{geomult}(1) = \dim(E_1) = 1$$

We have $\text{geomult}(0) = 1 = \text{algmult}(0)$, but $\text{geomult}(1) = 1 < 2 = \text{algmult}(1)$.

Equivalently, we have fewer than 3 vectors for our desired basis of eigenvectors.

Thus, A is not diagonalizable.

Problem 3 (20 points)

Let $V = \text{im}(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 3 & 4 & 5 \\ 3 & 3 & 3 \end{bmatrix}$$

Part 1 (10 points)

Find an orthonormal basis of V .

First, need to find a basis of V . Take RREF of A

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 3 & 4 & 5 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{This is RREF}$$

Thus, $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$ are a basis of $V = \text{im}(A)$

Gram-Schmidt:

$$\vec{w}_1 = \frac{1}{\|\vec{v}_1\|} \vec{v}_1 = \frac{1}{\sqrt{36}} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{aligned} \vec{v}_2^\perp &= \vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1) \vec{w}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} - \underbrace{\left(\frac{1}{6} + 3\left(\frac{1}{3}\right) + \frac{1}{3} + 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right) \right)}_6 \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\vec{w}_2 = \frac{1}{\|\vec{v}_2^\perp\|} \vec{v}_2^\perp = \frac{1}{\sqrt{4}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix} \right\}$

is an orthonormal basis of V .

Part 2 (5 points)

What is the dimension of $V^\perp$? Explain your answer.

For any subspace W of any $\mathbb{R}^n$, we have $\dim(W) + \dim(W^\perp) = n$, because $W = \text{im}(\text{proj}_W)$ and $W^\perp = \ker(\text{proj}_W)$ for the linear transformation $\text{proj}_W: \mathbb{R}^n \rightarrow \mathbb{R}^n$, then we apply the rank-nullity theorem.

V is a subspace of $\mathbb{R}^6$, and we found a basis of V with 2 elements, so $\dim(V) = 2$. Thus $\dim(V^\perp) = 6 - 2 = 4$.

Part 3 (5 points)

Find a basis of $V^\perp$.

Since $V = \text{im}(A)$, we have $V^\perp = \text{im}(A)^\perp = \ker(A^T)$.

Now we find a basis of the kernel, take RREF

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 3 \\ 1 & 3 & 1 & 2 & 4 & 3 \\ 1 & 4 & 0 & 1 & 5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -2 & -2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & 5 & 1 & 3 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Relations for columns without pivots:

$$4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis of $V^\perp$.

Problem 4 (15 points)

Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Part 1 (10 points)

Find the least-squares solutions of $A\vec{x} = \vec{b}$.

Least-squares solutions of $A\vec{x} = \vec{b}$ are exact solutions of $(A^T A)\vec{x} = (A^T \vec{b})$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 20 & 32 \\ 12 & 32 & 52 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 3 \\ 1 & 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 24 \end{bmatrix}$$

Solving $(A^T A)\vec{x} = (A^T \vec{b}) \rightarrow$ put augmented matrix in RREF

$$\left[\begin{array}{ccc|c} 4 & 8 & 12 & 8 \\ 8 & 20 & 32 & 16 \\ 12 & 32 & 52 & 24 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 8 & 20 & 32 & 16 \\ 12 & 32 & 52 & 24 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 4 & 8 & 0 \\ 0 & 8 & 16 & 0 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 16 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{This is RREF}$$

x_3 is a free variable

$$x_1 - x_3 = 2$$

$$x_1 = 2 + x_3$$

$$x_2 + 2x_3 = 0$$

$$x_2 = -2x_3$$

Thus, least-squares solutions are $\left\{ \begin{bmatrix} 2+r \\ -2r \\ r \end{bmatrix} : r \in \mathbb{R} \right\}$

Part 2 (5 points)

Which vector in $\text{im}(A)$ is the closest to $\vec{b}$?

For any subspace V of $\mathbb{R}^n$, and any vector $\vec{v} \in \mathbb{R}^n$,

the vector in V that is closest to $\vec{v}$ is always $\text{proj}_V(\vec{v})$.

(arguably, by definition)

Thus we want $\text{proj}_{\text{im}(A)}(\vec{b})$.

Least-squares solutions to $A\vec{x} = \vec{b}$ are exact solutions to $A\vec{x} = \text{proj}_{\text{im}(A)}(\vec{b})$.

