1. PRELIMINARIES ON DIFFERENTIABLE FUNCTIONS

Definition 1.1. Let U be an open subset of $\mathbb{R}^m$. A function $f : U \to \mathbb{R}^n$ is differentiable at $x \in U$ if there is a linear transformation $T : |R^n \to \mathbb{R}^m$ such that: for all $\epsilon > 0$ there is some $\delta > 0$ with the following property:

$$v \in \mathbb{R}^n, \|v\| < \delta \implies x + v \in U \text{ and } \|f(x+v) - f(x) - Tv\| \leq \epsilon\|v\|$$

There is only one linear transformation T with the above property and it is denoted by $f'(x)$.

Theorem 1.2. ***chain rule****: Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. Let $f : U \to V$ and $g : V \to W$ be functions that are differentiable at $x \in U$ and $y \in V$ respectively, Assume furthermore that $f(x) = y$. Then*

1. *the composite $g \circ f$ is differentiable at x and*
2. *$(g \circ f)'(x) = g'(f(x) \circ f'(x)$.*

Proof. left to the reader. See [1] or [10] □

Definition 1.3. $f : U \to \mathbb{R}$ is a function defined on an open subset $U \subset \mathbb{R}^n$. f is C^0 if f is continuous.

f is C^1 if all its partial derivatives $\partial_i f$ are defined and continuous on U. More generally, f is C^k (where $k \in \mathbb{N}$) if all its partial derivatives $\partial_i f$ are defined and are C^{k-1}.

f is C^∞ if f is C^k for every non-negative integer k.

The collection of C^k functions on U will be denoted by $C^k(U)$.

Proposition 1.4. (1) *If $f \in C^k(U)$ then $f \in C^{k-1}(U)$.*

(2) *If $f, g \in C^k(U)$ then $f + g$ and fg are also C^k. All constant functions are C^k.*

Proof. Part 2 follows from part (1) and the Liebniz rule. □

Lemma 1.5. *$U \subset \mathbb{R}^n$ is open. Let $f : U \to \mathbb{R}$ be C^1. Then f is differentiable (see the first definition) at every point $x \in U$. Its derivative $f'(x) : \mathbb{R}^n \to \mathbb{R}$ is given by*

$$f'(x)(a_1, a_2, ..., a_n) = a_1\partial_1 f(x) + a_2\partial_2 f(x) + ... + a_n\partial_n f(x) \, \forall (a_1, ..., a_n) \in \mathbb{R}^n$$

Proof. An iterated use of the mean-value theorem. See [1] or [10]. □

Definition 1.6. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Given $f : U \to V$ define $f_i : U \to \mathbb{R}$ for $i = 1, 2, ..., m$ by $f(x) = (f_1(x), ..., f_m(x))$ for every $x \in U$. If all the f_i are C^k then f is said to be C^k as well.

Theorem 1.7. *Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. If $f : U \to V$ and $g : V \to W$ are both C^k then so is $g \circ f$.*

Proof. Induction on k. The case $k = 0$ is OK.

Assume $k > 0$. In view of the lemma, one sees that "f is C^k" is equivalent to the reformulation "f is differentiable at all $x \in U$ and $x \mapsto f'(x)$ is a C^{k-1} function, denoted by f', from U to the space of $m \times n$ matrices."

The chain rule now tells us that $g \circ f$ is differentiable, and also that $(g \circ f)' = (g' \circ f).f'$ where . denotes matrix multiplication. Now both g' and f are C^{k-1}. By the induction hypothesis, $g' \circ f$ is C^{k-1}. In addition, f' is also C^{k-1}. It follows that

the matrix product $g' \circ f.f'$ is also C^{k-1}. We have shown that $(g \circ f)'$ is C^{k-1}. In view of the equivalent reformulation, the theorem follows.

