GLOBAL WELL-POSEDNESS, SCATTERING AND BLOW UP FOR THE ENERGY-CRITICAL, FOCUSING, NON-LINEAR SCHRÖDINGER AND WAVE EQUATIONS

by

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Lecture 1

In these lectures I will discuss recent joint works with F. Merle. In them we have developed an approach to the study of non-linear critical problems of dispersive type. The issues studied are global well-posedness and scattering. The approach works for both focusing and defocusing problems, but in these lectures I will concentrate on two focusing problems. The approach proceeds in steps, some of which are general and hence apply to "all problems" and some which are specific to each particular problem. The concrete problems to be discussed here are the energy critical, focusing non-linear Schrödinger equation and wave equation. I will try to separate both kinds of arguments in the exposition. I will start out by discussing (NLS).

Consider thus the Cauchy problem for the $\dot{H}^1$ critical non-linear Schrödinger equation

$$(CP) \qquad \begin{cases} i\partial_t u + \Delta u \pm |u|^{4/N-2} u = 0, x \in \mathbb{R}^N, t \in \mathbb{R}, \\ u|_{t=0} = u_0 \in \dot{H}^1. \end{cases}$$

The problem is called "critical" because if u is a solution and $\lambda > 0$, $u_\lambda(x,t) = \frac{1}{\lambda^{N-2/2}}\, u(\frac{x}{\lambda}, \frac{t}{\lambda^2})$ is also a solution and $||u_\lambda(-,0)||_{\dot{H}^1} = ||u_0||_{\dot{H}^1}$, $\forall \lambda > 0$. Here the $-$ sign corresponds to the defocusing problem and the $+$ sign to the focusing problem. The theory of the local Cauchy problem (Cazenave-Weissler 90, [**4**]) shows that if $||u_0||_{\dot{H}^1} \le \delta$, $\delta = \delta_N > 0$ is small (and $N \le 5$) then $\exists !$ solution to (CP) with $u \in C(\mathbb{R}; \dot{H}^1)$, $||u||_{L^{2(N+2)/N-2}_{x,t}} < \infty$ (*i.e.* the solution scatters). As we will see later this is equivalent to $\exists\, u_0^\pm \in \dot{H}^1$ s.t. $||u(-,t) - e^{it\Delta}\, u_0^\pm||_{\dot{H}^1} \xrightarrow[t\to\pm\infty]{} 0$. Also, the energy identity holds, *i.e.*

$$E(u(t)) = \frac{1}{2}\int |\nabla u(x,t)|^2\, dx \pm \frac{1}{2^*}\int |u(x,t)|^{2^*}\, dx = E(u_0)\,.$$

Here $\pm$ corresponds to the defocusing, focusing cases, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$ is the "Sobolev conjugate" exponent. Here we see the difference between the defocusing and focusing cases. In the

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defocusing case, Bourgain ([**3**], 1998) proved that for $N = 3, 4$, u_0 radial, the above result holds for $||u_0||_{\dot{H}^1} < \infty$. Bourgain's result was extended to u_0 radial, $N \geq 5$ by Tao ([**30**], 2004). For $N = 3$, general u_0 the same result was proved by Colliander-Keel-Staffilani-Takaoka-Tao ([**5**], 2004). This was extended to $N = 4$ by Ryckman-Visan ([**25**], 2005) and to $N \geq 5$ by Visan ([**33**], 2007). In the focusing case, these last results do not hold. In fact, a classical argument, based on the "virial identity" (Zakharov, Glassey) shows that if $\int |x|^2 |u_0(x)|^2 < \infty$ and $E(u_0) < 0$, then the solution must break-down in finite time (Glassey 77, [**11**]). Also,

$$W(x,t) = W(x) = \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{-(N-2)/2} \in \dot{H}^1$$

and solves the elliptic equation

$$\Delta W + |W|^{4/N-2}\, W = 0\,, \quad x \in \mathbb{R}^N,$$

and hence (NLS), but scattering does not occur, even though the solution is global in time. Our main result in this case is:

Theorem A **(K-Merle [16], 2006)**. — *For the energy critical, focusing (NLS),* $N = 3, 4, 5$, u_0 *radial with* $E(u_0) < E(W)$,

i) *if* $||u_0||_{\dot{H}^1} < ||W||_{\dot{H}^1}$ *the solution exists for all times and scatters.*
ii) *if* $||u_0||_{\dot{H}^1} > ||W||_{\dot{H}^1}$ *(and* $||u_0||_{L^2} < \infty$*) then the solution breaks down in finite time.*

Remark. — *The conditions* $E(u_0) < E(W)$ *and* $||\nabla u_0| = ||\nabla W||$ *are incompatible (from now on,* $||\ \ ||$ *is the* L^2 *norm).*

I will now turn to the proof of Theorem A. We start with a quick review of the local (CP) theory.

Theorem **(Cazenave-Weissler [4], 1990)**. — *Let* $u_0 \in \dot{H}^1(\mathbb{R}^N)$, $||u_0||_{\dot{H}^1} \leq A$. *Then, (for* $3 \leq N \leq 5$*)* $\exists \delta = \delta(A)$ *s.t. if* $||e^{it\Delta} u_0||_{S(I)} \leq \delta$, $0 \in \dot{I}$, *there exists a unique solution to (CP) in* $\mathbb{R}^N \times I$ *s.t.* $u \in C(I; \dot{H}^1)$, $\sup_{t \in I} ||u(t)||_{\dot{H}^1} + ||\nabla u||_{W(I)} \leq C(A)$ *and* $||u||_{S(I)} \leq 2\delta$. *(Here* $||f||_{S(I)} = ||f||_{I}^{2(N+2)/N-2}\, L_x^{2(N+2)/N-2}$; $||f||_{W(I)} = ||f||_{L_I^{2(N+2)/N-2} L_x^{2N(N+2)/N^2+4}}$*). Moreover,* $u_0 \mapsto u \in C(I; \dot{H}^1)$ *is Lipschitz.*

Sketch of the Proof. — The key ingredients are the Strichartz estimates (Strichartz 77, [**28**], Keel-Tao 98, [**15**]) $(N \geq 3)$

(S)

$$\text{i) } ||\nabla\, e^{it\Delta}\, u_0||_{W(-\infty,+\infty)} \leq C\, ||u_0||_{\dot{H}^1}$$

$$\text{ii) } ||\nabla \textstyle\int_0^t e^{i(t-t')\Delta} g(-,t')\, dt'||_{W(-\infty,+\infty)} \leq C\, ||\nabla g||_{L_t^2\, L_x^{2N/N+2}}$$

$$\text{iii) } \sup_t ||\nabla \int_0^t e^{i(t-t')\Delta}\, g(-,t')\, dt'|| \leq C\, ||\nabla g||_{L_t^2\, L^{2N/N+2}}$$

and the Sobolev embedding:

(Sob)

$$||v||_{L_I^{2(N+2)/N-2}\, L_x^{2(N+2)/N-2}} \leq C\, ||\nabla v||_{W(I)}\,.$$

We then have (with $f(u) = \pm|u|^{4/N-2}\,u$) to solve the integral equation (Duhamel's principle)

$$u(t) = e^{it\Delta}\,u_0 + \int_0^t e^{i(t-t')\Delta}\,f(u)\,dt'.$$

Let $B_{a,b} = \{v \in \mathbb{R}^N \times I : ||v||_{S(I)} \le a,\ ||\nabla v||_{W(I)} \le b\}$, $\Phi u_0(v) = e^{it\Delta}\,u_0 + \int_0^t e^{i(t-t')\Delta}\,f(v)\,dt'$. We will show that we can choose δ, a, b s.t.

$$\Phi u_0 : B_{a,b} \longrightarrow B_{a,b}$$

and is a contraction. From this, also using (S) iii), the theorem follows. But, by (S) i), ii)

$$||\nabla \Phi u_0(v)||_{(I)} \le C\,A + C\,||\nabla f(v)||_{L^2_I\,L_x^{2N/N+2}}.$$

But $|\nabla f(v)| \le C\,|\nabla v|\,|v|^{4/N-2}$, so Hölder gives that this is $\le C\,A + C\,||v||_{S(I)}^{4/N-2}\,||\nabla v||_{W(I)} \le C\,A + C\,a^{4/N-2}\,b$. By (Sob), $||\Phi u_0(v)||_{S(I)} \le \delta + C\,a^{4/N-2}\,b$. We then choose $b = 2AC$, a so that $C\,a^{4/N-2} \le \frac{1}{2}$, so that $||\nabla \Phi u_0(v)||_{W(I)} \le b$. If we now set $\delta = \frac{a}{2}$, $C\,a^{4/N-2-1}\,b \le \frac{1}{2}$ (which is possible if $N < 6$). We obtain $||\Phi u_0(v)||_{S(I)} \le a$. The contraction property is similar and the Theorem follows. □

Remark. — *Because of (S), (Sob), $\exists\,\tilde\delta$ s.t. if $||u_0||_{\dot H^1} < \tilde\delta$, the hypothesis of the Theorem holds for $I = (-\infty, +\infty)$. Moreover, given $u_0 \in \dot H^1$, $\exists\,I$ s.t. $||e^{it\Delta}\,u_0||_{S(I)} \le \delta$, so the Theorem applies on I. Note also that if $u^{(1)}$, $u^{(2)}$ are solutions of (CP) on I ($u \in C(I;\dot H^1)$, $\nabla u \in W(I)$, the integral equation holds with $u^{(1)}(t_0) = u^{(2)}(t_0)$, $t_0 \in I$), then $u^{(1)} \equiv u^{(2)}$ on I. This is because we can partition I into I_j's s.t. $||u^{(i)}||_{S(I_j)} \le a$, $||\nabla u^{(i)}||_{W(I_j)} \le b$; choosing $t_0 \in I_{j_0}$ using the uniqueness of the fixed point in I_{j_0} and then induction on j, our claim follows. Thus, there exists a maximal interval $I = I(u_0) = (-T_-(u_0), T_+(u_0))$ where the solution $u \in C(I', \dot H^1) \cap \{\nabla u \in W(I')\}$, $\forall\,I' \Subset I$, $I' \ne I$, is defined. We call I the maximal interval of existence. For $t \in I$ we have $E(u(t)) = E(u_0)$.*

Standard blow-up criterion. — If $T_+(u_0) < +\infty$, we must have $||u||_{S[0,T_+(u_0))} = +\infty$. If not, $M = ||u||_{S[0,T_+(u_0))} < \infty$. For $\varepsilon > 0$, to be chosen, partition $[0, T_+(u_0)) = \bigcup_{j=1}^{\gamma(\varepsilon,M)} I_j$, so that $||u||_{S(I_j)} \le \varepsilon$. If $I_j = [t_j, t_{j+1})$, using the integral equation and the proof of the Theorem above, we have

$$\sup_{t\in I_j} ||u(t)||_{\dot H^1} + ||\nabla u||_{W(I_j)} \le C\,||u(t_j)||_{\dot H^1} + C\,||u||_{S(I_j)}^{4/N-2}\,||\nabla u||_{W(I_j)}.$$

If $C\,\varepsilon^{4/N-2} \le \frac{1}{2}$ we can show inductively that

$$\sup_{t\in[0,T_+(u_0))} ||u(t)||_{\dot H^1} + ||\nabla u||_{W([0,T_+(u_0))} \le C(M).$$

Choose now $(t_n) \uparrow T_+(u_0)$ and show, again using the integral equation, that for n large, $||e^{i(t-t_n)\Delta}\,u(t_n)||_{S(t_n,T_+(u_0))} \le \frac{\delta}{2}$ (on $[t_n, T_+(u_0))$, $u(t) = e^{i(t-t_n)\Delta}\,u(t_n) + \int_{t_n}^t e^{i(t-t')\Delta}\,f(u)\,dt'$). But then, for same $\varepsilon_0 > 0$ we have

$$||e^{i(t-t_n)\Delta}\,u(t_n)||_{S(t_n,T_+(u_0)+\varepsilon_0)} \le \delta$$

which, by the Theorem contradicts the definition of $T_+(u_0)$.

Scattering. — If $T_+(u_0) = +\infty$ and $M = ||u||_{S(0,+\infty)} < \infty$ then u scatters at $+\infty$. In fact, by the integral equation, as before we show that $\sup_{t\in[0,+\infty)} ||u(t)||_{\dot{H}^1} + ||\nabla u||_{W(0,+\infty)} \le C(M)$. But then, since

$$u(t) = e^{it\Delta}\, u_0 + \int_0^t e^{i(t-t')\Delta}\, f(u)\, dt'$$

and if we set $u_0^+ = u_0 + \int_0^\infty e^{-it'\Delta}\, f(u)\, dt'$, so that by (S), $u_0^+ \in \dot{H}^1$ and $u(t) - e^{it\Delta}\, u_0^+ = e^{it\Delta}\int_t^\infty e^{-it'\Delta}\, f(u)\, dt' \to 0$ in $\dot{H}^1$ as $t \to +\infty$ from iii), so that we get scattering.

We now turn to a perturbation theorem which is an important step in what follows. The proof sketched in our original paper is incorrect. We are indebted to M. Visan and X. Zhang for pointing this out and suggesting the use of fractional derivatives to give a correct proof.

Perturbation Theorem. — Let $I = [0, L)$, $L \le +\infty$, let $\tilde{u}$ be defined on $\mathbb{R}^N \times I$ be such that

$$\sup_{t\in I} ||\tilde{u}(t)||_{\dot{H}^1} \le A\,, \quad ||\tilde{u}||_{S(I)} \le M\,, \quad ||\nabla\tilde{u}||_{W(I)} < \infty$$

verify in the sense of the integral equation

$$i\,\partial_t \tilde{u} + \Delta\tilde{u} + f(\tilde{u}) = e$$

and let $u_0 \in \dot{H}^1$ be s.t. $||u_0 - \tilde{u}(0)||_{\dot{H}^1} \le A'$. Then $\exists\, \varepsilon_0 = \varepsilon_0(M, A, A')$ s.t. if $0 < \varepsilon \le \varepsilon_0$ and

$$||\nabla e||_{L^2_I L_x^{2N/N+2}} \le \varepsilon\,, \quad ||e^{it\Delta}\,[u_0 - \tilde{u}(0)]||_{\delta(I)} \le \varepsilon\,,$$

then $\exists\,!$ solution u on $\mathbb{R}^N \times I$, s.t.

$$||u||_{S(I)} \le C(A', A, M) \text{ and } \sup_{t\in I} ||u(t) - \tilde{u}(t)||_{\dot{H}^1} \le C(A, A', M)(A' + \varepsilon + \varepsilon')$$

where $\varepsilon' = \varepsilon^\beta$ for some $\beta > 0$.

