

$C^\infty(V)$ denotes the sheaf of C^∞ sections of a vector bundle V on B . This is a sheaf of C_B^∞ -modules on B . Given C^∞ vector bundles $p_i : V_i \rightarrow B$ for $i = 1, 2$ a homomorphism $f : V_1 \rightarrow V_2$ is a C^∞ map such that (i) $p_2 \circ f = p_1$ and (ii) for all $x \in B$ the map

$$f|_{p_1^{-1}(x)} : p_1^{-1}(x) \rightarrow p_2^{-1}(x)$$

is a linear transformation.

Given an open subset $U \subset B$ let $P(U)$ denote the set of homomorphisms from $(V_1)|_U$ to $(V_2)|_U$.

If U, U' are both open and if U' is a subset of U then we have a restriction $P(U) \rightarrow P(U')$. This gives P the structure of a presheaf. The lemmas below are straightforward.

Lemma 0.1. *P is a sheaf*

Lemma 0.2. *Let F_1 and F_2 be sheaves on B . Let R be a sheaf of rings on B (we are concerned mainly with $R = C_B^\infty$). Let $Q(U)$ be the collection of $g : (F_1)|_U \rightarrow (F_2)|_U$ (here the g is required to be homomorphism of sheaves of $R|_U$ -modules). The lemma states that presheaf Q on B is a sheaf.*

Lemma 0.3. *Let A be a ring. Every left A -module homomorphism $h : A \rightarrow A$ is of right multiplication by $h(1)$.*

Let R be a sheaf of rings on a space X . Every homomorphism h of sheaves of left R -modules from R to itself is right multiplication by $h(1) \in R(X)$.

Homomorphisms of sheaves of R -modules from R^k to R^l are given by $(k \times l)$ matrices with coefficients in the ring $R(X)$.

Definition 0.4. A C^∞ homomorphism $f : V_1 \rightarrow V_2$ gives rise to a homomorphism of sheaves of C_B^∞ -modules $C^\infty(f) : C^\infty(V_1) \rightarrow C^\infty(V_2)$. Indeed if s is a C^∞ section of V_1 on an open subset U of B then $f \circ s$ is a section of V_2 defined on U . Thus $s \mapsto f \circ s$ gives rise to a homomorphism of presheaves, and that homomorphism is defined to be $C^\infty(f)$.

We put $F_i = C^\infty(V_i)$ and $R = C_B^\infty$ in the second lemma and obtain the sheaf Q on B .

Proposition 0.5. *With notation as above, $f \mapsto C^\infty(f)$ gives a bijection $P(B) \rightarrow Q(B)$.*

Proof. In fact the “ $f \mapsto C^\infty(f)$ construction” is valid for every open subset U of B . In other words, we get a homomorphism of presheaves $P \rightarrow Q$. Let $\mathcal{U}$ be the collection of open subsets $U \subset B$ for which the vector bundles $(V_i)|_U$ are trivial bundles for both $i = 1, 2$.

Now the three statements

- (a) $\mathcal{U}$ forms a basis for the topology of B
 - (b) both P and Q are sheaves
 - (c) $P(U) \rightarrow Q(U)$ is a bijection for all $U \in \mathcal{U}$ (this is the last assertion in lemma 3)
- imply that $P(U) \rightarrow Q(U)$ is a bijection for all open subsets $U \subset B$, and in particular, for $U = B$.

□