□

Proposition 1.8. *Let $f : U \times V \to \mathbb{R}$ be C^k where $k > 0$ and $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$ are both open subsets. Assume furthermore that $0 \in U$. Then there is a C^{k-1} function $g : U \times V \to \mathbb{R}$ such that*

$$f(x, y) = f(0, y) + xg(x, y) \text{ for all } x \in U, y \in V \tag{1}$$

Proof. The above equation in fact defines a C^k function g on the domain $(U \setminus \{0\}) \times V$. We have $a > 0$ such that the open interval $(-a, a)$ is contained in U. Now let $0 < |x| < a, y \in V$ and let $\gamma(t) = (tx, y)$ for $0 \leq t \leq 1$. An application of the fundamental theorem of calculus to the function $f \circ \gamma$ shows that

$$g(x, y) = \int_0^1 \partial_1 f(tx, y)dt \text{ whenever } 0 < |x| < a, y \in V \tag{2}$$

By assumption, $\partial_1 f$ is C^{k-1} on $U \times V$. It follows easily that the expression on the right in equation (2) defines a C^{k-1} function on $U \times V$, once again denoted by g by abuse of notation. The desired equation (1) holds on $(U \setminus \{0\}) \times V$ and continuity shows that it holds on $U \times V$ as well. □

Corollary 1.9. *Let $a, b > 0$, let $X = (-a, a)^r \times (-b, b)^s$ and let $f : X \to \mathbb{R}$ be a C^∞ function that vanishes on $0 \times (-b, b)^s$. (here $0 = (0, 0, ..., 0) \in (-a, a)^r$). Then there are C^∞ functions $g_1, ...g_r$ defined on X such that*

$$f = x_1 g_1 + x_2 g_2 + ... + x_r g_r$$

2. Inverse Function Theorem

Theorem 2.1. ***Inverse function theorem*** *(a) Let U be an open subset of $\mathbb{R}^n$, let $p \in U$ and let $f : U \to R^n$ be a C^1 function such that the linear transformation $f'(p) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then $f|_V$ is a one-to-one open map for some nbhd V of p in U. Thus $f(V)$ is open and there is a homeomorphism $h : V \to f(V)$ such that $f(x) = h(x)$ for all $x \in V$. (b) If the above f is C^k for some $k \geq 1$ then $h^{-1} : f(V) \to V$ is also C^k.*

Part (b) of the theorem is deduced from part (a) in exactly the same manner in all the sources (that I've seen) and is omitted. We concentrate on part (a). The proof by the Contraction Principle in [7, 9, 10] is valid even for Banach spaces. The proof of part (a) in [1] is interesting and different.

Problem 2.2. Deduce part (a) of the inverse function theorem from 'one-variable-calculus' (Intermediate Value thm. and Mean-Value thm. precisely) when $f(x_1, ..., x_n) = (g(x_1, ..., x_n), x_2, x_3, ..., x_n)$ under the assumption that $\partial_1 g(p) \neq 0$.

Problem 2.3. Assume thet the f in the statement of IFT is given by $f = (f_1, ..., f_n)$ where the $f_i : U \to \mathbb{R}$ are C^1. Prove that there is a permutation σ such that if $g_i = f_{\sigma(i)}$ for all i, then F_i defined by

$$F_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, g_i, g_{i+1}, ..., g_n)$$

satisfies the hypothesis of the inverse function theorem at p for all $i = 1, 2, ...n$.

Problem 2.4. Deduce the inverse function thm. from the previous two problems.

Problem 2.5. Assume that $f : U \to \mathbb{R}^n$ is continuous everywhere, and differentiable at a given point $p \in U$. Assume that the linear transformation $f'(p)$ is invertible. Prove that there is a nbhd V of p in U such that $f(V)$ contains a neighbourhood of $f(p)$.

Hint: prove for all sufficiently small and positive r, that
(i) $\|x - p\| = r$ implies $f(x) \neq f(p)$
(ii) the map from the sphere of radius r to $\mathbb{R}^n \setminus \{f(p)\}$ given by $x \mapsto f(x)$ is homotopic to $x \mapsto f(p) + f'(p)(x - p)$.