In the proof it suffices to give a priori estimates for u, assuming that it exists. The (CP) theory gives the rest. We will need 2 new ingredients:

$$(F) \qquad \Big\| \int_0^t e^{i(t-t')\Delta}\, h(t')\, dt' \Big\|_{L_t^q\, L_x^r} \le C||h||_{L_t^{\tilde{q}'}\, L_x^{\tilde{r}'}}$$

(Foschi [**6**], 2003, Vilela [**32**], 2007) holds, provided

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{N}{2}\Big[1 - \frac{1}{r} - \frac{1}{\tilde{r}}\Big] \text{ and } \frac{1}{q} < N\Big(\frac{1}{2} - \frac{1}{r}\Big),$$

$$\frac{1}{\tilde{q}} < N\Big(\frac{1}{2} - \frac{1}{\tilde{r}}\Big)\,, \quad \frac{1}{q} + \frac{1}{\tilde{q}} < 1\,, \quad \frac{N-2}{r} < \frac{N}{\tilde{r}}\,, \quad \frac{N-2}{\tilde{r}} < \frac{N}{r}.$$

Notice that $(q, r) = (\frac{2(N+2)}{N-2}, \frac{2(N+2)}{(N-2)})$, $(\tilde{q}, \tilde{r}) = (2, \frac{2N}{N-2})$ verify the conditions. For $0 < \alpha < 1$, α near 1, let $\frac{1}{r} = \frac{N-2}{2(N+2)} + \frac{\alpha}{N}$ so that

$$||f||_{L_x^{2(N+2)/(N-2)}} \le C||D^\alpha f||_{L_x^r}\,,$$

and with $q = \dfrac{2(N+2)}{N-2}$ we have

$$||f||_{S(I)} \le C\,||D^\alpha f||_{L^q_I L^r_x} \le C\,||\nabla f||_{W(I)}$$

and, by interpolation,

$$||D^\alpha f||_{L^q_I L^r_x} \le C\,||f||^{(1-\alpha)}_{S(I)}\,||\nabla f||^{\alpha}_{W(I)}.$$

Set $\tilde q = 2$, $\dfrac{1}{\tilde r} = \dfrac{N^2 - 2(\alpha-1)N - 4\alpha}{2N(N+2)}$, so that $\dfrac{1}{\tilde r'} = \dfrac{1}{r} + \dfrac{2}{(N+2)}$. Note then that, for α close to 1, (F) is verified. By interpolation we have $||D^\alpha e^{it\Delta}[u_0 - \tilde u(0)]||_{L^q_I L^r_x} \le C(A')\,\varepsilon'$, $\varepsilon' = \varepsilon^{(1-\alpha)}$. Moreover, by Hölder,

$$||\,|u|^{4/N-2}\,D^\alpha u||_{L^{q'}_I L^{\tilde r'}_x} \le ||u||^{4/N-2}_{S(I)}\,||D^\alpha u||_{L^q_I L^r_x}.$$

The second ingredient is the chain rule and Leibniz rule for fractional derivatives ([**19**], 93]): in this case,

$$\begin{aligned}||D^\alpha\big[f(\tilde u + w) - f(\tilde u)\big]||_{L^{\tilde q'}_t L^{\tilde r'}_x} &\le C\big[||\tilde u||^{4/n-2}_S + ||w||^{4/N-2}_S\big]||D^\alpha w||_{L^q_t L_x}\\ &+ C||w||_s\big[||\tilde u||^{(6-N)/N-2}_S + ||w||^{6-N/N-2}_S\big]\big[||D^\alpha \tilde u||_{L^q_t L^r_x} + ||D^\alpha w||_{L^q_t L^r_x}\big].\end{aligned}$$

To carryout the proof, we write $u = \tilde u + w$, so that the equation for w is $i\partial_t w + \Delta w = f(\tilde u + w) - f(\tilde u) - e$, $w|_{t=0} = u_0 - \tilde u(0)$. Note that by the integral equation for $\tilde u$, splitting into sub-intervals we obtain $||\nabla \tilde u||_{W(I)} \le \tilde M = \tilde M(M, A)$, so that, by interpolation, $||D^\alpha \tilde u||_{L^q_I L^r_x} \le M_1 = M_1(M, A)$. We then split $I = \overset{J}{\underset{j=1}{\cup}} I_j$, $J = J(M, A, \eta)$ so that on each I_j we have $||D^\alpha \tilde u||_{L^q_{I_j} L^r_x} \le \eta$, $\eta > 0$ to be chosen. Let $I_j = [a_j, a_{j+1}]$, $a_0 = 0$, $a_{J+1} = L$. By the integral equation on I_j

$$w(t) = e^{i(t-a_j)\Delta}\,w(a_j) + \int_{a_j}^t e^{i(t-t')\Delta}\,[f(\tilde u + w) - f(\tilde u)]\,dt' + \int_{a_j}^t e^{i(t-t')\Delta}\,e(t')\,dt'.$$

By (F) (and (Sob) and (S)) we have

$$\begin{aligned}||D^\alpha w||_{L^q_{I_j} L^r_x} &\le ||D^\alpha\,e^{i(t-a_j)\Delta}\,w(a_j)||_{L^q_{I_j} L^r_x} + C\,\varepsilon_0 + C\,||D^\alpha[f(\tilde u + w) - f(\tilde u)]||_{L^{\tilde q'}_{I_j} L^{\tilde r'}_x}\\ &\le \Big(||D^\alpha\,e^{i(t-a_j)\Delta}\,w(a_j)||_{L^q_{I_j} L^r_x} + C\,\varepsilon_0\Big) + C\,\eta^{4/N-2}\,||D^\alpha w||_{L^q_{I_j} L^r_x}\\ &\quad + C\,||D^\alpha w||^{(N+2)/N-2}_{L^q_{I_j} L^r_x}.\end{aligned}$$

Thus, if $C\,\eta^{4/N-2} \le \frac{1}{3}$, we get

$$||D^\alpha w||_{L^q_{I_j} L^r_x} \le \frac{3}{2}\,\gamma_j + \tilde C\,||D^\alpha w||^{(N+2)/(N-2)}_{L^q_{I_j} L^r_x},$$

where

$$\gamma_j = \Big[||D^\alpha\,e^{i(t-a_j)\Delta}\,w(a_j||_{L^q_t L^r_x} + C\,\varepsilon_0\Big].$$

Note that η depends only on N. From this a standard continuity argument shows that there exists $C_0 = C_0(\tilde C)$ s.t. if $\gamma_j \le C_0$, we have

$$a)\ ||D^\alpha w||_{L^q_{I_j} L^r_x} \le 3\gamma_j. \qquad b)\ \tilde C\,||D^\alpha w||^{(N+2)/N_2}_{L^q_{I_j} L^r_x} \le 3\gamma_j\,.$$

Hence, $||D^\alpha w||_{L^q_{I_j} L^r_x} \le 3[||D^\alpha e^{i(t-a_j)\Delta} w(a_j)||_{L^q_t L^r_x} + C\,\varepsilon_0]$. To continue in the iteration, put $t = a_{j+1}$ in the integral formula, apply $e^{i(t-a_{j+1})\Delta}$ to obtain:

$$e^{i(t-a_{j+1})\Delta} w(a_{j+1}) = e^{i(t-a_j)\Delta} + \int_{a_j}^{a_{j+1}} e^{i(t-t')\Delta}[f(\tilde u + w) - f(\tilde u)]\,dt' + \int_{a_j}^{a_{j+1}} e^{i(t-t')\Delta} e(t')\,dt'.$$

By the same argument we get:

$$\begin{aligned} ||D^\alpha e^{i(t-a_{j+1})\Delta} w(a_{j+1})||_{L^q_t L^r_x} \le & ||D^\alpha e^{i(t-a_j)\Delta} w(a_j)||_{L^q_t L^r_x} \\ & + C\,\varepsilon_0 + C\,\eta^{4/N-2}\,||D^\alpha w||_{L^q_{I_j} L^r_x} + \tilde C\,||D^\alpha w||_{L^q_{I_j} L^r_x}. \end{aligned}$$

Again, taking η small we see that $\gamma_{j+1} \le 10\gamma_j$ provided $\gamma_j \le C_0$. Recall that by assumption we have $\gamma_0 \le \varepsilon_0' + C\,\varepsilon_0$. Iterating, $\gamma_j \le 10^j(\varepsilon_0' + C\,\varepsilon_0)$, if $\gamma_j \le C_0$. If we have 10^{J+1} $(\varepsilon_0' + C\,\varepsilon_0) \le C_0$, this always holds. Repeating the argument we obtain

$$||D^\alpha w||_{L^q_I L^r_x} \le 3(J+1)\,10^{(J+1)}\,(\varepsilon' + C\,\varepsilon),$$

for ε_0 small. Hence, by Sobolev $||w||_{S(I)} \le C(\varepsilon' + \varepsilon)$. The rest of the argument follows similarly.

Some useful corollaries:

Corollary 1. — *Let $K \subset \dot H^1$ be s.t. $\overline{K}$ is compact. Then $\exists\, T_K^+, T_K^-$ s.t. $\forall\, u_0 \in K$, $T^+(u_0) \ge T_K^+$, $T^-(u_0) \ge T_K^-$.*

Choose $M = 1$, $\tilde A = \sup_{u_0 \in K} ||u_0||_{\dot H^1}$, $A = C(\tilde A)$ as in (CP), $\varepsilon_0 = \varepsilon_0(1, A, 1)$ as in Perturbation Theorem, $\varepsilon_1 \le \min(\varepsilon_0, 1)$. Cover K by balls $B(u_{0,j}, \varepsilon_1)$, $1 \le j \le J$ (compactness of $\overline{K}$). Consider $\tilde T_j^+, \tilde T_j^-$ s.t. $||u_j||_{S[-\tilde T_j^-, \tilde T_j^+]} \le 1$ and $T^+ = \min_{1\le j\le J} T_j^+$, $T^- = \min_{1\le j\le J} T_j^-$. Then, if $u_0 \in B(u_{0,j}, \varepsilon_1)$ for some j, the solution exists in $[-T^-, T^+]$ by Perturbation Theorem.

Corollary 2. — *Let $\tilde u_0 \in \dot H^1$, $||\tilde u_0||_{\dot H^1} \le A$, $\tilde u$ solution in $(-T_-(\tilde u_0), T_+(\tilde u_0))$. If $u_{0,n} \to \tilde u_0$ in $\dot H^1$, then $T_-(\tilde u_0) \ge \liminf T_-(u_{0,n})$; $T_+(\tilde u_0) \le \liminf T_+(u_{0,n})$ and $\forall\, t \in (-T_-(\tilde u_0), T_+(\tilde u_0))$ we have $u_n(t) \to \tilde u(t)$.*

In fact, if $I' \subset\subset I = (-T_-(\tilde u_0), T(\tilde u_0))$, $\sup_{t\in I'} ||\tilde u(t)||_{\dot H^1} \le C(A, I')$, $||\tilde u||_{S(I')} \le M$. Apply the Perturbation Theorem with $u = u_n$, $u_0 = u_{0,n}$ on I'. If $\varepsilon_0 = \varepsilon_0(M, C(A, I'), 1)$ and n is so large that $||u_{0,n} - \tilde u_0||_{\dot H^1} \le 1$, $||e^{it\Delta}[u_{0,n} - \tilde u_0]||_S \le \varepsilon_0$, we have u_n exists on I' and $\sup_{t\in I'} ||u_n(t) - \tilde u(t)||_{\dot H^1} \le C(A, M)\{||u_{0,n} - \tilde u_0||^\beta_{\dot H^1}\}$, $\beta > 0$, so the claim follows.

From now on we concentrate on the focusing case,

$$\begin{cases} i\,\partial_t u + \Delta u + |u|^{4/N-2}\,u = 0, \\ u|_{t=0} = u_0 \in \dot H^1. \end{cases}$$

We start out with a review of Glassey's blow-up result: assume that $\int |x|^2\,|u_0(x)|^2\,dx < \infty$, $u_0 \in \dot H^1$, $E(u_0) < 0$, $I = (-T_-(u_0), T_+(u_0))$. Let $y(t) = \int |x|^2\,|u(x,t)|^2\,dx$. A calculation shows that $y''(t) = 8\int |\nabla u(x,t)|^2 - |u(x,t)|^{2^*}\,dx$ (the same calculation also gives $y(t) < \infty$, $\forall\, t \in I$). Since $E(u(t)) = E(u_0) < 0$, $\frac12 \int |\nabla u(t))|^2 - |u(t)|^{2^*} = E(u(t)) + (\frac{1}{2^*} - \frac12)\int |u(t)|^{2^*} \le E(u(t)) = E(u_0) < 0$, $y''(t) < 16\,E(u_0) < 0$. But, since $y \ge 0$, I cannot be infinite. The next step is to establish some variational estimates. Recall that $W(x) = (1 + \frac{|x|^2}{N(N-2)})^{-(N-2)/2}$ is a stationary solution of (CP), $\in \dot{H}^1$ and solves the elliptic equation

$$\Delta W + |W|^{4/N-2}\, W = 0.$$

$W \geq 0$ and is radially decreasing. By the invariances of the equation, $W_{\theta_0,\lambda_0,x_0}(x) = e^{i\theta_0}\lambda_0^{N-2/2} W(\lambda_0(x-x_0))$ is still a solution. Aubin and Talenti (76) gave the following variational characterization of W: let C_N be the best constant in the Sobolev embedding $||u||_{L^{2^*}} \leq C_N ||\nabla u||_{L^2}$. Then $||u||_{L^{2^*}} = C_N\, ||\nabla u||_{L^2}$, $u \not\equiv 0 \Leftrightarrow u = W_{(\theta_0,\lambda_0,x_0)}$ for some $(\theta_0, \lambda_0, x_0)$. Note that by the elliptic equation, $\int |\nabla W|^2 = \int |W|^{2^*}$. Also $C_N ||\nabla W|| = ||W||_{L^{2^*}}$ so that $C_N^2 ||\nabla W||^2 = \big(\int |\nabla W|^2\big)^{(N-2)/N}$. Hence, $\int |\nabla W|^2 = \frac{1}{C_N^N}$. Moreover

$$E(W) = (\frac{1}{2} - \frac{1}{2^*}) \int |\nabla W|^2 = \frac{1}{N C_N^N}.$$

Lemma. — *Assume that* $||\nabla v|| < ||\nabla W||$ *and that* $E(v) \leq (1-\delta_0)\, E(W)$, $\delta_0 > 0$. *Then* $\exists\, \overline{\delta} = \overline{\delta}(\delta_0, N)$ *s.t.*

i) $||\nabla v||^2 \leq (1-\overline{\delta})\, ||\nabla W||^2$
ii) $\int |\nabla v|^2 - |v|^{2^*} \geq \overline{\delta}\, ||\nabla v||^2$
iii) $E(v) \geq 0$.

