1. PRELIMINARIES ON DIFFERENTIABLE FUNCTIONS

Definition 1.1. Let U be an open subset of $\mathbb{R}^m$. A function $f : U \to \mathbb{R}^n$ is differentiable at $x \in U$ if there is a linear transformation $T : |R^n \to \mathbb{R}^m$ such that: for all $\epsilon > 0$ there is some $\delta > 0$ with the following property:

$$v \in \mathbb{R}^n, \|v\| < \delta \implies x + v \in U \text{ and } \|f(x+v) - f(x) - Tv\| \leq \epsilon\|v\|$$

There is only one linear transformation T with the above property and it is denoted by $f'(x)$.

Theorem 1.2. ***chain rule****: Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. Let $f : U \to V$ and $g : V \to W$ be functions that are differentiable at $x \in U$ and $y \in V$ respectively, Assume furthermore that $f(x) = y$. Then*

1. *the composite $g \circ f$ is differentiable at x and*
2. *$(g \circ f)'(x) = g'(f(x) \circ f'(x)$.*

Proof. left to the reader. See [1] or [4] □

Definition 1.3. $f : U \to \mathbb{R}$ is a function defined on an open subset $U \subset \mathbb{R}^n$. f is C^0 if f is continuous.

f is C^1 if all its partial derivatives $\partial_i f$ are defined and continuous on U. More generally, f is C^k (where $k \in \mathbb{N}$) if all its partial derivatives $\partial_i f$ are defined and are C^{k-1}.

f is C^∞ if f is C^k for every non-negative integer k.

The collection of C^k functions on U will be denoted by $C^k(U)$.

Proposition 1.4. (1) *If $f \in C^k(U)$ then $f \in C^{k-1}(U)$.*

(2) *If $f, g \in C^k(U)$ then $f + g$ and fg are also C^k. All constant functions are C^k.*

Proof. Part 2 follows from part (1) and the Liebniz rule. □

Lemma 1.5. *$U \subset \mathbb{R}^n$ is open. Let $f : U \to \mathbb{R}$ be C^1. Then f is differentiable (see the first definition) at every point $x \in U$. Its derivative $f'(x) : \mathbb{R}^n \to \mathbb{R}$ is given by*

$$f'(x)(a_1, a_2, ..., a_n) = a_1\partial_1 f(x) + a_2\partial_2 f(x) + ... + a_n\partial_n f(x) \,\forall (a_1, ..., a_n) \in \mathbb{R}^n$$

Proof. An iterated use of the mean-value theorem. See [1] or [4]. □

Definition 1.6. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open. Given $f : U \to V$ define $f_i : U \to \mathbb{R}$ for $i = 1, 2, ..., m$ by $f(x) = (f_1(x), ..., f_m(x))$ for every $x \in U$. If all the f_i are C^k then f is said to be C^k as well.

Theorem 1.7. *Let U, V, W be open subsets of $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p$ respectively. If $f : U \to V$ and $g : V \to W$ are both C^k then so is $g \circ f$.*

Proof. Induction on k. The case $k = 0$ is OK.

Assume $k > 0$. In view of the lemma, one sees that "f is C^k" is equivalent to the reformulation "f is differentiable at all $x \in U$ and $x \mapsto f'(x)$ is a C^{k-1} function, denoted by f', from U to the space of $m \times n$ matrices."

The chain rule now tells us that $g \circ f$ is differentiable, and also that $(g \circ f)' = (g' \circ f).f'$ where . denotes matrix multiplication. Now both g' and f are C^{k-1}. By the induction hypothesis, $g' \circ f$ is C^{k-1}. In addition, f' is also C^{k-1}. It follows that

the matrix product $g' \circ f.f'$ is also C^{k-1}. We have shown that $(g \circ f)'$ is C^{k-1}. In view of the equivalent reformulation, the theorem follows.