Now deduce that $\{f(x) : \|x - p\| \leq r\}$ contains the connected component of $f(p)$ in the complement of the compact set $\{f(x) : \|x - p\| = r\}$.

3. the definition of a C^∞ manifold

It is a good idea to read the definition of a presheaf (also a sheaf) on a topological space. Even though the little that is required for the moment has been spelt out below.

Notation 3.1. Recall that $C^0(Y)$ denotes the collection of continuous functions $Y \to \mathbb{R}$. Let $U \subset V \subset Y$ both be open subsets of Y. If $f \in C^0(V)$ then $f|_U \in C^0(U)$. Denote $f \mapsto f|_U$ by $Res(U, V) : C^0(V) \to C^0(U)$.

Let $f : Y \to Z$ be continuous. For every open $V \subset Z$ and for every $g \in C^0(V)$ we obtain $f^*g \in C^0(f^{-1}V)$ giiven by $(f^*g)x = g(f(x))$ for all $x \in f^{-1}V$.

Definition 3.2. Let Y be a topological space. A *subpresheaf* (or a presheaf of subsets) R of C^0_Y consists of

(1) the data: a subset $R(U) \subset C^0(U)$ for every open subset U of Y
subject to the condition:
(2) $Res(U, V)R(V) \subset R(U)$ whenever $U \subset V$ are both open subsets of Y.

R is a *subsheaf* of C^0_Y if in addition
(3) $f \in C^0(V), V = \cup\{V_i : i \in I\}, \forall i \in I\, Res(V_i, V)f \in R(V_i) \implies f \in R(V)$.

A *sheaf of* $\mathbb{R}$*-subalgebras* of C^0_Y is a subsheaf R of C^0_Y such that $R(U)$ is a $\mathbb{R}$-subalgebra of $C^0(U)$ for every open $U \subset Y$.

Definition 3.3. A *ringed space*[1] is a pair (Y, R) where Y is a topological space and R is a sheaf of $\mathbb{R}$-subalgebras of C^0_Y.

If (Y, R) is a ringed space and if $U \subset Y$ is open, then we obtain *the ringed space* (U, R_U) by setting $R_U(V) = R(V)$ for all open $V \subset U$.

A morphism of ringed spaces $f : (Y, R) \to (Y', R')$ is simply a continuous map $f : Y \to Y'$ with the property: U' open in Y' and $g \in R'(U')$ implies $f^*g \in R(f^{-1}(U'))$. See notation above for f^*g.

In addition, if f is a homeomorphsim and if its inverse $f^{-1} : Y \to Y'$ gives rise to a morphism $(Y', R') \to (Y, R)$ of ringed spaces then f is said to be an *isomorphism* of ringed spaces.

[1] the term 'ringed space' in mathematical literature covers a more general situation than encountered in these notes

Example 3.4. Let $0 \leq k \leq \infty$ and let U be open in $\mathbb{R}^n$. We have the sheaf of $\mathbb{R}$-subalgebras C^k_U given by $C^k_U(V) = C^k(V)$ for all open subsets $V \subset U$. See Prop.1.4(2).

Definition 3.5. A C^k-manifold is a ringed space (M, R) with the property that M is covered by its open subsets U for which (U, R_U) is isomorphic to (Ω, C^k_Ω) where Ω is open in $\mathbb{R}^n$ for some n.

For a C^k-manifold (M, R) it is traditional to denote R by C^k_M.

Given C^k manifolds (M, C^k_M) and (N, C^k_N), a map $f : M \to N$ is C^k if it is a morphism $(M, C^k_M) \to (N, C^k_N)$ of ringed spaces. In other words,

(1) $f : M \to N$ is continuous and
(2) For every open $V \subset N$ and for every $g \in C^k_N(V)$, the continuous function f^*g defined on $f^{-1}V$ belongs to $C^k_M(f^{-1}V)$.