Proof. — Let $f(y) = \frac{1}{2}\, y - \frac{C^{2^*}}{2^*}\, y^{2^*/2}$, $\overline{y} = ||\nabla v||^2$. Note that $f(0) = 0$, $f(y) > 0$ for y near 0, $y > 0$ and that $f'(y) = \frac{1}{2} - \frac{C_N^{2^*}}{2}\, y^{2^*/2-1}$, so that $f'(y) = 0$ iff $y = y_0 = \frac{1}{C_N^N} = ||\nabla W||^2$. Also, $f(y_0) = \frac{1}{N C_N^N} = E(W)$. Since $0 \leq \overline{y} < y_c$, $f(\overline{y}) \leq (1-\delta_0)\, f(y_c)$ and f is non negative and strictly increasing between 0 and y_c, and $f'(y_c) \neq 0$, we obtain $0 \leq f(\overline{y})$, $\overline{y} \leq (1-\overline{\delta})\, y_c = (1-\overline{\delta}) ||\nabla W||^2$. This shows (i), (iii). For (ii), note that

$$\int |\nabla v|^2 - |v|^{2^*} \geq \int |\nabla v|^2 - C_N^{2^*}\Big(\int |\nabla v|^2\Big)^{2^*/2} = \int |\nabla v|^2 \Big[1 - C_N^{2^*}\Big(\int |\nabla v|^2\Big)^{2/(N-2)}\Big]$$
$$\geq \int |\nabla v|^2 \Big[1 - C_N^{2^*}(1-\overline{\delta})^{2/(N-2)}\Big(\int |\nabla W|^2\Big)^{2/(N-2)}\Big] = \int |\nabla v|^2 [1 - (1-\overline{\delta})^{2/(N-2)}]$$

which gives (iii). □

Remark. — *If* $||\nabla u_0|| < ||\nabla W||$, $E(u_0) \geq 0$.

From this static Lemma, we obtain dynamic consequences.

***Corollary* (Energy trapping)**. — *Let* u *be a solution of (CP) with maximal interval* I, $||\nabla u_0||^2 < ||\nabla W||^2$, $E(u_0) < E(W)$. *Choose* $\delta_0 > 0$ *s.t.* $E(u_0) \leq (1-\delta_0) E(W)$. *Then, for each* $t \in I$, *we have for* $\overline{\delta} = \overline{\delta}(\delta_0)$,

i) $||\nabla u(t)||^2 \leq (1-\overline{\delta}) ||\nabla W||^2$, $E(u(t)) \geq 0$
ii) $\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq \overline{\delta} \int |\nabla u(t)|^2$
iii) *(Coercivity and uniform bound)*

$$E(u(t)) \simeq ||\nabla u(t)||^2 \simeq ||\nabla u_0||^2\,,$$

with comparability constants which depend only on δ_0.

Proof. — From the continuity of the flow, conservation of energy and the previous Lemma. ☐

Remark. — *Let* $u_0 \in \dot{H}^1$, $E(u_0) < E(W)$ *but* $||\nabla u_0||^2 > ||\nabla W||^2$. *If* δ_0 *is chosen so that* $E(u_0) \leq (1-\delta_0)\,E(W)$, *we can conclude, in the same way that* $\int |\nabla u(t)|^2 \geq (1+\overline{\delta}) \int |\nabla W|^2$, $t \in I$.

But then, notice that:

$$\begin{aligned}\int |\nabla u(t)|^2 - |u(t)|^{2^*} &= 2^*\,E(u_0) - \frac{2}{(N-2)} \int |\nabla u(t)|^2 \\ &\leq 2^*\,E(W) - \frac{2}{(N-2)}\,\frac{1}{C_N^N} - \frac{2\overline{\delta}}{(N-2)}\,\frac{1}{C_N^N} = \frac{-2\overline{\delta}}{(N-2)C_N^N} < 0\,.\end{aligned}$$

Hence, if $\int |x|^2\,|u_0(x)|^2\,dx < \infty$, Glassey's proof shows that I cannot be finite. If u_0 is radial, $u_0 \in L^2$, using "local virial identities" one can see that the some holds. We now turn to the next step in the proof:

Concentration - Compactness Procedure. — We now turn to the proof of the positive result in Theorem A. Recall that by the coercitivity-uniform bound estimate, if $E(u_0) < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$, if δ_0 is s.t. $E(u_0) \leq (1-\delta_0)\,E(W)$, $E(u(t)) \simeq ||\nabla u(t)||^2 \simeq ||\nabla u_0||^2$, $t \in I$, and that if $||\nabla u_0||^2 < ||\nabla W||^2$, we have $E(u_0) \geq 0$. It now follows from (CP) that if $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) \leq \eta_0$, η_0 small, then $I = (-\infty, +\infty)$ and u scatters. Hence by considering $G = \{E : 0 < E < E(W)$: if $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) < E$, then $||u||_{S(I)} < \infty\}$ and $E_c = \sup G$, we find E_c with $\eta_0 \leq E_c \leq E(W)$ s.t. if $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) < E_c$, then $I = (-\infty, +\infty)$ and u scatters, and E_c is optimal with this property. Theorem A is the assertion $E_c = E(W)$. Assume then $E_c < E(W)$ and we will reach a contradiction. Note that if $0 \leq E < E_c$, $||\nabla u_0||^2 < ||\nabla W||^2$ and $E(u_0) < E$, then $||u||_{S(I)} < \infty$, while if $E_c < E < E(W)$, $\exists\, u_0$ s.t. $||\nabla u_0||^2 < ||\nabla W||^2$, $E_c \leq E(u_0) \leq E < E(W)$ and $||u||_{S(I)} = +\infty$. We will use a concentration-compactness argument to deduce some consequences of this that will eventually lead to a contradiction.

Proposition 1. — *There exists* $u_{0,c} \in \dot{H}^1$, $||\nabla u_{0,c}||^2 < ||\nabla W||^2$, *with* $E(u_{0,c}) = E_c(< E(W))$ *s.t. if* u_c *is the corresponding solution then* $||u_c||_{S(I)} = +\infty$.

Proposition 2. — *For any* u_c *as in Proposition 1, with (say)* $||u_c||_{S(I_+)} = +\infty$ $(I_+ = I \cap [0, +\infty))$, *there exist* $x(t) \in \mathbb{R}^N$, $\lambda(t) \in \mathbb{R}^+$, $t \in I_+$ *such that*

$$K = \Big\{v(x,t) : \frac{1}{\lambda(t)^{(N-2)/2}}\; u_c\Big(\frac{x - x(t)}{\lambda(t)}\,, t\Big) : t \in I^+\Big\}$$

has compact closure in $\dot{H}^1$.

The proof of Propositions 1 and 2 uses the coercitivity and uniform bound estimates, in conjunction with the "profile decomposition" of Keraani ([**20**], 2001), which describes the defect of compactness in the estimate

$$||e^{it\Delta}\,u_0||_S \leq C\,||u_0||_{\dot{H}^1},$$

which combines Strichartz (S) (i) with Sobolev (Sob). This is based on the "improved inequality" ($N = 3$)

$$||h||_{L^6(\mathbb{R}^3)} \leq C\, ||\nabla h||^{1/3}_{L^2(\mathbb{R}^3)}\, ||\nabla h||^{2/3}_{\dot{B}^0_{2,\infty}},$$

where $\dot{B}^0_{2,\infty}$ is the standard Besov space (see [**7**]).

***Theorem* (Profile decomposition, Keraani 2001).** — *Let* $\{v_{0,n}\} \in \dot{H}^1$, $||v_{0,n}||_{\dot{H}^1} \leq A$, $||e^{it\Delta} v_{0,n}|| \geq \delta > 0$. *Then, there exists a subsequence of* $\{v_{0,n}\}$ *and* $\{V_{0,j}\}_{j=1}^{\infty}$ *in* $\dot{H}^1$ *and triples* $(\lambda_{j,n}; x_{jn}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$, *with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \xrightarrow[n\to\infty]{} \infty\,, j \neq j'$$

(the triple is orthogonal), s.t.

i) $||V_{0,1}||_{\dot{H}^1} \geq \alpha_0(A) > 0$.

ii) *If* $V_j^\ell = e^{it\Delta}\, V_{0,j}$, *then we have, for each* J

$$v_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{N-2/2}} V_j^\ell\Big(\frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t_{j,n}}{\lambda_{j,n}^2}\Big) + w_n^J,$$

where $\liminf_{n\to\infty} ||e^{it\Delta}\, w_n^J||_S \xrightarrow[J\to\infty]{} 0$, *and for each* $J \geq 1$ *we have:*

(iii a) $||\nabla v_{0,n}||^2 = \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 + ||\nabla w_n^j||^2 + o(1)$, *as* $n \to \infty$
and

(iii b) $E(v_{0,n}) = \sum_{j=1}^{J} E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) + E(w_n^J) + o(1)$
as $n \to \infty$.

Lecture 2

In order to apply Keraani's Theorem to our non-linear problem, we need the notion of a "non-linear profile" : let $v_0 \in \dot{H}^1$, $v = e^{it\Delta} v_0$, $\{t_n\}$ a sequence with $\lim_{n\to\infty} t_n = \bar{t} \in [-\infty, +\infty]$. We say that $u(x,t)$ is a non linear profile associated with $(v_0, \{t_n\})$ if $\exists$ an interval I with $\bar{t} \in I$ (if $\bar{t} = \pm\infty$, $I = [a, +\infty)$, $(-\infty, a]$ respectively) such that u is a solution of (CP) on I and $\lim_{n\to\infty} ||u(-,t_n) - v(-,t_n)||_{\dot{H}^1} = 0$. There always exists a non-linear profile: if $\bar{t} \in (-\infty, +\infty)$ we solve (CP) with data at $\bar{t} = v(x, \bar{t})$. If $\bar{t} = +\infty$ (say), we solve integral equation

$$u(t) = e^{it\Delta}\, v_0 + \int_t^\infty e^{i(t-t')\Delta}\, f(u)\, dt',$$

in $\mathbb{R}^N \times [t_{n_0}, +\infty)$, where n_0 is so large that $||e^{it\Delta}\, v_0||_{S(t_{n_0}+\infty)} < \delta$. Then, $u(t_n) - v(t_n) = \int_{t_n}^\infty e^{i(t-t')\Delta}\, f(u)\, dt'$, which $\to 0$ in $\dot{H}^1$, since $\nabla f(u) \in L^2_{(t>t_{n_0})}\, L_x^{2N/N+2}$. It is easy to see that if $u^{(1)}, u^{(2)}$ are non-linear profiles associated to $(v_0, \{t_n\})$ in $I \ni \bar{t}$, then $u^{(1)} = u^{(2)}$ on I. Hence, there exists a maximal interval of existence I for the non-linear profile. Clearly, near finite end points of I, the S norm is infinite. These concepts are used in the following:

Proposition 3. — *Let* $\{z_{0,n}\} \in \dot{H}^1$, $||\nabla z_{0,n}||^2 < ||\nabla W||^2$ *and* $E(z_{0,n}) \to E_c(< E(W))$, $||e^{it\Delta}\, z_{0,n}||_{S(-\infty,+\infty)} \geq \delta > 0$. *Let* $(V_{0,j})_{j=1}^{\infty}$ *be as in the profile decomposition. Assume that one of*

a) $\liminf\limits_{n\to\infty} E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) < E_c$

or

b) $\liminf\limits_{n\to\infty} E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) = E_c$

and for $s_n = -\frac{t_{1,n}}{\lambda_1^2}$, *after passing to a subsequence so that* $s_n \to \bar{s} \in [-\infty,+\infty]$ *and* $E(V_1^\ell(-s_n)) \to E_c$, *and if* U_1 *is the non-linear profile associated to* $(V_{0,1},\{s_n\})$ *then* $I = (-\infty,+\infty)$, $||U_1||_{S(-\infty,+\infty)} < \infty$.

Then, (after passing to a subsequence) if z_n *solves (CP) for* $(z_{0,n})$, $||z_n||_{S(-\infty,+\infty)} < \infty$, *for* n *large. (In fact it is uniformly bounded in* n*.)*

We will first assume Proposition 3, use it to prove Propositions 1, 2, then prove Proposition 3.