Thus, for any least-squares solution $\vec{x}$, we can find $\text{proj}_{\text{im}(A)}(\vec{b})$ by multiplying by A .

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2+r \\ -2r \\ r \end{bmatrix} = \begin{bmatrix} (2+r) - 2r + r \\ (2+r) - 6r + 5r \\ (2+r) - 2r + r \\ (2+r) - 6r + 5r \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \text{proj}_{\text{im}(A)}(\vec{b})$$

Problem 5 (10 points)

Find a value of α where the parallelogram in $\mathbb{R}^2$ formed by $D_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $D_{2\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has area 1.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \quad \sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha) \quad \sin^2(\alpha) + \cos^2(\alpha) = 1$$

$$D_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha) - \sin(\alpha) \\ \sin(\alpha) + \cos(\alpha) \end{bmatrix}$$

$$D_{2\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(2\alpha) - \sin(2\alpha) \\ \sin(2\alpha) + \cos(2\alpha) \end{bmatrix}$$

Area of parallelogram formed by $D_\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $D_{2\alpha} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is $|\det(A)|$

$$\text{Where } A = \begin{bmatrix} \cos(\alpha) - \sin(\alpha) & \cos(2\alpha) - \sin(2\alpha) \\ \sin(\alpha) + \cos(\alpha) & \sin(2\alpha) + \cos(2\alpha) \end{bmatrix}$$

$$\det(A) = (\cos(\alpha) - \sin(\alpha))(\sin(2\alpha) + \cos(2\alpha)) - (\sin(\alpha) + \cos(\alpha))(\cos(2\alpha) - \sin(2\alpha))$$

$$= (\cos(\alpha)\sin(2\alpha) + \cancel{\cos(\alpha)\cos(2\alpha)} - \cancel{\sin(\alpha)\sin(2\alpha)} - \sin(\alpha)\cos(2\alpha))$$

$$- (\sin(\alpha)\cos(2\alpha) - \cancel{\sin(\alpha)\sin(2\alpha)} + \cancel{\cos(\alpha)\cos(2\alpha)} - \cos(\alpha)\sin(2\alpha))$$

$$= 2\cos(\alpha)\sin(2\alpha) - 2\sin(\alpha)\cos(2\alpha)$$

$$= 4\sin(\alpha)\cos^2(\alpha) - 2\sin(\alpha)\cos^2(\alpha) + 2\sin^3(\alpha)$$

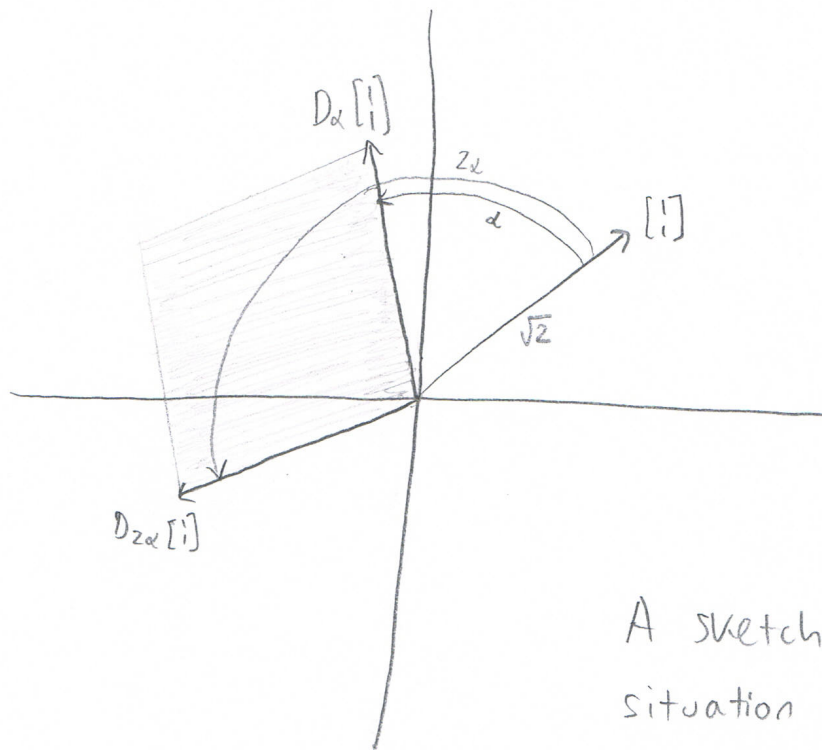
$$= 2\sin^3(\alpha) + 2\sin(\alpha)\cos^2(\alpha)$$

$$= 2\sin(\alpha)(\sin^2(\alpha) + \cos^2(\alpha))$$

$$= 2\sin(\alpha)$$



Thus, we need an α with $|2\sin(\alpha)| = 1$, i.e. $\sin(\alpha) = \pm \frac{1}{2}$. $\alpha = 30^\circ$ works.