□

Proposition 1.8. *Let $f : U \times V \to \mathbb{R}$ be C^k where $k > 0$ and $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^n$ are both open subsets. Assume furthermore that $0 \in U$. Then there is a C^{k-1} function $g : U \times V \to \mathbb{R}$ such that*

$$f(x, y) = f(0, y) + xg(x, y) \text{ for all } x \in U, y \in V \tag{1}$$

Proof. The above equation in fact defines a C^k function g on the domain $(U \setminus \{0\}) \times V$. We have $a > 0$ such that the open interval $(-a, a)$ is contained in U. Now let $0 < |x| < a, y \in V$ and let $\gamma(t) = (tx, y)$ for $0 \leq t \leq 1$. An application of the fundamental theorem of calculus to the function $f \circ \gamma$ shows that

$$g(x, y) = \int_0^1 \partial_1 f(tx, y)dt \text{ whenever } 0 < |x| < a, y \in V \tag{2}$$

By assumption, $\partial_1 f$ is C^{k-1} on $U \times V$. It follows easily that the expression on the right in equation (2) defines a C^{k-1} function on $U \times V$, once again denoted by g by abuse of notation. The desired equation (1) holds on $(U \setminus \{0\}) \times V$ and continuity shows that it holds on $U \times V$ as well. □

Corollary 1.9. *Let $a, b > 0$, let $X = (-a, a)^r \times (-b, b)^s$ and let $f : X \to \mathbb{R}$ be a C^∞ function that vanishes on $0 \times (-b, b)^s$. (here $0 = (0, 0, ..., 0) \in (-a, a)^r$). Then there are C^∞ functions $g_1, ...g_r$ defined on X such that*

$$f = x_1 g_1 + x_2 g_2 + ... + x_r g_r$$

2. Inverse Function Theorem

Theorem 2.1. ***Inverse function theorem*** *(a) Let U be an open subset of $\mathbb{R}^n$, let $p \in U$ and let $f : U \to R^n$ be a C^1 function such that the linear transformation $f'(p) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then $f|_V$ is a one-to-one open map for some nbhd V of p in U. Thus $f(V)$ is open and there is a homeomorphism $h : V \to f(V)$ such that $f(x) = h(x)$ for all $x \in V$. (b) If the above f is C^k for some $k \geq 1$ then $h^{-1} : f(V) \to V$ is also C^k.*

Part (b) of the theorem is deduced from part (a) in exactly the same manner in all the sources (that I've seen) and is omitted. We concentrate on part (a). The proof by the Contraction Principle in [2, 3, 4] is valid even for Banach spaces. The proof of part (a) in [1] is interesting and different.

Problem 2.2. Deduce part (a) of the inverse function theorem from 'one-variable-calculus' (Intermediate Value thm. and Mean-Value thm. precisely) when $f(x_1, ..., x_n) = (g(x_1, ..., x_n), x_2, x_3, ..., x_n)$ under the assumption that $\partial_1 g(p) \neq 0$.

Problem 2.3. Assume thet the f in the statement of IFT is given by $f = (f_1, ..., f_n)$ where the $f_i : U \to \mathbb{R}$ are C^1. Prove that there is a permutation σ such that if $g_i = f_{\sigma(i)}$ for all i, then F_i defined by

$$F_i(x_1, ..., x_n) = (x_1, ..., x_{i-1}, g_i, g_{i+1}, ..., g_n)$$

satisfies the hypothesis of the inverse function theorem at p for all $i = 1, 2, ...n$.

Problem 2.4. Deduce the inverse function thm. from the previous two problems.

Problem 2.5. Assume that $f : U \to \mathbb{R}^n$ is continuous everywhere, and differentiable at a given point $p \in U$. Assume that the linear transformation $f'(p)$ is invertible. Prove that there is a nbhd V of p in U such that $f(V)$ contains a neighbourhood of $f(p)$.

Hint: prove for all sufficiently small and positive r, that
(i) $\|x - p\| = r$ implies $f(x) \neq f(p)$
(ii) the map from the sphere of radius r to $\mathbb{R}^n \setminus \{f(p)\}$ given by $x \mapsto f(x)$ is homotopic to $x \mapsto f(p) + f'(p)(x - p)$.

Now deduce that $\{f(x) : \|x - p\| \leq r\}$ contains the connected component of $f(p)$ in the complement of the compact set $\{f(x) : \|x - p\| = r\}$.

3. the definition of a C^∞ manifold

It is a good idea to read the definition of a presheaf (also a sheaf) on a topological space. Even though the little that is required for the moment has been spelt out below.

Notation 3.1. Recall that $C^0(Y)$ denotes the collection of continuous functions $Y \to \mathbb{R}$. Let $U \subset V \subset Y$ both be open subsets of Y. If $f \in C^0(V)$ then $f|_U \in C^0(U)$. Denote $f \mapsto f|_U$ by $Res(U, V) : C^0(V) \to C^0(U)$.

Let $f : Y \to Z$ be continuous. For every open $V \subset Z$ and for every $g \in C^0(V)$ we obtain $f^*g \in C^0(f^{-1}V)$ giiven by $(f^*g)x = g(f(x))$ for all $x \in f^{-1}V$.