Proof of Proposition 1. — Find $u_{0,n} \in \dot{H}^1$, $\int |\nabla u_{0,n}|^2 < \int |\nabla W|^2$, $E(u_{0,n}) \to E_c$, $||e^{it\Delta}\, u_{0,n}||_{S(-\infty,+\infty)} \geq \delta$, $||u_n||_{S(I_n)} = +\infty$, I_n a maximal interval. Since $E_c < E(W)$, for n large $E(u_{0,n}) \leq (1-\delta_0)\, E(W)$. By energy trapping, $\exists\, \bar{\delta}$ s.t. $||\nabla u_n(t)||^2 < (1-\bar{\delta})||\nabla W||^2$, $t \in I_n$. Fix $J \geq 1$ and apply the profile decomposition to $\{u_{0,n}\}$. We have

$$(\dagger) \qquad ||\nabla u_{0,n}||^2 = \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 + ||\nabla w_n^J||^2 + o(1),$$

$$(\ddagger) \qquad E(u_{0,n}) = \sum_{j=1}^{J} E\Big(V_j^\ell\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}\Big)\Big) + E(w_n^J) + o(1).$$

For n large, we have, from (†) that $||\nabla w_n^J||^2 \leq (1-\frac{\bar{\delta}}{2})||\nabla W||^2$ and $||\nabla V_{0,j}||^2 \leq (1-\frac{\bar{\delta}}{2})||\nabla W||^2$, $1 \leq j \leq J$. Hence, for n large $E(w_n^J) \geq 0$, $E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2}))) \geq 0$. Thus, $E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) \leq E(u_{0,n}) + o(1)$ by (‡), so that $\varliminf\limits_{n\to\infty} E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) \leq E_c$. Assume first that we have strict inequality. Then Proposition 3 a) gives a contradiction for large n. Thus, $\liminf_{n\to\infty} E(V_1^\ell(s_n)) = E_c$. Let U_1 be the non-linear profile associated to $(V_{0,1},\{s_n\})$. The first observation is that $V_{0,j} = 0$, $j > 1$. Indeed, by (‡) and the facts that $E(u_{0,n}) \to E_c$, $E(V_1^\ell(s_n)) \to E_c$ (after passing to a subsequence), we see that $E(w_n^J) \to 0$, $E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) \to 0$, $j = 2,\ldots,J$, But then, by coercitivity, we see that

$$\sum_{j=2}^{J} \Big\|\nabla V_j^\ell\Big(-\frac{t_{j,n}}{\lambda_{j,n}^2}\Big)\Big\|^2 + ||\nabla w_n^J||^2 \to 0\,.$$

But, $||\nabla V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})||^2 = ||\nabla V_{0,j}||^2$, establishing the claim, and in addition showing that $||\nabla w_n^J|| \to 0$, so that

$$u_{0,n} = \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^\ell\Big(\frac{x - x_{1,n}}{\lambda_{1,n}}, s_n\Big) + w_n^J .$$

Renormalize, setting $v_{0,n} = \lambda_{1,n}^{N-2/2} \; u_{0,n}(\lambda_{1,n}(x + x_{1,n}))$, which has the same properties as $u_{0,n}$, and so that

$$v_{0,n} = V_1^\ell(s_n) + \tilde{w}_n^J , \quad ||\nabla \tilde{w}_n^J|| \to 0 .$$

Let $I_1 = \max \operatorname{int}$ of U_1. By definition of non-linear profile, $E(U_1(s_n)) = E(V_1^\ell(s_n)) + o(1) = E_c + o(1)$, $||\nabla U_1(s_n)||^2 = ||\nabla V_1^\ell(s_n)||^2 + o(1) = ||\nabla V_{1,0}||^2 + o(1) = ||\nabla u_{0,n}||^2 + o(1) < ||\nabla W||^2$ for n large. Now fix $\bar{s} \in I_1$, so that $E(U_1(\bar{s})) = E(U_1(s_n)) \to E_c$, so that $E(U_1(\bar{s})) = E_c$. Also, $||\nabla U_1(s_n)||^2 < ||\nabla W||^2$ for n large, so that, by energy trapping, $||\nabla U_1(\bar{s})||^2 < ||\nabla W||^2$. If $||U_1||_{S(I_1)} < \infty$, Proposition 3 b) gives a contradiction. Hence $||U_1||_{S(I_1)} = +\infty$, we take $u_c = U_1$. □

Proof of Proposition 2. — (by contradiction) Let $u(x,t) = u_c(x,t)$. If not, $\exists\, \eta_0 > 0$, $\{t_n\}_{n=1}^\infty$, $t_n \geq 0$ s.t. $\forall\, \lambda_0 \in \mathbb{R}^+$, $x_0 \in \mathbb{R}^N$ we have (after rescaling)

$$(*) \qquad \Big\| \frac{1}{\lambda_0^{(N-2)/2}}\, u\Big(\frac{x - x_0}{\lambda_0}, t_n\Big) - u(x, t_n) \Big\|_{\dot{H}^1} \geq \eta_0 , \quad n \neq n'.$$

After passing to a subsequence, $t_n \to \bar{t} \in [0, T_+(u_0)]$, so that $\bar{t} = T_+(u_0)$ by continuity of the flow. We can also assume, by (CP) that $||e^{it\Delta}\, u(t_n)||_{S(0,+\infty)} \geq \delta$. We now apply the profile decomposition to $v_{0,n} = u(t_n)$. We have $E(u(t)) = E(u_0) = E_c < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$, so that $||\nabla u(t)||^2 \leq (1 - \bar{\delta}).\ ||\nabla W||^2$, $t \in I_+$. But then $\liminf_{n\to\infty} E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) \leq E_c$. If we have strict inequality, Proposition 3 a) gives a contradiction. Hence we have equality and as before $V_{0,j}$, $j = 2, \ldots, J$, are all 0 and $||\nabla w_n^J|| \to 0$. Thus, we have

$$u(t_n) = \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^\ell\Big(\frac{x - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}^2}\Big) + w_n^J$$

$||w_n^J||_{\dot{H}^1} \to 0$. We next claim that $s_n = -\frac{t_{1,n}}{\lambda_{1,n}^2}$ must be bounded. In fact, if $\frac{t_{1,n}}{\lambda_{1,n}^2} \leq -C_0$, C_0 a large positive constant, for n large we have $||e^{it\Delta} w_n^J||_{S(-\infty,+\infty)} \leq \frac{\delta}{2}$ and

$$\Big\| \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^\ell\Big(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{\lambda_{1,n}^2}\Big) \Big\|_{S(0,+\infty)} \leq ||V_1^\ell||_{S((C_0,+\infty))} \leq \frac{\delta}{2}$$

for C_0 large, a contradiction.

If, on the other hand $\frac{t_{1,n}}{\lambda_{1,n}^2} \geq C_0$, C_0 large positive, for n large, we have

$$\Big\| \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^\ell\Big(\frac{x - x_{1,n}}{\lambda_{1,n}}, \frac{t - t_{1,n}}{\lambda_{1,n}^2}\Big) \Big\|_{S(-\infty,0)} \leq ||V_1^\ell||_{S((-\infty,-C_0))} \leq \frac{\delta}{2}.$$

Thus, for n large, $||e^{it\Delta}\, u(t_n)||_{S(-\infty,0)} \leq \delta$, so that (CP) gives $||u||_{S(-\infty,t_n)} \leq 2\delta$. But $t_n \uparrow T_+(u_0)$, a contradiction. Hence $|\frac{t_{1,n}}{\lambda_{1,n}^2}| \leq C_0$, so that, after passing to a subsequence $\frac{t_{1,n}}{\lambda_{1,n}^2} \to$

$t_0 \in (-\infty, +\infty)$. But then by $(*)$, $w_n^J \to 0$ gives for $n \neq n'$, both large

$$\Big\| \frac{1}{(\lambda_0)^{N-2/2}} \frac{1}{(\lambda_{1,n})^{N-2/2}} V_1^\ell\Big(\frac{x - x_0\,\lambda_0 - x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}^2}\Big) - \frac{1}{(\lambda_{1,n'})^{N-2/2}} V_1^\ell\Big(\frac{x - x_{1,n'}}{\lambda_{1,n'}}, -\frac{t_{1,n'}}{\lambda_{1,n'}^2}\Big)\Big\|_{\dot H^1} \geq \frac{\eta_0}{2}$$

for all λ_0, x_0. After changing variables this gives, for all $\lambda_0, \tilde{x}_0$ that

$$\Big\|\Big(\frac{\lambda_{1,n'}}{\lambda_0\,\lambda_{1,n'}}\Big)^{(N-2/2)} V_1^\ell\Big(\frac{\lambda_{1,n'}y}{\lambda_0\,\lambda_{1,n'}} + x_{n,n'} - \tilde{x}_0\,, -\frac{t_{1,n}}{(\lambda_{1,n})^2}\Big) - V_1^\ell\Big(y,\, \frac{t_{1,n'}}{\lambda_{1,n'}^2}\Big)\Big\|_{\dot H^1} \geq \frac{\eta_0}{2}.$$

Choosing now $\lambda_0, \tilde{x}_0$ suitably this is a contradiction since $\frac{t_{1,n'}}{\lambda_{1,n'}^2} \to t_0$, $\frac{t_{1,n}}{\lambda_{1,n}^2} \to t_0$. □

Proof of Proposition 3. — Assume first that $\liminf E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) = E_c$. Fix $J \geq 1$ and note that as in the proof of Proposition 1, we have $V_{0,j} = 0$, $j > 1$, and $||\nabla w_n^J|| \to 0$. Moreover, if $v_{0,n} = \lambda_{1,n}^{(N-2)/2} z_{0,n}(\lambda_{1,n}(x + x_{1,n}))$, $\tilde{w}_n = \lambda_{1,n}^{N-2/2}\, w_n^J(\lambda_{1,n}(x + x_{1,n}))$, we have $||\nabla \tilde{w}_n|| \to 0$, $v_{0,n} = V_1^\ell(s_n) + \tilde{w}_n$, with $||\nabla v_{0,n}||^2 < ||\nabla W||^2$, $E(v_{0,n}) \to E_c < E(W)$. By definition of the non-linear profile, $||\nabla(V_1^\ell(s_n) - U_1(s_n))|| \to 0$, so that $v_{0,n} = U_1(s_n) + \tilde{\tilde{w}}_n$, $||\nabla \tilde{\tilde{w}}_n|| \to 0$. From this we see that $E(U_1) = E_c < E(W)$ and so, by energy-trapping $\sup_{t\in I}||\nabla U_1(t)||^2 < ||\nabla W||^2$. Since $||\nabla \tilde{\tilde{w}}_n|| \to 0$ the Perturbation Theorem gives this case, under assertion b). Assume next that $\liminf E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) < E_c$ and passing to a subsequence that $\lim E(V_1^\ell(-\frac{t_{1,n}}{\lambda_{1,n}^2})) < E_c$. We next show that $\liminf E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) < E_c$, $j = 2, \cdots, J$. In fact, $||\nabla z_{0,n}||^2 = \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 + ||\nabla w_n^J||^2 + o(1)$ and since $E(z_{0,n}) \to E_c < E(W)$, for n large $E(z_{0,n}) \leq (1 - \delta_0)E(W)$. Since $||\nabla z_{0,n}||^2 < ||\nabla W||^2$, energy trapping gives that $||\nabla z_{0,n}||^2 \leq (1 - \bar{\delta})||\nabla W||^2$. Thus, for all n large $E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) \geq 0$, $E(w_n^J) \geq 0$. Coercitivity shows that $E(V_1^\ell(-s_n)) \geq C\,\alpha_0 = \bar{\alpha}_0 > 0$, for n large. Then, $E(z_{0,n}) \geq \bar{\alpha}_0 + \sum_{j=2}^{J} E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) + o(1)$, so our claim follows from $E(z_{0,n}) \to E_c$. Next, note that if U_j is the non-linear profile associated to $(V_{0,j}, \{-\frac{t_{j,n}}{\lambda_{j,n}^2}\})$ (after passing to a subsequence in n) then U_j exists for all time and $||U_j||_{S(-\infty,\infty)} < \infty$, $1 \leq j \leq J$. In fact, for n large, $E(V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})) < E_c$, so $E(U_j) < E_c$ by definition of non-linear profile. Moreover, $||\nabla V_j^\ell(-\frac{t_{j,n}}{\lambda_{j,n}^2})||^2 \leq ||\nabla z_{0,n}||^2 + o(1) \leq (1 - \bar{\delta})||\nabla W||^2 + o(1)$, so by energy trapping we have $||\nabla U_j(t)|| < ||\nabla W||$, $\forall\, t \in I_j$. But then, by definition of E_c, $I_j = (-\infty, +\infty)$, $||U_j||_{S(-\infty+\infty)} < \infty$. Next, note that $\exists\, j_0$ s.t. for $j \geq j_0$ we have

$$||U_j||^2_{S(-\infty,+\infty)} \leq C||\nabla V_{0,j}||^2\,.$$

In fact, J fixed, choosing n large we have

$$\sum_{j=1}^{J} ||\nabla V_{0,j}||^2 \leq ||\nabla z_{0,n}||^2 + o(1) \leq 2||\nabla W||^2.$$

Hence, for $j \geq j_0$, $||\nabla V_{0,j}|| \leq \tilde{\delta}$, $\tilde{\delta}$ so small that $||e^{it\Delta}\, V_{0,j}||_{S(-\infty,+\infty)} \leq \delta$, which shows that $||U_j||_{S(-\infty,+\infty)} \leq 2\delta$, $\sup_t ||U_j(t)||_{\dot H^1} + ||\nabla U_j||_{W(-\infty,+\infty)} \leq C||V_{0,j}||_{\dot H^1}$. But then,

$||U_j||_{S(-\infty,+\infty)} \le C||V_{0,j}||_{\dot H^1}$ as desired. Next, for $\varepsilon_0 > 0$, to be chosen, define

$$H_{n,\varepsilon_0} = \sum_{j=1}^{J(\varepsilon_0)} \frac{1}{\lambda_{j,n}^{N-2/2}} U_j\Big(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}^2}\Big).$$

Then $||H_{n,\varepsilon_0}||_{S(-\infty,+\infty)} \le C_0$ uniformly in ε_0, for $n \ge n(\varepsilon_0)$:

$$\begin{aligned}
||H_{n,\varepsilon_0}||^{2(N+2)/(N-2)}_{S(-\infty,+\infty)} &= \iint \Big[\sum_{j=1}^{J(\varepsilon_0)} \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\Big(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}^2}\Big)\Big]^{2(N+2)/(N-2)} \\
&\le \sum_{j=1}^{J(\varepsilon_0)} \iint \Big|\frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\Big(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}^2}\Big)\Big|^{2(N+2)/(N-2)} \\
&\quad + C_{j(\varepsilon_0)} \sum_{j\ne j'} \iint \Big|\frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\Big(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}^2}\Big)\Big| \cdot \\
&\quad \cdot \Big|\frac{1}{\lambda_{j',n}^{(N-2)/2}} U_{j'}\Big(\frac{x-x_{j',n}}{\lambda_{j',n}}, \frac{t-t_{j',n}}{\lambda_{j',n}^2}\Big)\Big|^{(N+6)/(N-2)} = I + II.
\end{aligned}$$

