

Find the determinant of $A = \begin{bmatrix} 3 & 5 & 2 \\ 1 & 8 & 4 \\ 6 & 9 & 7 \end{bmatrix}$ using Laplace cofactors.

$$\det \begin{bmatrix} 3 & 5 & 2 \\ 1 & 8 & 4 \\ 6 & 9 & 7 \end{bmatrix} = 3 \cdot \det \begin{bmatrix} 8 & 4 \\ 9 & 7 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 1 & 8 \\ 6 & 9 \end{bmatrix}$$

$$= 3(56 - 36) - 5(7 - 24) + 2(9 - 48)$$

$$= 3 \cdot 20 - 5 \cdot (-17) + 2 \cdot (-39)$$

$$= 60 + 85 - 78$$

$$= 67$$

Find $\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 0 & 4 & -2 & 1 \end{bmatrix}$ using Laplace cofactors.

$$1 \cdot \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 4 & -2 & 1 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & -2 & 1 \end{bmatrix} + 3 \cdot \det \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 4 & -2 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 2 \\ 4 & -2 & 1 \end{bmatrix} = 0 \cdot \underbrace{\det \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}}_7 - 1 \cdot \underbrace{\det \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}}_{(-7)} + 1 \cdot \underbrace{\det \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}}_{(-14)} = 0 + 7 - 14 = \boxed{-7}$$

$$\det \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & -2 & 1 \end{bmatrix} = (-1) \cdot \underbrace{\det \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}}_7 - 1 \cdot \underbrace{\det \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}}_2 + 1 \cdot \underbrace{\det \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}}_{(-4)} = (-7) - 2 + (-4) = \boxed{-13}$$

$$\det \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 4 & 1 \end{bmatrix} = (-1) \cdot \underbrace{\det \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}}_{(-7)} - 0 \cdot \underbrace{\det \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}}_2 + 1 \cdot \underbrace{\det \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}}_8 = 7 - 0 + 8 = \boxed{15}$$

$$\det \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \\ 0 & 4 & -2 \end{bmatrix} = (-1) \cdot \underbrace{\det \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}}_{(-14)} - 0 \cdot \underbrace{\det \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}}_{(-4)} + 1 \cdot \underbrace{\det \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}}_8 = 14 - 0 + 8 = \boxed{22}$$

$$1 \cdot (-7) - 2 \cdot (-13) + 3 \cdot (15) - 4 \cdot (22) = (-7) + 26 + 45 - 88 = \boxed{-24}$$

Diagonalize the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix}$, if possible.

Need eigenvalues

Eigenvalues of A = roots of characteristic polynomial $f_A(x)$

$$f_A(x) = \det(A - xI) = \det\left(\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}\right) = \det\begin{bmatrix} -x & 1 & 1 \\ -4 & 4-x & 2 \\ -2 & 1 & 3-x \end{bmatrix} =$$

$$(-x) \cdot \det\begin{bmatrix} 4-x & 2 \\ 1 & 3-x \end{bmatrix} - 1 \cdot \det\begin{bmatrix} -4 & 2 \\ -2 & 3-x \end{bmatrix} + 1 \cdot \det\begin{bmatrix} -4 & 4-x \\ -2 & 1 \end{bmatrix}$$

Using Laplace
cofactor expansion

$$= (-x)(x^2 - 7x + 10) - 1(4x - 8) + 1(-2x + 4)$$

$$= -x^3 + 7x^2 - 10x - 4x + 8 - 2x + 4$$

$$= -x^3 + 7x^2 - 16x + 12$$

By rational root test, can find roots 2, 3. Check factorization:

$$f_A(x) = -x^3 + 7x^2 - 16x + 12 = -(x-2)^2(x-3)^1$$

So eigenvalues of A are 2 and 3.

$$\text{algmult}(2) = 2 \quad \text{algmult}(3) = 1$$

Now need to find eigenspaces $E_\lambda = \ker(A - \lambda I)$ for each eigenvalue
(or more particularly, a basis of each eigenspace)

$$E_2 = \ker(A - 2I) = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = \ker\begin{bmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\text{RREF of } \begin{bmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ -2 & 1 & 1 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Relations for each column without a pivot give a basis of kernel:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

a basis of E_2 is
 $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

$$E_3 = \ker(A - 3I) = \ker\left(\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right) = \ker\begin{bmatrix} -3 & 1 & 1 \\ -4 & 1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\text{RREF of } \begin{bmatrix} -3 & 1 & 1 \\ -4 & 1 & 2 \\ -2 & 1 & 0 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Relations for each column with out pivot:

$$1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leadsto \text{a basis of } E_3 \text{ is}$$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Can check that these are eigenvectors (though unnecessary):

$$\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \checkmark$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \checkmark$$

Since E_2 has a basis with 2 vectors, and E_3 has a basis of 1 vector,

$$\text{geomult}(2) = \dim(E_2) = 2 \quad \text{algmult}(3) = \dim(E_3) = 1$$

Since $\text{geomult}(\lambda) = \text{algmult}(\lambda)$ for all eigenvalues λ , A is diagonalizable

The basis we've found for $\mathbb{R}^3$ in which A is diagonal is gotten by putting together the bases of each eigenspace:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

eigenvector eigenvector eigenvector
for 2 for 2 for 3

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}^{-1}}_{S^{-1}} \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ -2 & 1 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}}_S = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{diagonal}}$$