Problem 3.6. Let U and V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. They are C^k manifolds in a natural manner. Show that the two definitions of "$f : U \to V$ is C^k" given in 3.5 and 1.6 are equivalent to each other.

Problem 3.7. Let U be an open subset of C^k manifold M. Let $f_1, f_2, ..., f_r$ be real-valued functions defined on U. Denote by $f : U \to \mathbb{R}^r$ the map $x \mapsto (f_1(x), ..., f_r(x))$. Prove that $f : M \to \mathbb{R}^r$ is C^k in the sense of definition 3.5 if and only if $f_i \in C^k_M(U)$ for all $i = 1, 2, ..., r$.

Problem 3.8. $f : M \to N$ is a local homeomorphism and (N, C^k_N) is a C^k manifold. Show that M acquires the structure of a C^k manifold for which $f : M \to N$ is C^k in the sense of 3.5.

Problem 3.9. In the previous problem, is there only one C^k structure on M for which f is C^k?

Problem 3.10. Define real analytic manifolds and complex analytic manifolds (the latter are referred to as complex manifolds). State and prove the analog of the previous problem in this context.

Problem 3.11. It is well known that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \exp(-1/t)$ for $t > 0$ and
$f(t) = 0$ otherwise
is C^∞. Using this function alone (and 1.4,1.7, 3.5) show that if $p \in U \subset M$ where M is a C^k manifold and U is open in M there is a C^k function $\phi : M \to \mathbb{R}$ which (a) vanishes on the complement of U and (b) satisfies $\phi(p) = 1$.

Problem 3.12. Let M and N be C^k manifolds. Let $f : M \to N$ be any function. Show that f is C^k if and only if

$$g : N \to \mathbb{R} \text{ is } C^k \implies g \circ f \text{ is } C^k$$

4. Tangent-spaces and cotangent-spaces

There are plenty of books on Differential geometry that contain the basic concepts and definitions: tangent bundles, one-parameter groups, the Frobenius theorem on sub-bundles of the tangent bundle:[4, 2, 6, 9, 12] are some of them.

The definition given of the tangent space of a C^∞ manifold in most sources e.g. [4, 9, 12] is the same. In Eucliden space, given a vector v we may compute the directional derivative $D_v f(p)$ of a function f defined on a neighbourhood U of $p \in \mathbb{R}^n$. Now $E(f) = D_v f(p)$ is a linear functional that satisfes the Liebniz rule:

$$E(fg) = f(p)E(g) + g(p)E(f)$$

The domain of the linear functional is the $\mathbb{R}$-algebra of germs(see [9, 12] of C^∞ functions at p is denoted by $C^\infty_{\mathbb{R}^n,p}$.

It is a simple exercise to deduce from Corollary 1.9 that every linear functional that satisfies the Liebniz rule is of the type $f \mapsto D_v f(p)$ for a unique $v \in \mathbb{R}^n$.

Definition 4.1. The tangent-space T_pM of a C^∞ manifold M at p is the collection of linear functionals $E : C^\infty_{M,p} \to \mathbb{R}$ that satisfies the Leibniz rule. By the above remarks, we see that if $\dim(M) = n$ then T_pM is a vector space of rank n.

If $f : (M, p) \to (N, q)$ is C^∞ then we get a $\mathbb{R}$-algebra homomorphism $f^* : C^\infty_{N,q} \to C^\infty_{M,p}$. If $E \in T_pM$ then the composite

$$C^\infty_{N,q} \xrightarrow{f^*} C^\infty_{M,p} \xrightarrow{E} \mathbb{R}$$

also satisfies the Leibniz rule. The corresponding element of T_qN is denoted by $f'(p)E$. This gives rise to the linear transformation $f'(p) : T_pM \to T_qN$.

Definition 4.2.