For n large, $II \to 0$ by orthogonality of $(\lambda_{j,n}, x_{j,n}, t_{j,n})$. Thus, for n large $II \le I$. But

$$\begin{aligned}
I &\le \sum_{j=1}^{j_0} ||U_j||^{2(N+2)/(N-2)}_{S(-\infty,+\infty)} + \sum_{j=j_0+1}^{J_0(\varepsilon)} ||U_j||^{2(N+2)/(N-2)}_{S(-\infty,+\infty)} \\
&\le \sum_{j=1}^{j_0} ||U_j||^{2(N+2)/(N-2)}_{S(-\infty,+\infty)} + C\sum_{j=j_0+1}^{J_0(\varepsilon)} ||\nabla V_{0,j}||^{2(N+2)/(N-2)} \\
&\le \sum_{j=1}^{j_0} ||U_j||^{2(N+2)/(N-2)}_{S(-\infty,+\infty)} + C\sup_{j>j_0} ||\nabla V_{0,j}||^{(2\frac{(N+2)}{N-2}-2)} \sum_{j>j_0}^{J(\varepsilon_0)} ||\nabla V_{0,j}||^2 \le \frac{C_0}{2}.
\end{aligned}$$

Define now $R_{n,\varepsilon_0} = |H_{n,\varepsilon_0}|^{4/(N-2)} H_{n,\varepsilon_0} - \sum_{j=1}^{J(\varepsilon_0)} |\tilde U_{j,n}|^{4/(N-2)} \tilde U_{j,n}$, where

$$\tilde U_{j,n}(x,t) = \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j\Big(\frac{x-x_{j,n}}{\lambda_{j,n}}, \frac{t-t_{j,n}}{\lambda_{j,n}^2}\Big).$$

We then have $||\nabla R_{n,\varepsilon_0}||_{L^2_t L_x^{2N/(N-2)}} \to 0$ as $n \to +\infty$. This uses orthogonality, $||U_j||_{S(-\infty,+\infty)} < \infty$, $||\nabla U_j||_{W(-\infty,+\infty)} < \infty$. Let now $\tilde u = H_{n,\varepsilon_0}$, $e = R_{n,\varepsilon_0}$. Choose now $J(\varepsilon_0)$ so large that for n large $||e^{it\Delta} w_n^{J(\varepsilon_0)}||_{S(-\infty,+\infty)} \le \frac{\varepsilon_0}{2}$. Note that by the profile decomposition and the definition of non-linear profile, we have, for n large

$$z_{0,n} = H_{n,\varepsilon_0}(0) + \tilde w_n^{J(\varepsilon_0)}$$

where $||e^{it\Delta}\tilde w_n^{J(\varepsilon_0)}||_S \le \varepsilon_0$. Also, arguments as above show also that $\sup_t ||\nabla H_{n,\varepsilon_0}(t)|| \le \tilde C_0$ uniformly in ε_0, for n large and $||\nabla \tilde w_n^{J(\varepsilon_0)}|| \le 2||\nabla W||$. Now choose $\varepsilon_0 < \varepsilon_0(C_0, \tilde C_0, 2||\nabla W||)$ as in Perturbation Theorem, and n so large that $||\nabla R_{n,\varepsilon_0}||_{L^2_t L^{2N/N-2}} \le \varepsilon_0$. Then the Perturbation Theorem gives us Proposition 3 a). □

An important Corollary of the above arguments is (Keraani [**20**], 2001, Bahouri-Gérard [**1**],1999).

Lemma. — *There exists a function* $g : (0, E_c] \to [0, +\infty)$, $g \downarrow$ *s.t.* $\forall u_0$ *with* $||\nabla u_0||^2 < ||\nabla W||^2$ *and* $E(u_0) \le E_c - \eta$, *then* $||u||_{S(-\infty,+\infty)} \le g(\eta)$.

Proof. — If not, $\exists\, \eta_0 > 0$ and a sequence $u_{0,n}$ s.t. $||\nabla u_{0,n}||^2 < ||\nabla W||^2$, $E(u_{0,n}) \le E_c - \eta_0$ and $||u_n||_{S(-\infty,+\infty)} \to +\infty$. For n large we must have $||e^{it\Delta} u_{0,n}||_{S(-\infty,+\infty)} \ge \delta$. But if we now apply the proof of Proposition 3, case a), we reach a contradiction. □

Remark. — *In the profile decomposition, if all the* $v_{0,n}$ *are radial the* $V_{0,j}$ *can be chosen radial and* $x_{n,j} = 0$. *We can then repeat our procedure restricted to radial function and conclude the analogs of Propositions 1, 2 with* u_c *radial,* $x(t) \equiv 0$.

Remark. — *Because of the continuity of* $u(t)$, $t \in I$ *in* $\dot{H}^1$, *in Proposition 2 we can construct* $\lambda(t)$, $x(t)$ *continuous in* $[0, T_+(u_0))$, *with* $\lambda(t) > 0$ *for each* $t \in [0, T_+(u_0))$. *To see this, first one can construct piecewise constant (with small jump)* $\lambda_1(t)$, $x_1(t)$ *so that the corresponding set* K_1 *is contained in* $\tilde{K}_1 = \{w(t)$ *solution of (CP) with initial data in* $\bar{K}, t \in [0, t_0], t_0$ *small*$\}$. *It is clear that* $\tilde{K}_1$ *is compact. We can then construct continuous* $\lambda_2(t)$, $x_2(t)$ *s.t.* K_2 *is contained in the precompact set* $\{\lambda_0^{-(N-2)/2} w((x - x_0)\lambda_0^{-1})$, $w \in \tilde{K}_1$, $\frac{1}{2} \le \lambda_0 \le 2$, $|x_0| \le 1\}$.

We now turn to further properties of critical elements.

Lemma. — *Let* u_c *be as in Proposition 2, with* $T_+(u_0) < \infty$. *(After scaling we can assume* $T_+(u_0) = 1$*). Then* $\exists\, C_0 = C_0(K) > 0$ *s.t.* $\lambda(t) \ge \frac{C_0(K)^{1/2}}{(1-t)^{1/2}}$.

Proof. — Consider $t_n \uparrow 1$, $u_{0,n} = \frac{1}{\lambda(t_n)^{N-2/2}}\, u(\frac{x - x(t_n)}{\lambda(t_n)}, t_n)$. We know that $\exists\, C_0 = C_0(\bar{K})$ s.t. $T_+(u_{0,n}) \ge C_0$. Note that $u(x, t_n) = \lambda(t_n)^{N-2/2}\, u_{0,n}(\lambda(t_n)x + x(t_n))$, hence by uniqueness in (CP), for $t_n + t < T_+(u_0) = 1$, we have $u(x, t + t_n) = \lambda(t_n)^{N-2/2}\, u_n(\lambda(t_n)\, x + x(t_n), \lambda^2(t_n)t)$. Hence, $t_n + t \le 1$ for all $0 < \lambda^2(t_n)t \le C_0$. With $t = \frac{C_0}{\lambda^2(t_n)}$, we get $t_n + \frac{C_0}{\lambda^2(t_n)} \le 1$ or $\lambda^2(t_n) \ge \frac{C_0}{(1-t_n)}$ as desired. □

Lemma. — *Let* u_c *be a critical element as in Proposition 2, with* $T_+(u_0) = +\infty$. *Then, there is a (possibly different) critical element* v_c, *with a corresponding* $\tilde{\lambda}$, *and* $A_0 > 0$, *with* $\tilde{\lambda}(t) \ge A_0 > 0$, *for* $t \in [0, T_+(v_{0,c}))$.

Proof. — Recall that $E(u_c) = E_c \ge \eta_0$. By a previous remark, $\exists\, t_n$, $t_n \uparrow +\infty$ s.t. $\lambda(t_n) \to 0$, or the result holds for u_c. After possibly redefining $\{t_n\}_{n=1}^{c_0}$, we can assume that $\lambda(t_n) \le \inf_{[0,t_n]} \lambda(t)$. By compactness of $\bar{K}$, $w_{0,n}(x) = \frac{1}{\lambda(t_n)^{N-2/2}} u_c(\frac{x - x(t_n)}{\lambda(t_n)}, t_n) \to w_0$ in $\dot{H}^1$. Hence, $E(w_0) = E_c \ge \eta_0 > 0$. Moreover, $||\nabla w_0||^2 < ||\nabla W||^2$ by the corresponding property of u_c and energy trapping ($E_c < E(W)$). Let $w(x, \tau)$, $\tau \in (-T_-(w_0), 0]$ be the corresponding solution of (CP). If $T_-(w_0) < \infty$, we let $v_c(x, t) = \bar{w}(x, -t)$ and Proposition 2, and the previous lemma, give the result. If $T_-(w_0) = +\infty$, let $w_n(x, \tau)$ be the solution of (CP) with data $w_{0,n}$, $\tau \in (-T_-(w_{0,n}), 0]$. By semicontinuity we have $\liminf T_-(w_{0,n}) = +\infty$ and for every $\tau \in (-\infty, 0]$, $w_n(x, \tau) \to w(x, \tau)$ in $\dot{H}^1$. By uniqueness in (CP), for $0 \le t_n + \frac{\tau}{\lambda(t_n)^2}$, we have $w_n(x,\tau) = \frac{1}{\lambda(t_n)^{N-2/2}} u_c\big(\frac{x-x(t_n)}{\lambda(t_n)},\ t_n + \frac{\tau}{\lambda(t_n)^2}\big)$. Define now $\tau_n = -\lambda(t_n)^2\, t_n$. Note that $\liminf_n(-\tau_n) = \liminf_n(\lambda(t_n)^2\, t_n) = +\infty$. In fact, if $-\tau_n \to -\tau_0 < \infty$, $w_n(x,-\tau_n) = \frac{1}{\lambda(t_n)^{N-2/2}} u_c(\frac{x-x(t_n)}{\lambda(t_n)}, 0)$ would converge to $w_0(x,-\tau_0)$ in $\dot{H}^1$, with $\lambda(t_n)\to 0$, a contradiction to $E(w_0)\neq 0$, so $w_0 \not\equiv 0$. Hence, for all $\tau\in(-\infty,0]$, for n large we have $0\le t_n + \frac{\tau}{\lambda(t_n)^2}\le t_n$. Note also that we must have $||w||_{S(-\infty,0)} = +\infty$. Otherwise, by The Perturbation Theorem we would have, for n large, $T_-(w_{0,n}) = +\infty$, $||w_n||_{S(0,\infty)}\le M$, which contradicts $||u_c||_{S(0,+\infty)} = +\infty$. Fix $\tau\in(-\infty,0]$, n so large that $t_n + \frac{\tau}{\lambda(t_n)^2}\ge 0$ and $\lambda(t_n + \frac{\tau}{\lambda(t_n)^2})$ is defined. Then,

$$\frac{1}{\lambda(t_n+\frac{\tau}{\lambda(t_n)^2})^{(N-2)/2}}\, u_c\Big(\frac{x - x(t_n + \frac{\tau}{\lambda^2(t_n)})}{\lambda(t_n + \frac{\tau}{\lambda^2(t_n)})}, t_n + \frac{\tau}{\lambda^2(t_n)}\Big) = \frac{1}{\tilde\lambda_n(\tau)^{(N-2)/2}}, w_n\Big(\frac{x-\tilde x_n(\tau)}{\tilde\lambda(\tau)}, \tau\Big)\in K\,,$$

with

$$\tilde\lambda_n(\tau) = \frac{\lambda(t_n + \frac{\tau}{\lambda(t_n)^2})}{\lambda(t_n)} \ge 1\,,\ \ \tilde x_n(\tau) = x(t_n + \frac{\tau}{\lambda^2(t_n)}) - \frac{x(t_n)}{\tilde\lambda(t_n)}\,.$$

Since $\frac{1}{\lambda_n^{N/2}}\vec{v}(\frac{x-x_n}{\lambda_n}) \underset{n\to\infty}{\longrightarrow} \vec{\bar{v}}$ in L^2 with either $\lambda_n\to 0$ or ∞ or $|x_n|\to\infty$ implies that $\vec{\bar{v}}\equiv 0$, we can assume, after passing to a subsequence that $\tilde\lambda_n(\tau)\to\tilde\lambda(\tau)$, $1\le\tilde\lambda(\tau)<\infty$ $\tilde x_n(\tau)\to\tilde x(\tau)\in\mathbb{R}^N$. But then, $\frac{1}{\tilde\lambda(\tau)^{N-2/2}}\, w\big(\frac{x-\tilde x(\tau)}{\tilde\lambda(\tau)}, \tau\big)\in\bar{K}$ as desired. □

We now conclude the proof of Theorem A, by establishing:

***Theorem* (Rigidity Theorem)**. — *Let $u_0\in\dot{H}^1$, $E(u_0)<E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$. Let u be the corresponding solution of (CP) with maximal interval $I = (-T_-(u_0), T_+(u_0))$. Assume $\exists\,\lambda(t)>0$, defined for $t\in[0,T_+(u_0))$ s.t.*

$$K = \Big\{v(x,t) = \frac{1}{\lambda(t)^{N-2/2}}\, u\Big(\frac{x}{\lambda(t)}, t\Big), t\in[0,T_+(u_0))\Big\}$$

has compact closure in $\dot{H}^1$. Assume that if $T_+(u_0)<\infty$, $\lambda(t)\ge\frac{C_0(K)^{1/2}}{(T_+-t)^{1/2}}$ and if $T_+(u_0) = +\infty$, $\lambda(t)\ge A_0>0$. Then we must have $T_+(u_0)=+\infty$, $u_0 = 0$.

Proof. — **Case 1 :** $T_+(u_0)<+\infty$ so that $\lambda(t)\to+\infty$ as $t\to T_+(u_0)$. Fix φ radial, $\varphi\in C_0^\infty$, $\varphi\equiv 1$ on $|x|\le 1$, supp$\varphi\subset\{|x|<2\}$ set $\varphi_R(x) = \varphi(\frac{x}{R})$.

