

1. THE JACOBSON RADICAL

Let A be an arbitrary ring. Given a left A -module M , and $m \in M$, define $\text{Ann}_M(m) = \{a \in A \mid a \cdot m = 0\}$. This is a left ideal in A . The map $a \mapsto a \cdot m$ gives a morphism $A \rightarrow M$, of left A -modules. Hence, we obtain an A -module embedding $A/\text{Ann}_M(m) \rightarrow M$.

We put $\text{Ann}_M = \bigcap_{m \in M} \text{Ann}_M(m)$. Equivalently, Ann_M is the kernel of the ring homomorphism $A \rightarrow \text{End}_{\mathbb{Z}} M$, $a \mapsto \text{act}_M(a)$. Thus, Ann_M is a two-sided ideal in A .

Let S_A be the set of isomorphism classes of simple A -modules. The following result gives various equivalent definitions of the Jacobson Radical:

Theorem 1. *For any ring A , the following 7 sets are equal:*

- (1): $\{a \in A \mid \forall x, y \in A, 1 + xay \text{ is invertible}\}$
- (2L) $\{a \in A \mid \forall x \in A, 1 + xa \text{ has a left inverse}\}$
- (2R) $\{a \in A \mid \forall x \in A, 1 + ax \text{ has a right inverse}\}$
- (3L) *The intersection of all maximal left ideals in A .*
- (3R) *The intersection of all maximal right ideals in A .*
- (4L) $\bigcap_{L \in S_A} \text{Ann}_L$
- (4R) *The intersection of the annihilators of all simple right A -modules.*

The set $J(A)$ defined by these conditions is a two-sided ideal in A , thanks to (4L)-(4R), called the Jacobson Radical of A .

Proof. We will prove $4L \iff 3L \iff 2L \iff 1$. The corresponding claims for ‘R’ will follow.

(4L $\iff$ 3L): Let $\mathcal{M} = \{(M, m) \mid M \in S_A, m \in M \setminus \{0\}\}$. Define a map from $\mathcal{M}$ to the set of maximal left ideals by $(M, m) \mapsto \text{Ann}_M(m)$. This map is surjective since simple A -modules are precisely the A -modules of the form A/I for a maximal left ideal I of A . Hence, the intersection of all maximal left ideals of A is equal to

$$\bigcap_{(M, m) \in \mathcal{M}} \text{Ann}_M(m) = \bigcap_{M \in S_A} \bigcap_{m \in M \setminus \{0\}} \text{Ann}_M(m) = \bigcap_{M \in S_A} \text{Ann}_M.$$

We note that the rightmost intersection above is clearly a two-sided ideal.

(3L $\iff$ 2L): $a \in A$ has no left inverse iff it is contained in a proper left ideal iff it is contained in a maximal left ideal.

Let $\mathcal{L}$ be the set of elements $a \in A$ such that $1 + xa$ has a left inverse for all $x \in A$. Let $a \in \mathcal{L}$ and assume there is a maximal left ideal I not containing a . Then $Aa + I = A \Rightarrow 1 = xa + y$ (for some $x \in A, y \in I$) $\Rightarrow y = 1 + xa$. Since $a \in \mathcal{L}$, then y has a left inverse, which is impossible since it is in I . Therefore, $\mathcal{L} \subseteq I$ for all maximal left ideals I .

Conversely, assume a is in all maximal left ideals, but there is x such that $1 + xa$ has no left inverse. Then $1 + xa$ is in some maximal left ideal I . But xa is also in this ideal, so 1 is in this ideal, a contradiction.

From above, we deduce that the set $\mathcal{L}$ is a two-sided ideal. We will use this in the next step:

(2L $\iff$ 1): Let $a' \in \mathcal{L}$ and for arbitrary $x, y \in A$ set $a = xa'y$. Then $a \in \mathcal{L}$ since $\mathcal{L}$ is a two-sided ideal. Hence, $1 + a$ has a left inverse b , so $b(1 + a) = 1 \Rightarrow b = 1 - ba$. Since $a \in \mathcal{L}$, then b has a left inverse c , so that $cb = 1$. But $c = c \cdot 1 = c(b(1 + a)) = cb(1 + a) = 1 + a$. Thus $(1 + a)b = 1$, so $1 + a$ has a two-sided inverse, whence (2L) $\Rightarrow$ (1). The reverse inclusion is clear. $\square$

Lemma 2 (Nakayama lemma). *Let M be a finitely-generated A -module such that $J(A) \cdot M = M$. Then $M = 0$.*

Proof. Find a minimal generating set $m_1, \dots, m_n$. Then we have $M = J(A) \cdot M = \sum J(A)m_i$. Then we may write $m_n = j_1m_1 + \dots + j_{n-1}m_{n-1} + j_nm_n$ for $j_i \in J(A)$. But then we have $(1 - j_n)m_n = j_1m_1 + \dots + j_{n-1}m_{n-1}$. Since $j_n \in J(A)$, then $1 - j_n$ is invertible, so that m_n is actually in the span of $\{m_i \mid i < n\}$, so we have a smaller generating set, a contradiction. $\square$