5. differentiable submanifolds

What does it mean for an arbitrary subset A of a C^∞ manifold M to be a C^∞ submanifold?

For this purpose, we define a function $f : A \to \mathbb{R}$ to be admissible if every point $p \in A$ has a neighborhood U_p in M and a C^∞ function $f_p : U_p \to \mathbb{R}$ that agrees with f on the intersection $A \cap U_p$. The collection of admissible functions $f : A \to \mathbb{R}$ will be denoted by $R(A)$. This is clearly a $\mathbb{R}$-subalgebra of $C^0(A)$.

It is clear that if B is any subset of A and if $f \in R(A)$, then $f|_B \in R(B)$. The only subsets B we are interested in, however, are the open subsets (in the relative topology) of A. The $B \mapsto R(B)$ (for such B) defines a sheaf of $\mathbb{R}$-subalgebras of C^0_A.

Definition 5.1. *A is a C^∞ submanifold of M* if the ringed space(A, R) is a C^∞ manifold.

Given a point $p \in A$ we will say that *A is smooth at p* if there is a neighbourhood U of p in A such that the ringed space (U, R_U) is a C^∞ manifold (see prop 5.6).

Thus a subset A is a C^∞ submanifold iff A is smooth at every point $p \in A$.

Example 5.2. Let U_1 and U_2 be open in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively. Let $y_0 \in U_2$. Then $U_1 \times \{y_0\}$ is a C^∞ submanifold of $U_1 \times U_2$.

One frequently obtains C^∞ submanifolds through the two lemma below, both of which are simple consequences of the Inverse fUnction Thm.

Lemma 5.3. ***Parametric form.*** *Let U and V be open subsets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and let $f : (U, q) \to (V, p)$ be a C^∞ function. Assume that the linear transformation $f'(q) : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one. Then there is*

a neighbourhood U_1 of q in U,
a neighbourhood U_2 of zero in $\mathbb{R}^{n-m}$,
a neighbourhood Ω of p in V, and
a diffeomorphism $\phi : U_1 \times U_2 \to \Omega$
such that $\phi(x, 0) = f(x)$ for all $x \in U_1$.

In view of example 5.2, it follows that $f(W)$ is a C^∞ submanifold of V for all W open in U_1–and in particular this is the case for $W =$ all sufficiently small nbhds of q in U.

Lemma 5.4. ***Implicit form.*** *If U, V are open in $\mathbb{R}^n, \mathbb{R}^{n-m}$ respectively, if $f : (U, p) \to (V, q)$ is C^∞ and if $f'(p) : T_p\mathbb{R}^n \to T_q\mathbb{R}^{n-m}$ is surjective, then*

there are nbhds Ω of p in U, Ω_2 of q in V and Ω_1 of 0 in $\mathbb{R}^m$, and a diffeomorphism $\phi : \Omega \to \Omega_1 \times \Omega_2$ such that $f(x) = p_2\phi(x)$ for all $x \in \Omega$. Here $p_2 : \Omega_1 \times \Omega_2 \to \Omega_2$ is the projection to the second factor.

In view of example 5.2, we see that $f^{-1}(q)$ is smooth at p. Furthermore, $T_p f^{-1}(q)$ is the kernel of $f'(p)$.

Remark 5.5. In view of the fact that $f'(p) : T_pM \to T_qN$ has been defined for every C^∞ map $f : (M, p) \to (N, q)$ we may re-state the second statement of the above lemmas as follows:

(1) If $f'(p)$ is one-to-one, then $f(W)$ is a C^∞ submanifold of N for all sufficiently small nbhds W of p in M.
(2) If $f'(p)$ is onto, then $f^{-1}(q)$ is smooth at p. Furthermore the tangent-space of $f^{-1}(q)$ at p is the kernel of the linear transformation $f'(p)$.