Define $y_R(t) = \int |u(x,t)|^2\,\varphi_R(x)dx$, $t\in[0,T_+)$. A classical computation shows that

$$y_R'(t) = 2\,\mathrm{Im}\int \bar{u}\,\nabla u\,\nabla\varphi_R\,.$$

Note that $\nabla\varphi_R = \frac{1}{R}\nabla\varphi(\frac{x}{R})$ is supported in $R\le|x|\le 2R$. Then,

$$|y_R'(t)|\le\tilde C_N\Big(\int|\nabla u|^2\Big)^{1/2}\Big(\int\frac{|u|^2}{|x|^2}\Big)^{1/2}\le\tilde C_N\Big(\int|\nabla u|^2\Big)\le\tilde C_N||\nabla W||$$

where we have used Hardy's inequality and energy trapping. Next, we will show that, for all $R>0$,

$$\lim_{t\uparrow T_+(u_0)}\int_{|x|\le R}|u(x,t)|^2\,dx = 0\,.$$

In fact, $u(x,t) = \lambda(t)^{N-2/2} v(\lambda(t)x, t)$, where v is compact. Then,

$$\begin{aligned}\int_{|x|<R} |u(x,t)|^2\,dx =& \lambda(t)^{-2} \int_{|y|<R\lambda(t)} |v(y,t)|^2\,dy \\ =& \lambda(t)^{-2} \int_{|y|<\varepsilon T\lambda(t)} |v(y,t)|^2\,dy + \lambda(t)^{-2} \int_{\varepsilon R\lambda(t)\le|y|\le R\lambda(t)} |v(y,t)|^2\,dy \\ =& A + B,\end{aligned}$$

where $\varepsilon > 0$ is at our disposal. By Hölder, we have

$$A \le \lambda(t)^{-2} (\varepsilon R \lambda(t))^{N-2/N} ||v(t)||^2_{L^{2^*}} \le C\,\varepsilon^2\,R^2\,||\nabla W||^2$$

which, for fixed R is small with ε

$$B \le \lambda(t)^{-2}\,(R\lambda(t))^{N-2/N}\,||v(t)||^2_{L^{2^*}(|y|\ge\varepsilon R\lambda(t))} \xrightarrow[t\to T_+]{} 0$$

by the compactness of v, since $\lambda(t) \uparrow +\infty$.

Now, using that $|y_R'(t)| \le C$ and the fundamental theorem of calculus, we have

$$y_R(0) \le \lim_{t\uparrow T_+} y_R(t) + C\,T_+(u_0) = C\,T_+(u_0)\,.$$

Letting $R \to \infty$, we conclude that $u_0 \in L^2$. Fix now $\varepsilon > 0$ and choose α so small that

$$\int_{T_+(u_0)-\alpha}^{T_+(u_0)} |y_R'| \le C\,\alpha \le \frac{\varepsilon}{2}$$

for all $R > 0$. By invariance of the L^2 norm (and this is a fundamental point here), we have:

$$||u_0||^2_{L^2} = ||u(T_+(u_0) - \alpha)||^2_{L^2}\,.$$

For α fixed as above, choose R so large that

$$||u(T_+(u_0) - \alpha)||^2_{L^2} \le ||u(T_+(u_0) - \alpha)||^2_{L^2(|x|<R)} + \frac{\varepsilon}{2}\,.$$

We then have

$$||u_0||^2_{L^2} \le y_R(T_+(u_0) - \alpha) + \frac{\varepsilon}{2} \le \lim_{t\uparrow T^+} - \int_{T_+(u_0)-\alpha}^{t} y_R' + \frac{\varepsilon}{2} \le \varepsilon\,.$$

Since this is true for each $\varepsilon > 0$, $||u_0||_{L^2} = 0$, which contradicts $T_+ < \infty$. This ends the proof in Case 1. □

Lecture 3

To conclude the proof of Theorem A, we need to treat the

Proof. — **Case 2 :** $T_+(u_0) = +\infty$, $\lambda(t) \ge A_0 > 0$.

Note first that the compactness of $\bar{K}$, together with $\lambda(t) \ge A_0 > 0$ gives that $\forall\,\varepsilon > 0$, $\exists\,R(\varepsilon) > 0$ s.t. $\forall\,t \in [0,\infty)$, we have

$$\int_{|x|>R(\varepsilon)} |\nabla u|^2 + |u|^{2^*} + \frac{|u|^2}{|x|^2} \le \varepsilon\,.$$

In fact, since $u(x,t) = \lambda(t)^{N-2/2}\, v(\lambda(t)x,t)$ a change of variables shows that the integral equals

$$\int_{|y|>R(\varepsilon)\lambda(t)} |\nabla v|^2 + |v|^{2^*} + \frac{|v|^2}{|y|^2} \leq \int_{|y|>A_0 R(\varepsilon)} \leq \varepsilon$$

for $R(\varepsilon)$ large by the compactness of $\bar{K}$.

To continue with the proof, pick δ_0 s.t. $E(u_0) < (1-\delta_0)E(W)$. Then, $\exists R_0 > 0$ s.t. for $R > R_0$, $t \in [0,\infty)$ we have (if $||\nabla u_0|| \neq 0$)

$$\int_{|x|<R} |\nabla u|^2 - |u|^{2^*} \geq C_{\delta_0} ||\nabla u_0||^2.$$

In fact, by our coercitivity estimate we have, for all $t \in [0,\infty)$, $\int |\nabla u|^2 - |u|^{2^*} \geq C_{\delta_0}||\nabla u_0||^2$, but, by the first claim, we can make the tails smaller than $\frac{C_{\delta_0}}{2}\,||\nabla u_0||^2$. Next, choose $\psi \in C_0^\infty$, radial, with $\psi(x) = |x|^2$ for $|x| \leq 1$, $\psi(x) \equiv 0$ for $|x| \geq 2$. Define

$$z_R(t) = \int |u(x,t)|^2\, R^2\, \psi\Big(\frac{x}{R}\Big)\, dx.$$

The computations that we used in Glassey's blow-up proof to yield the "virial identity" now give:

$$z_R'(t) = 2R\ \mathrm{Im} \int \bar{u}\,\nabla\, u\, \nabla\, \psi\Big(\frac{x}{R}\Big),$$

$$z_R''(t) = 4\sum_{\ell,j} \mathrm{Re} \int \partial_{x_\ell x_j}\psi\Big(\frac{x}{R}\Big) \cdot \partial_{x_\ell} u \cdot \partial_{x_j}\bar{u} - \frac{1}{R^2}\int \Delta^2\psi\Big(\frac{x}{R}\Big)|u|^2 - \frac{4}{n}\int \Delta\psi\Big(\frac{x}{R}\Big)|u|^{2^*}.$$

From these formulas, we deduce:

$$\begin{aligned} |z_R'(t)| \leq & C\, R \int_{|x|\leq 2R} |u|\,|\nabla u| \leq C\, R^2 \Big(\int_{|x|\leq 2R} \frac{|u|^2}{|x|^2}\Big)^{1/2} \Big(\int |\nabla u|^2\Big)^{1/2} \\ \leq & C\, R^2 \int |\nabla u|^2 \leq C_{\delta_0} R^2 ||\nabla W||^2. \end{aligned}$$

On the other hand,

$$z_R''(t) \geq 8\Big[\int_{|x|\leq R} |\nabla u|^2 - |u|^{2^*}\Big] - \tilde{C}_N \Big[\int_{R\leq|x|\leq 2R} |\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*}\Big],$$

which, for R large is bounded below by $\tilde{C}_{\delta_0,N}||\nabla u_0||^2$. Integrating in t, we obtain

$$z_R'(t) - z_R'(0) \geq \tilde{C}_{\delta_0,N}\, t\, ||\nabla u_0||^2,$$

$$|z_R'(t) - z_R'(0)| \leq 2\, C_{\delta_0} R^2\, ||\nabla u_0||^2,$$

which is a contradiction for large t. □

Remark. — *In the defocusing case, for $N = 3, 4, 5$, this approach (in a simplifed form since the variational estimates are not needed) provides an alternative proof of the result of Bourgain, Tao for radial functions in the defocusing case.*

***Corollary* (focusing case).** — *$u_0 \in \dot{H}^1$, radial, $E(u_0) < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$, $N = 3,4,5$. Then $I = (-\infty,+\infty)$, $||u||_{S(-\infty,+\infty)} < \infty$, $\exists\, u_0^{\pm} \in \dot{H}^1$ s.t. $||u(t) - e^{it\Delta}u_0^{\pm}||_{\dot{H}^1} \xrightarrow[t\to\pm\infty]{} 0$. Also, if $E(u_0) \leq (1-\delta_0)E(W)$, $||u||_{S(-\infty,+\infty)} \leq g(\delta_0)$.*

Remark. — *The result admits the following strengthening: if $u_0 \in \dot{H}^1$ is s.t. $\forall\, t \in (-T_-(u_0), T_+(u_0))$ we have $||\nabla u(t)||^2 \leq ||\nabla W||^2 - \delta_0$, for some $\delta_0 > 0$, then $I = (-\infty,+\infty)$ and $||u||_{S(-\infty,+\infty)} < \infty$. For a detailed proof, see the arguments in* [**18**].

This remark and our Theorem A have consequences for the concentration of finite time blow-up solutions (see [**17**] for the details of the proof):

Corollary. — *Let $u_0 \in \dot{H}^1$ be radial (no size restriction). Assume $T_+(u_0) < \infty$ and $\sup_{t\in[0,T_+(u_0))} ||\nabla u(t)|| < \infty$ (type II blow-up). Then, for all $R > 0$ we have:*

$$\limsup_{t\uparrow T_+(u_0)} \int_{|x|\leq R} |\nabla u(t)|^2 \geq \int |\nabla W|^2,$$

$N = 3,4,5$.

Remark. — *For $N \geq 4$, u_0 radial, $T_+(u_0) < \infty$, u not a finite blow-up solution of type II, one can show that if $\int |\nabla u(t_n)|^2 \to +\infty$, then $\forall\, R > 0$, $\int_{|x|<R} |\nabla u(t_n)|^2 \to +\infty$. For $N = 3$ this is likely false, in light of examples like those of P. Raphael* [**24**] *for $N = 2$, which should give a radial solution, blowing-up on a sphere.*

We now turn our attention to the non-linear wave equation (NLW).

$$\begin{cases} \partial_t^2 u - \Delta u = \pm |u|^{4/N-2}\, u\,, \quad x \in \mathbb{R}^N\,,\ \ t \in \mathbb{R} \\ u|_{t=0} = u_0 \in \dot{H}^1 \\ \partial_t u|_{t=0} = u_1 \in L^2\,. \end{cases}$$

Here the $-$ sign corresponds to the defocusing case, the $+$ sign to the focusing case. The problem is energy critical because if $u(x,t)$ is a solution, $\lambda > 0$, then $u_\lambda(x,t) = \frac{1}{\lambda^{N-2/2}}\, u(\frac{x}{\lambda}, \frac{t}{\lambda})$ is also a solution and the norm in $\dot{H}^1 \times L^2$ of the initial data remains unchanged.

The defocusing case has been studied for many years, going back to work of Struwe (radial)[**29**], Grillakis (general)[**12**], Shatah-Struwe [**27, 26**], Bahouri-Shatah [**2**], Kapitansky [**14**], Bahouri-Gérard [**1**], Ginibre-Velo [**10**], Ginibre-Soffer-Velo [**9**], etc. (mid to late 80's, mid 90's). The energy here is

$$E((u_0,u_1)) = \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{2}\int (u_1)^2 \mp \frac{1}{2^*}\int |u_0|^{2^*}$$

which is constant in time, with $-$ in the focusing case, and $+$ in the defocusing case, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$. In the defocusing case, Shatah-Struwe and Bahouri-Shatah showed that for any data $(u_0,u_1) \in \dot{H}^1 \times L^2$ we have global well-posedness and scattering in the energy space. In the focusing case, this does not hold. In 1974, H. Levine [**22**] showed (by obstruction) that if $(u_0,u_1) \in \dot{H}^1 \times L^2$, $u_0 \in L^2$, $E((u_0,u_1)) < 0$, there is always break-down in finite time. Very recently (2007) Krieger-Schlag-Tataru [**21**] have constructed explicit radial examples which break-down in finite time. Also, $W(x) = (1 + \frac{|x|^2}{N(N-2)})^{-(N-2)/2}$ solves the elliptic equation

$\Delta W + |W|^{4/N-2}W = 0$, $W \in \dot{H}^1$, is radial and so $W(x,t) = W(x)$ solves (CP) with data $(W,0)$, globally iin time, but does not scatter. We now turn to our study in the focusing case.

Theorem B. — *Assume that $E((u_0,u_1)) < E((W,0))$.*

i) *If $\int |\nabla u_0|^2 < \int |\nabla W|^2$, we have g.w.p., scattering.*
ii) *If $\int |\nabla u_0|^2 > \int |\nabla W|^2$, there is break-down in finite time.*

The condition $\int |\nabla u_0|^2 = \int |\nabla W|^2$ is not compatible with $E((u_0,u_1)) < E((W,0))$. Note that no radial assumption is made in the Theorem, which has been proved for $3 \le N \le 5$.