A sketch of the
situation in Problem 5

Problem 6 (10 points)

Let A be the 2×2 matrix that represents scaling $\mathbb{R}^2$ by a factor of 5.

Let B be the 2×2 matrix that represents orthogonal projection of $\mathbb{R}^2$ onto the line $L = \{ \begin{bmatrix} r \\ r \end{bmatrix} : r \in \mathbb{R} \}$.

Let $C = AB$.

Find the eigenvalues of C . For each eigenvalue, find a basis of the corresponding eigenspace.

(One approach is to simply compute what C is and do the necessary calculations to find the answer. However, I would also accept an answer using geometric intuition. Make sure to explain yourself as fully as possible if you choose the latter approach.)

Direct, algebraic approach: $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

To find B : an orthonormal basis of L is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$

$$\text{Thus } B = \text{proj}_L = QQ^T = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\text{Thus } C = AB = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix}$$

Eigenvalues of C = roots of char poly $f_C(x) = \det(C - xI)$

$$\det \begin{bmatrix} 5/2 - x & 5/2 \\ 5/2 & 5/2 - x \end{bmatrix} = x^2 - 5x = (x-0)(x-5)$$

Thus eigenvalues of C are 0, 5

$$\text{Eigenspace of } 0 = E_0 = \ker(C - 0I) = \ker \begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix}$$

RREF of $\begin{bmatrix} 5/2 & 5/2 \\ 5/2 & 5/2 \end{bmatrix}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, relation for column without pivot is

$1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, thus $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis of $E_0 = \ker(C - 0I)$

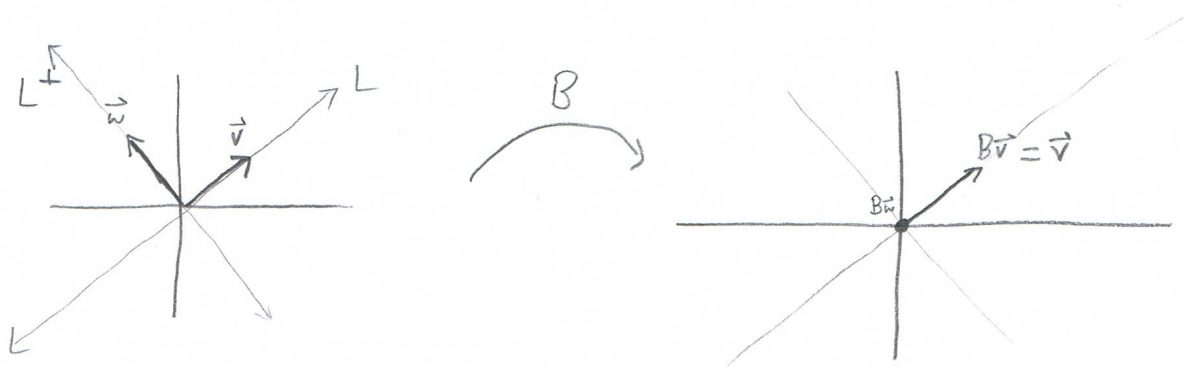
$$\text{Eigenspace of } 5 = E_5 = \ker(C - 5I) = \ker \begin{bmatrix} -5/2 & 5/2 \\ 5/2 & -5/2 \end{bmatrix}$$

RREF of $\begin{bmatrix} -5/2 & 5/2 \\ 5/2 & -5/2 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, relation for column without pivot is

$1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, thus $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis of $E_5 = \ker(C - 5I)$

Geometric approach:

