Reeb Stability

If \( \mathcal{F} \) is a codimension one foliation of a compact manifold \( M^n \). Suppose that \( L \) is a compact leaf of \( \mathcal{F} \) at \( \overline{\mathcal{F}}(L) \) is finite. Then all leaves are diffeomorphic with \( L \). We assume here that if \( M^n \) has boundary, \( M^n \) is a union of leaves.

and the leaves of \( \mathcal{F} \) are the fibers of a fibration of \( M^n \) over \( S^1 \) or \( I \).

Rmk: \( H(L; \mathbb{R}) = 0 \) is necessary. On \( L \times S^1 \), the one form \( \alpha = \omega + A \) is

Definition of foliation: \( \phi_i \) : \( U_i \rightarrow \mathbb{R}^n \) manifold structure. \( \phi_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

Coadim \( p \) folation

\[ \begin{align*}
\phi_{ij} : \mathbb{R}^n &\rightarrow \mathbb{R}^{n-p} \\
\phi_{ji} : \mathbb{R}^{n-p} &\rightarrow \mathbb{R}^n 
\end{align*} \]

Holonomy: \( H \rightarrow \text{germs} \ Diff(\mathbb{R}^{n-p}) \)

\( dH \rightarrow GL(p, \mathbb{R}) \)

*Thm*: If codim \( k \) foliation, \( L \) compact leaf of \( \mathcal{F} \). Then either:

1. \( dH \) is nontrivial
2. \( H(L; \mathbb{R}) \) is nontrivial
3. \( H \) is trivial and \( \overline{\mathcal{F}}(L) \) has a product structure.
$K \subset G$ a subset of $G$. $\varepsilon > 0$ $(K,\varepsilon)$-coycle with values in $\mathbb{R}^K$ is an $\mathbb{R}^K$-valued function on $K$ sat $\|s^r(a,b)\| \leq \varepsilon$

$s^r(a,b) = r(ab) - r(a) - r(b)$

$\text{Normal } (K,\varepsilon)\text{-coycle is sat } \max_{\beta \in B} \|r(\beta)\| = 1$

Suppose $B$ generates $G$

$B \subset K$

Let $B^t = \text{set of products of at most } t \text{ elements of } B$. 

**Lemma:** Nontrivial cocycles exist iff normal $(B^t,\varepsilon)$-cocycles exist for every $\varepsilon > 0$ and $t$.

**Pf:** The set of normal $(B^t,\varepsilon)$-cocycles is compact space.

$(B^t,\varepsilon)$-cocycles $\supset (B^{t'},\varepsilon')$ cocycles if $t' \geq t$ and $0 < \varepsilon' \leq \varepsilon$

$(\bigcap_{\varepsilon} B^t) = B^t \Rightarrow B$ has nontrivial element.

Ways to construct cocycles.

Therefore we need to construct $(B^t,\varepsilon)$-cocycles.

$y_x(a) = \frac{1}{m} (H^t(a) - x)$

where $m = \max_{a \in B} \|H^t(a) - x\|$

if $H$ is nontrivial and $\partial H$ is trivial

**Lemma:** $\forall t, \varepsilon > 0 \exists \delta$ so if $\|x\| < \delta$ $y_x$ is a $(B^t,\varepsilon)$-cocycle.

Induction on $t$. 

$\Box$
Cor: \( G \) top gp acting on a manifold \( V \) with a fixed pt \( p \).

Either

1. \( \mathbb{R}^G \to \mathbb{R}^{GL(K)} \) is nontrivial

or

2. \( H^c(G, \mathbb{R}) \neq 0 \)

or

3. trivial action globally.

The same proof.

Application:

\[
\text{Diff}(\mathbb{R}^n) \quad \Downarrow \quad \text{no section}
\]

\[
[B_n, B_n] \subset B_n
\]

\[
H^c([B_n, B_n], \mathbb{R}) = 0
\]

\[
[B_n, B_n] \quad \rightarrow \quad GL(2, \mathbb{R})
\]

representation problem