Proposition 5.6. *Given $p \in A \subset M$ where M is a C^∞ manifold, then A is smooth at p if and only if there is a diffeomeorphism $U_1 \times U_2 \to W$ (where W is a nbhd of p in M, and U_1, U_2 are open in $\mathbb{R}^m, \mathbb{R}^{n-m}$ respectively) such that $W \cap \phi^{-1}A$ is simply $U_1 \times \{c\}$ for some point $c \in U_2$*

Proof. $\Leftarrow$ has already been remarked in example 5.2.

Now for the converse. The question being local at p, we may assume that M is an open subset of $\mathbb{R}^n$. Because A is smooth at p there is an isomorphism of ringed spaces

$$g : (M_1 \cap A, R_{M_1 \cap A}) \to (U, C^\infty_U)$$

where M_1 is a nbhd of p in M and U is an open subset of $\mathbb{R}^m$. Put $q = g(p)$. Denote the inverse of g by f. The composite

$$U \xrightarrow{f} M_1 \cap A \hookrightarrow M_1 \hookrightarrow \mathbb{R}^n$$

is then of the type $(f_1, ..., f_n)$. Denote the i-the co-ordinate function on $\mathbb{R}^n$ by $p_i : \mathbb{R}^n \to \mathbb{R}$. Because the p_i are C^∞, the $p_i|_{M_1 \cap A}$ belong to $R(M_1 \cap A)$. And because f is a morphism of ringed spaces these pull back to C^∞ functions of U. In other words, all the f_i are C^∞ functions on U.

Note next that g is given by $(g_1, ..., g_m)$ where the $g_i \in R(M_1 \cap A)$. It follows that there is a nbhd of M_2 of p in M_1 and C^∞ functions $\widetilde{g}_i$ defined on M_2 such that $\widetilde{g}_i(x) = g_i(x)$ for all $x \in M_2 \cap A$. Putting $\widetilde{g} = (\widetilde{g}_1, ..., \widetilde{g}_m)$ we obtain a C^∞ map $\widetilde{g} : M_2 \to \mathbb{R}^m$ such that $\widetilde{g}(x) = g(x)$ for all $x \in M_2 \cap A$.

Summarising, we have open subsets $U \subset \mathbb{R}^m$ and $M_2 \subset \mathbb{R}^n$ and C^∞ maps $f : U \to \mathbb{R}^n$ and $\widetilde{g} : M_2 \to \mathbb{R}^m$. Furthermore $\widetilde{g}(f(x)) = x$ for all $x \in f^{-1}(M_2 \cap A)$. The chain rule shows that $f'(q)$ is one-to-one. The claim now follows from lemma 5.3. $\square$

Problem 5.7. A subset of a topological space is locally closed if it is the intersection of an open subset and a closed subset. Prove that a submanifold A of a manifold M is a locally closed subset. (No differentiability assumptions anywhere).

Problem 5.8. let A' denote the transpose of a matrix A. Let $X(resp.Y)$ be the space of $(k \times n)(resp.symmetric(k \times k))$ matrices with entries in $\mathbb{R}$. Define $F : X \to Y$ by $F(A) = A'A$ for all $A \in X$. Deduce from the implicit function theorem that the Stiefel variety $V_k(\mathbb{R}^n) = F^{-1}id_{k\times k}$ is a C^∞ submanifold of X of codimension $k(k+1)/2$.

Problem 5.9. Assume that $0 \in \mathbb{R}$ is a regular value (see dfn 6.1) of C^∞ function $f : M \to \mathbb{R}$. Define $h(x,t) = f(x) - t^2$ for all $x \in M, t \in \mathbb{R}$. Prove that zero is a regular value of h as well.

remark Let $W = \{x \in M : f(x) \geq 0\}$, let $\partial(W) = \{x \in M : f(x) = 0\}$. *Then* W *is a manifold with boundary* $\partial(W)$ and $h^{-1}(0)$ is referred to as the *double* of W because it is obtained from $W \times \{\pm 1\}$ by identifying $(x,-1)$ with $(x,1)$ for all $x \in \partial(W)$.