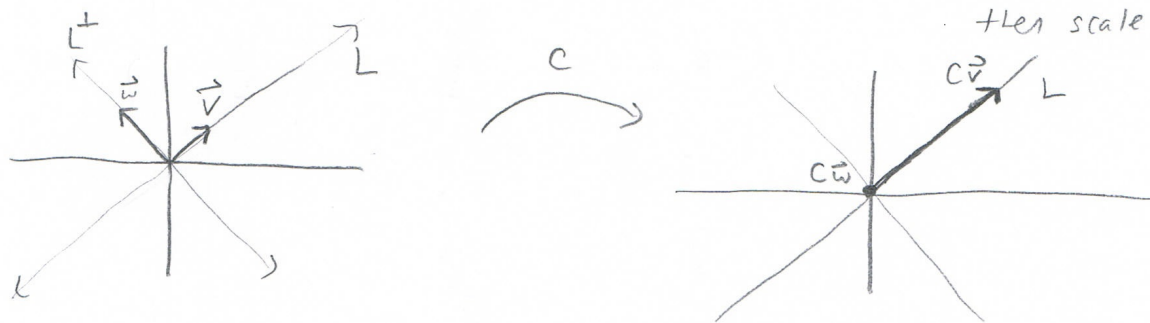
Orthogonal projection $B = \text{proj}_L$ does nothing to vectors that are in L , and "smushes" vectors in $L^\perp$ to $\vec{0}$.



Thus, vectors in L are eigenvectors of B with eigenvalue 1, and vectors in $L^\perp$ are eigenvectors of B with eigenvalue 0. Since B is 2×2 , there can't be more eigenvalues.

Since C is just AB , we have

$$C\vec{x} = AB\vec{x} = A(\vec{x}, \text{projected onto } L) = \vec{x}, \text{ projected onto } L, \text{ then scaled by 5}$$



Thus $L = \left\{ \begin{bmatrix} r \\ 1 \end{bmatrix} : r \in \mathbb{R} \right\} = \text{span} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ is now the eigenspace for eigenvalue 5, though $L^\perp = \left\{ \begin{bmatrix} -r \\ 1 \end{bmatrix} : r \in \mathbb{R} \right\} = \text{span} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$ is still the eigenspace for eigenvalue 0.

Problem 7 (5 points)

Mark the following statements as true or false. **Read and think very carefully.**

The kernel of a 4×6 matrix A is a subspace of $\mathbb{R}^4$.

T ☒ F

If A is a 3×3 matrix, then $\det(2A) = 8 \det(A)$.

☒ T F

If A is a 2×5 matrix, then the columns of A must be linearly dependent.

☒ T F

The rank of a 3×7 matrix A can be equal to 4.

T ☒ F

The rotation matrix D_{43° is an orthogonal matrix.

☒ T F

- The kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$, not $\mathbb{R}^m$.

$$m \begin{bmatrix} A \end{bmatrix} \cdot n \begin{bmatrix} \vec{x} \end{bmatrix} = n \begin{bmatrix} \vec{0} \end{bmatrix}$$

- $\det \begin{bmatrix} \frac{2\vec{v}_1}{2\vec{v}_2} \\ \frac{2\vec{v}_2}{2\vec{v}_3} \end{bmatrix} = 2 \det \begin{bmatrix} \frac{\vec{v}_1}{2\vec{v}_2} \\ \frac{2\vec{v}_2}{2\vec{v}_3} \end{bmatrix} = 4 \det \begin{bmatrix} \frac{\vec{v}_1}{\vec{v}_2} \\ \frac{\vec{v}_2}{2\vec{v}_3} \end{bmatrix} = 8 \det \begin{bmatrix} \frac{\vec{v}_1}{\vec{v}_2} \\ \frac{\vec{v}_2}{\vec{v}_3} \end{bmatrix}$

- More than n vectors in $\mathbb{R}^n$ are always linearly dependent (p.136)

Another way: columns of A are linearly dependent $\iff \ker(A) \neq \{\vec{0}\}$

Since rank-nullity says $\dim(\text{im}(A)) + \dim(\ker(A)) = 5$ and $\dim(\text{im}(A)) \leq 2$, must have $\dim(\ker(A)) \geq 3$, so certainly $\ker(A) \neq \{\vec{0}\}$.

- A 3×7 matrix can have at most 3 pivots (since at most one pivot per row, and at most one pivot per column.)

- The vectors $\begin{bmatrix} \cos(43^\circ) \\ \sin(43^\circ) \end{bmatrix}$ and $\begin{bmatrix} -\sin(43^\circ) \\ \cos(43^\circ) \end{bmatrix}$ are orthonormal.