Definition 3.2. Let Y be a topological space. A *subpresheaf* (or a presheaf of subsets) R of C^0_Y consists of

(1) the data: a subset $R(U) \subset C^0(U)$ for every open subset U of Y
subject to the condition:
(2) $Res(U, V)R(V) \subset R(U)$ whenever $U \subset V$ are both open subsets of Y.

R is a *subsheaf* of C^0_Y if in addition
(3) $f \in C^0(V), V = \cup\{V_i : i \in I\}, \forall i \in I\, Res(V_i, V)f \in R(V_i) \implies f \in R(V)$.

A *sheaf of* $\mathbb{R}$*-subalgebras* of C^0_Y is a subsheaf R of C^0_Y such that $R(U)$ is a $\mathbb{R}$-subalgebra of $C^0(U)$ for every open $U \subset Y$.

Definition 3.3. A *ringed space*[1] is a pair (Y, R) where Y is a topological space and R is a sheaf of $\mathbb{R}$-subalgebras of C^0_Y.

If (Y, R) is a ringed space and if $U \subset Y$ is open, then we obtain *the ringed space* (U, R_U) by setting $R_U(V) = R(V)$ for all open $V \subset U$.

A morphism of ringed spaces $f : (Y, R) \to (Y', R')$ is simply a continuous map $f : Y \to Y'$ with the property: U' open in Y' and $g \in R'(U')$ implies $f^*g \in R(f^{-1}(U'))$. See notation above for f^*g.

In addition, if f is a homeomorphsim and if its inverse $f^{-1} : Y \to Y'$ gives rise to a morphism $(Y', R') \to (Y, R)$ then f is said to be an isomorphism of ringed spaces.

[1] the term 'ringed space' in mathematical literature covers a more general situation than encountered in these notes

Example 3.4. Let $0 \leq k \leq \infty$ and let U be open in $\mathbb{R}^n$. We have the sheaf of $\mathbb{R}$-subalgebras C^k_U given by $C^k_U(V) = C^k(V)$ for all open subsets $V \subset U$. See Prop.1.4(2).

Definition 3.5. A C^k-manifold is a ringed space (M, R) with the property that M is covered by its open subsets U for which (U, R_U) is isomorphic to (Ω, C^k_Ω) where Ω is open in $\mathbb{R}^n$ for some n.

For a C^k-manifold (M, R) it is traditional to denote R by C^k_M.

Given C^k manifolds (M, C^k_M) and (N, C^k_N), a map $f : M \to N$ is C^k if it is a morphism $(M, C^k_M) \to (N, C^k_N)$ of ringed spaces. In other words,

(1) $f : M \to N$ is continuous and
(2) For every open $V \subset N$ and for every $g \in C^k_N(V)$, the continuous function f^*g defined on $f^{-1}V$ belongs to $C^k_M(f^{-1}V)$.

Problem 3.6. Let U and V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. They are C^k manifolds in a natural manner. Show that the two definitions of "$f : U \to V$ is C^k" given in 3.5 and 1.6 are equivalent to each other.

Problem 3.7. Define real analytic manifolds and complex analytic manifolds (the latter are referred to as complex manifolds). State and prove the analog of the previous problem in this context.

Problem 3.8. It is well known that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = \exp(-1/t)$ for $t > 0$ and
$f(t) = 0$ otherwise
is C^∞. Using this function alone (and 1.4,1.7, 3.5) show that if $p \in U \subset M$ where M is a C^k manifold and U is open in M there is a C^k function $\phi : M \to \mathbb{R}$ which (a) vanishes on the complement of U and (b) satisfies $\phi(p) = 1$.

Problem 3.9. Let M and N be C^k manifolds. Let $f : M \to N$ be any function. Show that f is C^k if and only if

$$g : N \to \mathbb{R} \text{ is } C^k \implies g \circ f \text{ is } C^k$$

4. Tangent-spaces and cotangent-spaces

References

[1] Apostol, T.M."Mathematical analysis" Addison-Wesley series in mathematics, 1974.
[2] Lang, S. "Differential manifolds"Addison-Wesley 1972.
[3] Narasimhan, R. "Analysis on Real and Complex Manifolds" Advanced studies in pure mathematics, 1973.
[4] Rudin, W."Principles of mathematical analysis" McGraw-Hill,1976.
[5] Spivak, M. "Calculus on Manifolds"1965.