Problem 5.10.

Show that the graph of a function $f : \mathbb{R} \to \mathbb{R}$ s a C^∞ sub manifold of $\mathbb{R}^2$ off f is itself a C^∞ function.

Problem 5.11. $\|(x_1, ..., x_n)\| = \sqrt{x_1^2 + ... + x_n^2}$. Let $v_1, v_2, ..., v_g$ be distinct points of $\mathbb{R}^n$. Let $r_1, r_2, ..., r_g, R > 0$. Assume

(1) $r_i + r_j < \|v_i - v_j\|$ whenever $1 \leq i < j \leq g$, and
(2) $r_i + \|v_i\| < R$ whenever $1 \leq i \leq g$

Let $f(x) = (R^2 - \|x\|^2)\prod_{i=1}^{i=g}(\|x - v_i\|^2 - r_i^2)$ for all $x \in \mathbb{R}^n$. Let $h(x,t) = f(x) - t^2$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Prove that 0 is a regular value of h.

When $n = 2$*, the manifold* $h^{-1}(0)$ *as an orientable surface of genus* g*. In general, it is the connected sum of* g *copies of* $S^1 \times S^{n-1}$.

Problem 5.12. With f as in the previous problem, show that

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k : f(x) = \|y\|^2\}$$

is a C^∞ submanifold of $\mathbb{R}^n \times \mathbb{R}^k$ of codimension 1.

Problem 5.13. *map germs* Let M, N be C^∞ manifolds. Let $p \in M$ and $q \in N$. The set of map germs mapgrm$((M,p) \to (N,q))$ is the set of equivalence classes of C^∞ maps $f : (U,p) \to (N,q)$ where U is a nbhd of p in M. The equivalence relation $\sim$ is given as follows: given $f_i : (U_i, p) \to (N,q)$ for $i = 1, 2$,

$f_1 \sim f_2$ if and only if there is a nbhd U of p contained in $U_1 \cap U_2$ such that $f_1(x) = f_2(x)$ for all $x \in U$. The *germ of* f is the equivalence class of f, and will be denoted by $[f]$. The three subsets of mapgrm$((M,p) \to (N,q))$ consisting of germs of f such that $f'(p) : T_pM \to T_qN$ is (i) one-to-one (ii) onto and (iii) an isomorphism are denoted by (i) immgrm$((M,p) \to (N,q))$, (ii) sbmgrm$((M,p) \to (N,q))$ and (iii) diffgrm$((M,p) \to (N,q))$

autgrm(M,p) is defined to be diffgrm$((M,p) \to (M,p))$.

(1) For $[f] \in \text{mapgrm}((M,p) \to (N,q))$ and $g \in \text{mapgrm}((N,q) \to (S,r))$ define $[g] \circ [f] \in \text{mapgrm}((M,p) \to (S,r))$.
(2) Show that the above binary operation $\circ$ turns $\text{autgrm}(M,p)$ into a group.
(3) We have an action of the group $\text{autgrm}(N,q) \times \text{autgrm}(M,p)$ on the set $\text{mapgrm}((M,p) \to (N,q))$ given by

$$([g],[h]).[f] = [g] \circ [f] \circ [h]^{-1}$$

The subsets (i), (ii) and (iii) listed above, of $\text{mapgrm}((M,p) \to (N,q))$, are also taken into themselves by this group action. This group action restricts to an action of the subgroups $\text{autgrm}(N,q)$ and $\text{autgrm}(M,p)$.

There are nine questions:

which of the three groups $\text{autgrm}(N,q), \text{autgrm}(M,p), \text{autgrm}(N,q) \times \text{autgrm}(M,p)$ fails to act transitively on the sets (i) $\text{immgrm}((M,p) \to (N,q))$, (ii) $\text{sbmgrm}((M,p) \to (N,q))$ and (iii) $\text{diffgrm}((M,p) \to (N,q))$?