The general scheme of the proof follows the approach we described for (NLS). To describe the proof, I will start out by a review of the local Cauchy problem. Consider first the linear wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = h \ \text{ in } \mathbb{R}^N \times \mathbb{R} \\ w|_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t w|_{t=0} = w_1 \in L^2(\mathbb{R}^N) \end{cases}$$

whose solution is given by

$$w(x,t) = \cos(t\sqrt{-\Delta})w_0 + (-\Delta)^{-1/2} \sin(t\sqrt{-\Delta}w_1 + \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}}\, h(t')\, dt' .$$

Let $S(t)(w_0,w_1) = \cos(t\sqrt{-\Delta})\, w_0 + (-\Delta)^{-1/2} \sin(t\sqrt{-\Delta})\, w_1$. The relevant Strichartz estimates for us are:

$$\begin{aligned} &\sup_t ||(w(t),\partial_t w(t))||_{\dot{H}^1\times L^2} \\ &+ ||D^{1/2}w||_{L_t^{2(N+1)/N-1}L_x^{2(N+1)/N-1}} \\ &+ ||\partial_t\, D^{-1/2}w||_{L_t^{2(N+1)/N-1}L_x^{2(N+1)/N-1}} + ||w||_{L_t^{2(N+1)/N-2}L_x^{2(N+1)/N-2}} \\ &+ ||w||_{L_t^{(N+2)/N-2}L_x^{2(N+2)/N-2}} \\ &\le C\Big\{||(w,w_1)||_{\dot{H}^1\times L^2} + ||D^{1/2}h||_{L_t^{2(N+1)/N+3}L_x^{2(N+1)/N+3}}\Big\} . \end{aligned}$$

We then define

$$||\ \ ||_{S(I)} = ||\ \ ||_{L_I^{2(N+1)/N-2}L_x^{2(N+1)/N-2}}$$

and

$$||\ \ ||_{W(I)} = ||\ \ ||_{L_I^{2(N+1)/N-1}L_x^{2(N+1)/N-1}} .$$

We also need the Leibniz and chain rules for fractional derivatives ([**19**], 1993) in the following form: if $F(0) = F'(0) = 0$, $F \in C^2$ and for all a,b we have $|F'(a+b)| \le C\{|F'(a)| + |F'(b)|\}$ and $|F''(a+b)| \le C\{|F''(a)| + |F''(b)|\}$, we have, for $0 < \alpha < 1$:

$$||D^\alpha F(u)||_{L_x^p} \le C||F'(u)||_{L_x^{p_1}}\, ||D^\alpha u||_{L_x^{p_2}}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},$$

$$||D^\alpha(F(u)-F(v))||_{L^p_x} \le C\Big[||F'(u)||_{L^{p_1}_x} + ||F'(v)||_{L^{p_1}_x}\Big]||D^\alpha(u-v)||_{L^{p_2}_x}$$
$$+ C\Big[||F''(u)||_{L^{r_1}_x} + ||F''(v)||_{L^{r_1}_x}\Big] \cdot \Big[||D^\alpha u||_{L^{r_2}_x} + ||D^\alpha v||_{L^{r_2}_x}\Big] \cdot ||u-v||_{L^{r_3}_x},$$
$$\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Using these estimates and the argument in the study of (CP) for (NLS), one obtains (see also [**26**]):

Theorem. — *If* $(u_0, u_1) \in \dot H^1 \times L^2$, $||(u_0,u_1)||_{\dot H^1 \times L^2} \le A$, $0 \in I$, $\exists\, \delta(A) > 0$ *s.t. if* $||S(t)(u_0,u_1)||_{S(I)} \le \delta$, $\exists !$ *solution of (CP) on* $\mathbb{R}^N \times I$, *with* $(u, \partial_t u) \in C(I; \dot H^1 \times L^2)$, $||D^{1/2}u||_{W(I)} + ||\partial_t D^{1/2} u||_{W(I)} < \infty$, $||u||_{S(I)} \le 2\delta$, $||u||_{L_I^{N+2/N-2} L_x^{2(N+2)/N-2}} < \infty$, *and we have Lipschitz continuity dependence on the data* $(3 \le N \le 5)$.

Corollary. — $\exists\, \tilde\delta > 0$ *s.t. if* $||(u_0,u_1)||_{\dot H^1 \times L^2} \le \tilde\delta$, *the hypothesis is verified for* $I = (-\infty, +\infty)$. *Moreover, given* $(u_0,u_1) \in \dot H^1 \times L^2$, $\exists\, I \ni 0$ *s.t. the hypothesis is verified on* I.

We say that u solves (CP) for (u_0,u_1) on $I \ni 0$ if $(u_1, \partial_t u) \in C(I; \dot H^1 \times L^2)$, $D^{1/2}u \in W(I)$, $u \in S(I)$, $(u, \partial_t u)|_{t=0} = (u_0,u_1)$ and u solves the appropriate integral equation. It is easy to obtain uniqueness and one can then define a maximal interval of existence $I = (-T_-(u_0,u_1), T_+(u_0,u_1))$. One also has the standard blow-up criterion: if $T_+(u_0,u_1) < \infty$, then $||u||_{S(0,T_+(u_0,u_1))} = +\infty$. Also, if $T_+(u_0,u_1) = +\infty$ and $||u||_{S(0,+\infty)} < \infty$, u scatters at $+\infty$, *i.e.* $\exists\, u_0^+, u_1^+ \in \dot H^1 \times L^2$ s.t. $||(u(t), \partial_t(u)) - S(t)(u_0^+, u_1^+)||_{\dot H^1 \times L^2} \to 0$. Note that for $t \in I$, we have $E((u(t), \partial_t u(t))) = E((u_0,u_1))$. It turns out that there is another very important conservation law in the energy space. This will be crucial for us, in order to be able to treat non-radial data. It says that, for $t \in I$, we have

$$\int \nabla u(x,t) \cdot \partial_t u(x,t)\, dx = \int \nabla u_0 \cdot u_1$$

(conservation of momentum).

Finally, we mention that Foschi's estimates [**6**] also hold for the wave equation. One can then prove the analogue of the Perturbation Theorem for (NLS), for (NLW) and all its corollaries.

We conclude these remarks on (CP) by mentioning the finite speed of propagation property. Recall that if $R(t)$ is the forward fundamental solution for the linear wave equation, we can write the solution of the linear Cauchy problem (for $T > 0$) as

$$w(t) = \partial_t R(t) * w_0 + R(t) * w_1 - \int_0^t R(t-s) * h(s)\, ds\,.$$

The finite speed of propagation states that $\operatorname{supp} R(-,t) \subset \bar B(0,t)$, $\operatorname{supp} \partial_t R(t) \subset \bar B(0,t)$. Thus, if $\operatorname{supp} w_0 \subset {}^c\overline{B(x_0,a)}$, $\operatorname{supp} w_1 \subset {}^c\overline{B(x_0,a)}$, $\operatorname{supp} h \subset {}^c\big[\bigcup_{0\le t\le a} \overline{B(x_0,a-t)} \times t\big]$, $w \equiv 0$ on $\bigcup_{0 \le t \le a} [B(x_0,t) \times t)]$. This has consequences for solutions of (NLW). If (u_0,u_1), (u_0', u_1') are data s.t. $(u_0,u_1) = (u_0', u_1')$ on $\overline{B(x_0,a)}$, then, the corresponding solutions agree on

$\underset{0\le t\le a}{\cup}[B(x_0,t)\times(t)]\cap\mathbb{R}^N\times(I\cap I')$. This is because, for each u, if we define $u^{(n+1)}(x,t)=S(t)(u_0,u_1)+\int_0^t\frac{\sin((t-s\sqrt{-\Delta})}{\sqrt{-D}}f(u^{(n)}(s))\,ds$, $u^0(x,t)=S(t)(u_0,u_1)$, we have $u^n\to u$, $u'^n\to u'$ and they agree on the required set, by induction. Typical applications of this are: $\operatorname{supp}u_0\subset B(0,b)$, $\operatorname{supp}u_1\subset B(0,b)$, then $u(x,t)\equiv 0$ on $\{(x,t):|x|>b+t,\,t\ge 0,\,t\in I\}$. Similar statements hold for $t<0$. Thus, one can approximate solutions by regular, compactly supported solutions. The next step is to obtain energy trapping, coercivity and uniform bounds, by variational arguments, as in the case of (NLS). Recall that $W_{\theta_0,\lambda_0,x_0}(x)=e^{i\theta_0}\,\lambda_0^{(N-2)/2}\cdot W(\lambda_0(x-x_0))$ and that Aubin-Talenti showed that if C_N is the best constant in the Sobolev embedding ($||u||_{L^{2^*}}\le C_N||\nabla u||$) then $||u||_{L^{2^*}}=C_N||\nabla u||$, $u\not\equiv 0\Leftrightarrow u=W_{\theta_0,\lambda_0,x_0}$. Moreover, we showed that $||\nabla W||^2=\frac{1}{C_N^N}$, and if $\mathcal{E}(W)=\frac{1}{2}||\nabla W||^2-\frac{1}{2^*}\,||W||_{L^{2^*}}^{2^*}$, $\mathcal{E}(W)=\frac{1}{NC_N^n}$. Using our $t-$independent variational estimates we obtain:

Energy trapping. — If u is a solution of (NLW), with max int I, $(u,\partial_t u)|_{t=0}=(u_0,u_1)\in\dot H^1\times L^2$ and for $\delta_0>0$, $E((u_0,u_1))\le(1-\delta_0)\,E((W,0))$, $||\nabla u_0||^2<||\nabla W||^2$, then $\forall\,t\in I$ we have: $\exists\,\bar\delta=\bar\delta(\delta_0)$ s.t.

i) $||\nabla u(t)||^2\le(1-\bar\delta)||\nabla W||^2$
ii) $\int|\nabla u(t)|^2-|u(t)|^{2^*}\ge\bar\delta\int|\nabla u(t)|^2$
iii) $\mathcal{E}(u(t))\ge 0$ (and hence $E((u_t),\partial_t u))\ge 0$)
iv) $E((u(t),\partial_t u(t)))\simeq||(u(t),\partial_t u(t))||^2_{\dot H^1\times L^2}\simeq||(u_0,u_1)||^2_{\dot H^1\times L^2}$, with comparability constants depending on δ_0.

Also, as in the case of (NLS) we have: if $E((u_0,u_1))\le(1-\delta_0\,E((W,0))$; $||\nabla u_0||^2>||\nabla W||^2$, then, for $t\in I$ we have $||\nabla u(t)||^2\ge(1+\bar\delta)||\nabla W||^2$.

We next turn to the proof of ii) in Theorem B. We will show it in the case when $||u_0||_{L^2}<\infty$. The general case follows by using, in addition, localization and finite speed of propagation. We know that, in the situation of ii),

$$\int|\nabla u(t)|^2\ge(1+\bar\delta)\int|\nabla W|^2\,,\quad t\in I$$

$$E((W,0))\ge E((u(t),\partial_t u(t)))+\tilde\delta_0\,.$$

Then,

$$\frac{1}{2^*}\int|u(t)|^{2^*}\ge\frac{1}{2}\int(\partial_t u(t))^2+\frac{1}{2}\int|\nabla u(t)|^2-E((W,0))+\tilde\delta_0$$

or

$$\int|u(t)|^{2^*}\ge\frac{N}{N-2}\int(\partial_t u(t))^2+\frac{N}{N-2}\int|\nabla u(t)|^2-2^*\,E((W,0))+2^*\,\tilde\delta_0\,.$$

Let $y(t)=\int|u(t)|^2$, so that $y'(t)=2\int u(t)\partial_t u(t)$, $y''(t)=2\int\{(\partial_t u)^2-|\nabla u(t)|^2+|u|^{2^*}(t)\}$. Then,

$$y''(t)\ge 2\int(\partial_t u)^2+\frac{2N}{N-2}\int(\partial_t u)-2^*\,E((W,0))+\tilde{\tilde\delta}_0+\frac{2N}{N-2}\int|\nabla u(t)|^2-2\int|\nabla u(t)|^2$$
$$=\frac{4(N-1)}{(N-2)}\int(\partial_t u)^2+\frac{4}{(N-2)}\int|\nabla u|^2-\frac{4}{(N-2)}\int|\nabla W|^2+\tilde{\tilde\delta}_0\ge\frac{4(N-1)}{(N-2}\int(\partial_t u)^2+\tilde{\tilde\delta}_0\,.$$

If $I \cap [0,+\infty) = [0,+\infty)$, $\exists\, t_0 > 0$ s.t. $y'(t_0) > 0$ and $y'(t) > 0$, $\forall t > t_0$. For $t > t_0$ we have:

$$y''(t)\, y(t) \geq \frac{4(N-1)}{(N-2)} \int (\partial_t u)^2 \int u^2 \geq \Big(\frac{N-1}{N-2}\Big)\, y'(t)^2$$

or

$$\frac{y''(t)}{y'(t)} \geq \Big(\frac{N-1}{N-2}\Big) \frac{y'(t)}{y(t)}$$

or

$$y'(t) \geq C_0\, y(t)^{(N-1)/N-2}\,, \quad t > t_0\,.$$

But since $\frac{(N-1)}{N-2} > 1$ this leads to finite time blow-up, a contradiction.

We now turn to the proof of i) in Theorem B. We repeat the "concentration-compactness" procedure, replacing Keraani's work with the work of Bahouri-Gérard ([**1**], 1999) on high frequency approximation to solutions of the linear wave equation. We then obtain E_c, with $0 < \eta_0 \leq E_c \leq E((W,0))$ with the property that if $E((u_0,u_1)) < E_c$, $||\nabla u_0||^2 < ||\nabla W||^2$, we have $I = (-\infty,+\infty)$, $||u||_{S(-\infty,+\infty)} < \infty$ and E_c is optimal with this property. i) is the assertion $E_c = E((W,0))$. If not, $E_c < E((W,0))$, which will lead to a contradiction. Exactly as in the (NLS) case we have:

Proposition 1. — *$\exists\, (u_{0,c}, u_{1,c}) \in \dot H^1 \times L^2$, with $||\nabla u_{0,c}||^2 < ||\nabla W||^2$, $E((u_{0,c}, u_{1,c})) = E_c$ and s.t. for the solution u_c of (CP), with* max int I, we have $||u_c||_{S(I)} = +\infty$.

Proposition 2. — *For any u_c as in Proposition 1, s.t. (say) $||u_c||_{S(I_+)} = +\infty$, $\exists\, x(t) \in \mathbb{R}^N$, $\lambda(t) \in \mathbb{R}^+$, $t \in I^+$ s.t.*

$$K = \Big\{ v(x,t) = \Big(\frac{1}{\lambda(t)^{N-2/2}}\, u_c\Big(\frac{x - x(t)}{\lambda(t)}\,, t\Big)\,, \frac{1}{\lambda(t)^{N/2}}\, \partial_t\, u_c\Big(\frac{x-x(t)}{\lambda(t)}\,, t\Big)\Big)\Big\}$$

has compact closure in $\dot H^1 \times L^2$.

