Counting short geodesics on a flat torus

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Let

\[ A_k = \{(m, n) \in \mathbb{Z}^2 | \gcd(m, n) = k\} \subset \mathbb{Z}^2 \]

The goal is to prove the theorem.

**Theorem 0.1.**

\[ \frac{\#\{A_1 \cap B(R)\}}{\pi R^2} = \frac{6}{\pi^2}. \]

Let

\[ \mu_R(U) = \frac{\#\{RU \cap SL_2(\mathbb{Z})(0, 1)\}}{R^2} \]

We know that

\[ SL_2(\mathbb{Z})(0, 1) = A_1 \]

The key point is to prove that as \( R \to \infty \), we have that

\[ \mu_R \xrightarrow{\text{weak}} k\mu_{\text{leb}} \]

The proof will use the key fact that the actions of \( SL_2(\mathbb{Z}) \) on \( \mathbb{R}^2 \) is ergodic.

1 **Old method**

We know that \( \mathbb{Z}^2 = \cup_{k=1}^\infty A_k \). Let \( B(R) \) be the radius \( R \) disk on \( \mathbb{R}^2 \). Since the number of lattice points is the same as the area of the ball as the radius goes to infinity. We have the following.

\[ \lim_{R \to \infty} \frac{\#\{\mathbb{Z}^2 \cap B(R)\}}{\pi R^2} = 1. \]

By homogeneity, we have that

\[ \#\{A_k \cap B(R)\} = \#\{A_1 \cap B(R/k)\} \]

Therefore we have that

\[ \lim_{R \to \infty} \sum_k \frac{\#\{A_1 \cap B(R/k)\}}{\pi R^2} = 1. \]

Let

\[ \frac{\#\{A_1 \cap B(R)\}}{\pi R^2} = \lambda. \]

We would have that

\[ \frac{\#\{A_1 \cap B(R/k)\}}{\pi R^2} = \frac{\lambda}{k^2}. \]
By computation, we have that
\[ \sum_k \frac{\lambda_k}{k^2} = 1 \]
This gives that
\[ \lambda = \frac{6}{\pi^2} \]

2 Using moduli space

Then we know that the \( \mathbb{H}^2/SL_2(\mathbb{Z}) \) is the moduli space of unit flat metric on torus (or on \( \mathbb{R}^2 \)).

\( (1, \tau = x + yi) \rightarrow (1/\sqrt{y}, x/\sqrt{y} + \sqrt{y}i) \).

Instead of changing the lattice point on \( \mathbb{R}^2 \), we could change the metric on \( \mathbb{R}^2 \) instead by \( \tau \). In this case, the unit ball becomes like an ellipse.

Now we want to determine \( k \). Let \( B_\tau \) be the unit ball with respect to \( \tau \).

\[ \int_{\mathbb{H}^2/SL_2(\mathbb{Z})} \mu_R(B_\tau) \rightarrow k \int_{\mathbb{H}^2/SL_2(\mathbb{Z})} d\tau = k \frac{\pi}{6} \]

The number \( \frac{\pi}{6} \) is coming from the fact that

\[ \int_{\mathbb{H}^2/PSL_2(\mathbb{Z})} d\tau = \frac{\pi}{3} \]

Let

\[ M^\gamma = \{ \tau, \gamma \}|\tau \in \mathbb{H}^2/SL_2(\mathbb{Z}), \gamma \ \text{a lattices point} \} = \mathbb{H}^2/\text{Stab}(0,1) \]

Let \( \chi_R \) be the characteristic function.

\[ \int_{\mathbb{H}^2/SL_2(\mathbb{Z})} \mu_R(B_\tau) = \int_{\mathbb{H}^2/\text{Stab}(0,1)} \chi_R(l_{(0,1)}(\tau))d\tau = \int_0^1 \int_{\frac{1}{\pi^2}}^{+\infty} \frac{dx dy}{y^2} = R^2 \]

Therefore we have

\[ k \frac{\pi}{6} = R^2 \]

Generalizing this method, we could count geodesics on any hyperbolic surface.