Problem 5.14. Given (M,p) and (N,q) and a non-negative integer r consider the set of C^∞ maps $f : (U,p) \to (N,q)$ (where U is a nbhd of p in M) such that $\text{rank}(f'(x)) = r$ for all $x \in U$. Denote the set of germs of such maps by $\text{mapgrm}_r((M,p) \to (N,q))$. Which of the three groups $\text{autgrm}(N,q), \text{autgrm}(M,p), \text{autgrm}(N,q) \times \text{autgrm}(M,p)$ fails to act transitively on $\text{mapgrm}_r((M,p) \to (N,q))$? (The rank theorem in [10] is another corollary of the Inverse function thm, and this is useful here.)

Problem 5.15. Let $p : \widetilde{M} \to M$ be the universal covering space of a topological manifold M. Let Γ be the group of covering transformations. Assume that $\widetilde{M}$ has been given a C^∞ structure.

Then M itself acquires the structure of a ringed space: given an open subset U of M, we define $R(U) \subset C^0(U)$ by

$$\forall f \in C^0(U), f \in R(U) \iff p^* f \in C^\infty(p^{-1}(U))$$

Show that the ringed space (M,R) is a C^∞ manifold if and only if every covering transformation is a diffeomorphism of $\widetilde{M}$.

Problem 5.16. Let $p : \mathbb{R}^2 \to M$ be the universal cover of the Mobius band. The covering transformations are $(x,y) \mapsto x + m, (-1)^m y)$ for all $m \in \mathbb{Z}$. The previous problem defines a C^∞ structure on$M$. Let $A = p(\mathbb{R} \times \{0\})$. This A is a subset of M homeomorphic to a circle. Is there a C^∞ function $f : M \to \mathbb{R}$ for which $b \in \mathbb{R}$ is a regular value and $f^{-1}(b) = A$?

6. TRANSVERSALITY

[8, 3, 5] are all excellent references.

Definition 6.1. $M, N, f : M \to N$ are all C^∞ and A is a C^∞ submanifold of N.

f is said to be transverse to A at $p \in f^{-1}(A)$ if

$$T_q A + f'(p) T_p M = T_q N \text{ where } q = f(p).$$

If this condition is satisfied at all point p of $f^{-1}(A)$ then *f is said to be transverse to A*.

A point $a \in N$ is *a regular value of f* if $f : M \to N$ is transverse to the submanifold $\{a\}$ of N–in other words, $f'(p)$ is surjective whenever $f(p) = a$.

Proposition 6.2. *If f is transverse to A, then $f^{-1}(A)$ is a C^∞ submanifold. Furthermore (with notation as above, for all $p \in f^{-1}(A), q = f(p) \in A$,*

the linear transformation $f'(p) : T_pM \to T_qN$ takes $T_pf^{-1}(A)$ into T_qA and induces an isomorphism of quotient-spaces $\overline{f'(p)} : T_pM/T_pf^{-1}(A) \xrightarrow{\cong} T_qN/T_qA$

In particular, if $a \in N$ is a regular value of f then $f^{-1}(a)$ is a submanifold whose tangent-space at p equals the kernel of $f'(p)$.

A beautiful proof of Sard's thm is given in [8]. The situation for C^k maps is more delicate– see [9].

Theorem 6.3. ***Sard's theorem*** *If $f : M \to N$ is C^∞ then the complement in N of the set of regular values of f is a set of measure zero.*

Remark 6.4. If $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function, then the complement of the set of regular values is a finite set.

If $f \in \mathbb{C}[x_1, ..., x_n]$ then $f : \mathbb{C}^n \to \mathbb{C}$ given by $x \mapsto f(x)$ has its regular values as the complement of a finite subset of $\mathbb{C}$. The proof of such statements require some rudimentary algebraic geometry.

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