Remark. — *$x(t), \lambda(t)$ can be taken continuous. Moreover, if $T_+ < \infty$, $\lambda(t) \geq \frac{C_0(K)}{(T_+ - t)}$ (same proof as (NLS)). Also, if $T_+ = +\infty$, by possibly changing u_c, we can find one for which $\lambda(t) \geq A_0 > 0$.*

One can also show:

Lemma. — *$\exists\, g : (0, E_c] \to [0,\infty)$, $g \downarrow$ s.t. $\forall (u_0,u_1)$ with $E((u_0,u_1)) \leq E_c - \eta$, $||\nabla u_0||^2 < ||\nabla W||^2$, we have $||u||_{S(-\infty,+\infty)} \leq g(\eta)$.*

To proceed further, we need specific features of the problem. We now will develop some further properties of critical elements, specific to (NLW). We start out with some further consequences of the finite speed of propagation.

Lemma. — *Let $(u_0,u_1) \in \dot H^1 \times L^2$, $||(u_0,u_1)||_{\dot H^1 \times L^2} \leq A$. If for some $M > 0$, $\varepsilon > 0$, $0 < \varepsilon < \varepsilon_0 = \varepsilon_0(A)$, we have:*

$$\int_{|x| \geq M} |\nabla u_0|^2 + |u_1|^2 + \frac{|u_0|^2}{|x|^2} \leq \varepsilon\,,$$

then for $0 < t < T_+(u_0, u_1)$ *we have*

$$\int_{|x|\geq \frac{3}{2}M+t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 + |u(t)|^{2^*} + \frac{|u(t)|^2}{|x|^2} \leq C\varepsilon\,.$$

Proof. — Choose $\psi_M \equiv 1$, $|x| \geq \frac{3}{2}M$, $\psi_M \equiv 0$, $|x| \leq M$, $|\nabla \psi_M| \leq \frac{C}{M}$. Let $u_{0,M} = \psi_M u_0$, $u_{1,M} = \psi_M u_1$. Because of our assumption, $||(u_{0,M}, u_{1,M})||_{\dot H^1\times L^2} \leq C\varepsilon$. If ε_0 is so small that $C\varepsilon_0 < \delta$, then u_M solves (CP) in $I = (-\infty, +\infty)$ and $\sup\limits_{t\in(-\infty,+\infty)} ||(u_M(t), \partial_t u_M(t)||_{\dot H^1\times L^2} < 2C\varepsilon$. But by finite speed, $u_M = u$ for $|x| \geq \frac{3}{2}M + t$, $t > 0$, $t \in I$. □

Lemma. — *Let* u_c *be a critical element as in Proposition 2, with* $T_+((u_0, u_1)) < \infty$. *(Assume without loss of generality that* $T_+((u_0, u_1)) = 1$*). Then,* $\exists\, \bar x \in \mathbb{R}^N$ *s.t.*

$$\operatorname{supp} u_c(-, t), \partial_t u_c(-, t) \subset B(\bar x, 1 - t)\,, \quad 0 < t < 1\,.$$

Proof. — We first show, for each t, $0 < t < 1$, that there is a ball B_{1-t}, of radius $(1 - t)$ s.t. $\operatorname{supp} \nabla u$, $\operatorname{supp} \partial_t u \subset B_{1-t}$. If not, for a fixed t, $\exists \varepsilon_0 > 0$, $\eta_0 > 0$ s.t. $\forall\, x_0 \in \mathbb{R}^N$ we have

$$\int_{|x-x_0|\geq(1+\eta_0)(1-t)} |\nabla u(t)|^2 + (\partial_t u(t))^2 \geq \varepsilon_0 > 0\,.$$

Choose a sequence $t_n \uparrow 1$. Recall that $\lambda(t_n) \geq \frac{C_0(K)}{1-t_n}$. We claim that, given $R_0 > 0$, $M > 0$, for n large we have

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq R_0} |\nabla u(x, t_n)|^2 + |\partial_t u(x, t_n)|^2 + \frac{|u(x, t_n)|^2}{|x|^2} \leq \frac{\varepsilon_0}{M}.$$

Indeed, let $\vec v(x, t) = \frac{1}{\lambda(t)^{N/2}}\big(\nabla u(\frac{x-x(t)}{\lambda(t)}, t), \partial_t u(\frac{x-x(t)}{\lambda(t)}, t)\big)$ which is compact in $L^2(\mathbb{R}^N)^{N+1}$. Then

$$\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq R_0} |\nabla u(x, t_n)|^2 + |\partial_t u(x, t_n)|^2 = \int\limits_{|y|\geq\lambda(t_n)\,R_0} |\vec v(y, t_n)|^2$$

and the claim follows from the compactness of $\bar K$, $\lambda(t_n) \uparrow +\infty$. (The proof for the term $\frac{|u(x,t_n)|^2}{|x|^2}$ follows from a similar argument). From this claim and the previous Lemma, used backward in time, we conclude that $\forall\, t \in [0, t_n]$ we have

$$\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq\frac{3}{2}R_0+(t_n-t)} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 \leq \varepsilon_0\,.$$

But if R_0 is so small that $(1 + \eta_0)(1 - t) \geq \frac{3}{2}R_0 + (t_n - t)$, we reach a contradiction, proving the claim.

The next step is to show that $|\frac{x(t)}{\lambda(t)}| \leq M$, $0 \leq t < 1$. If not, $\exists\, t_n \uparrow 1$ s.t. $|\frac{x(t_n)}{\lambda(t_n)}| \uparrow +\infty$. Fix a ball $B(x_0, 1)$ s.t. $\operatorname{supp} \nabla u_0, u_1 \subset B(x_0, 1)$. For a fixed $R_0 > 0$, $\varepsilon_0 > 0$ given, the previous argument shows that, for n large,

$$\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq\frac{3}{2}R_0+(t_n)} |\nabla u_0|^2 + |u_1|^2 \leq \varepsilon_0\,.$$

But, if $|\frac{x(t_n)}{\lambda(t_n)}| \to +\infty$, $B(x_0, 1) \subset \{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq \frac{3}{2} R_0 + t_n\}$, for n large, so that ∇u_0, u_1 are identically 0, contradicting $T_+ = 1$. Let now $t_n \uparrow 1$ and choose a subsequence s.t. $\frac{-x(t_n)}{\lambda(t_n)} \to \bar{x}$. The same argument shows that for $0 < t < t_n$, n large we have

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq\frac{3}{2}R_0+(t_n-t)} |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \leq \varepsilon_0 \,.$$

Letting $n \to \infty$ we obtain

$$\int_{|x-\bar{x}|\geq\frac{3}{2}\,R_0+(1-t)} |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \leq \varepsilon,$$

so that $\operatorname{supp} \nabla u(-,t), \partial_t u(-,t) \subset B(\bar{x}, 1-t)$. If $\frac{-x(t_n)}{\lambda(t_n)} \to \bar{x}$, $\frac{-x(t_{n'})}{\lambda(t_{n'})} \to \bar{x}'$, $x \neq x'$ and we choose $1-t$ so small that $(1-t) < |x - x'|$, we must have $\nabla u(-,t), \partial_t u(-,t) \equiv 0$, which contradicts coercivity, $T_+ = 1$. □

Remark. — *After translation we can take* $\bar{x} = 0$.

Lecture 4

We next turn to a fundamental result that is crucial in the treatment of non-radial solutions.

Theorem **(Orthogonality for critical elements)**. — *Let* $(u_{0,c}, u_{1,c})$ *be as in Propositions 1,2,* $\lambda(t), x(t)$ *continuous,* $\lambda(t) > 0$. *Assume that either* $T_+((u_{0,c}, u_{1,c})) < +\infty$ *or* $T_+((u_{0,c}, u_{1,c})) = +\infty$, $\lambda(t) \geq A_0 > 0$. *Then*

$$\int \nabla u_{0,c} \cdot u_{1,c} = 0\,.$$

Note that in the radial case this is automatic. We first sketch the proof in the case $T_+ < \infty$. We need a further linear estimate.

Lemma **(Trace Theorem)**. — *Let*

$$\begin{cases} \partial_t^2 w - \Delta w = h \in L^1_t\, L^2_x, \\ w|_{t=0} = w_0 \in \dot{H}^1, \\ \partial_t w|_{t=0} = w_1 \in L^2. \end{cases}$$

Then, for $|\alpha| \leq \frac{1}{4}$, *we have:*

$$\sup_t \Big\|\nabla w\Big(\frac{x_1 - \alpha t}{\sqrt{1-\alpha^2}}\,, x', \frac{t - \alpha x_1}{\sqrt{1-\alpha^2}}\Big)\Big\|_{L^2(dx,dx')} + \sup_t \Big\|\partial_t w\Big(\frac{x_1 - \alpha t}{\sqrt{1-\alpha^2}}\,, x', \frac{t - \alpha x_1}{\sqrt{1-\alpha^2}}\Big)\Big\|_{L^2(dx,dx')}$$
$$\leq C\big\{||w_0||_{\dot{H}^1} + ||w_1||_{L^2} + ||h||_{L^1_t\,L^2_x}\big\}$$

Proof. — It suffices to consider $v(x,t) = U(t)f$, where $\hat{v}(\xi, t) = e^{it|\xi|}\hat{f}(\xi)$ and prove

$$\sup_t \Big\|v\Big(\frac{x_1 - \alpha t_1}{\sqrt{1-\alpha^2}}\,, x', \frac{t - \alpha x_1}{\sqrt{1-\alpha^2}}\Big)\Big\|_{L^2(dx,dx')} \leq C||f||_{L^2}.$$

Note that

$$v(x,t) = \int e^{ix_1\xi_1}\, e^{ix'\cdot\xi'}\, e^{it\sqrt{\xi_1^2+|\xi'|^2}}\, \hat{f}(\xi_1, \xi')\, d\xi_1\, d\xi',$$

so that

$$v\Big(\frac{x_1-\alpha t}{\sqrt{1-\alpha^2}}, x', \frac{t-\alpha x_1}{\sqrt{1-\alpha^2}}\Big)$$
$$= \int e^{ix_1(\xi_1-\alpha|\xi|)/\sqrt{1-\alpha^2}}\, e^{-i\alpha t\,\xi_1/\sqrt{1-\alpha^2}}\, e^{-i\alpha t\xi_1/\sqrt{1-\alpha^2}}\, e^{it|\xi|/\sqrt{1-\alpha^2}}\, e^{ix'\xi'}\, \hat f(\xi)\, d\xi_1 d\xi'$$
$$= \int e^{ix_1(\xi_1-\alpha|\xi|)/\sqrt{1-\alpha^2}}\, e^{ix'\cdot\xi'}\, \hat g_t(\xi)\, d\xi_1\, d\xi',$$

where $\hat g_t(\xi) = e^{-i\alpha t\xi_1/\sqrt{1-\alpha^2}}\, \hat f(\xi)$, so that $||g_t||_{L^2} = ||f||_{L^2}$. If we now let $\eta_1 = \frac{\xi_1-\alpha|\xi|}{\sqrt{1-\alpha^2}}$, $\eta' = \xi'$ and compute $|\frac{d\eta}{d\xi}| = \frac{(1-\alpha\xi_1||\xi|)}{\sqrt{1-\alpha^2}} \simeq 1$ for $|\alpha| \le \frac{1}{4}$, we see that the estimate follows from Plancherel. □

If u is a solution of (CP) with maximal interval I, $I' \Subset I$, recall that $u \in L_{I'}^{(N+2)/N-2}\, L_x^{2(N+2)/N-2}$, $\frac{4}{N-2} + 1 = \frac{N+2}{N-2}$, so that $|u|^{4/N-u}\, u \in L^1_{I'}\, L^2_x$. Hence, the conclusion of the previous lemma holds, provided the integrations are restricted to $(\frac{x_1-\alpha t}{\sqrt{1-\alpha^2}}, x', \frac{t-\alpha x_1}{\sqrt{1-\alpha^2}}) \in \mathbb{R}^N \times I'$.

Idea of the proof of Theorem 5 when $T_+((u_0,u_1)) = 1$. — Assume that $\int \partial x_1(u_{0,c}) \cdot u_{1,c} = \gamma > 0$. Recall that $\operatorname{supp} u_c, \partial_t u_c \subset B(0, 1-t)$, $0 < 0 < 1$. For convenience, set $u(x,t) = u_c(x, 1+t)$, $-1 \le t < 0$, supported in $B(0,|t|)$. For $0 < \alpha < \frac{1}{4}$ we consider the Lorentz transformation

$$z_\alpha(x_1, \bar x, t) = u\Big(\frac{x_1-\alpha t}{\sqrt{1-\alpha^2}}, x', \frac{t-\alpha x_1}{\sqrt{1-\alpha^2}}\Big)$$

and fix our attention on $-\frac{1}{2} \le t < 0$. In that region, the Lemma above and the remark following it, together with the support property of u, show that z_α is in the energy space and solves our equation. An easy calculation shows that $\operatorname{supp} z_\alpha(-,t) \subset B(0,|t|)$, $z_\alpha \not\equiv 0$, so that $T_+ = 0$ is the final time of existence for z_α. A long calculation shows that

$$\lim_{\alpha\downarrow 0} \frac{E(z_\alpha(-\frac{1}{2}), \partial_t z_\alpha(-\frac{1}{2})) - E((u_{0,c}, u_{1,c}))}{\alpha} = -\gamma$$

and that, for some $t_0 \in [-\frac{1}{2}, -\frac{1}{4}]$, $\int |\nabla z_\alpha(t_0)|^2 < \int |\nabla W|^2$, for α small. But, since $E((u_{0,c}, u_{1,c})) = E_c$, for α small this contradicts the definition of E_c, since the final time of existence is finite. □

Comments on the proof of Theorem 5 when $T_+ = +\infty$ ($\lambda(t) \ge A_0 > 0$)

The finiteness of the energy of z_α is now unclear, because of the lack of the support property. We then do a renormalization. We first rescale u_c and consider $u_R(x,t) = R^{(N-2)/2}\, u_c(Rx, Rt)$ for R large, and for α small

$$z_{\alpha,R}(x_1, \bar x, t) = u_R\Big(\frac{x_1-\alpha t}{\sqrt{1-\alpha^2}}, \bar x, \frac{t-\alpha x_1}{\sqrt{1-\alpha^2}}\Big).$$

We assume, as before, that $\int \partial_{x_1} u_{0,c} \cdot u_{1,c} = \gamma > 0$. We then prove (by integration in $t_0 \in (1,2)$) that if $h(t_0) = \theta(x)\, z_{\alpha,R}(x_1, \bar x, t_0)$, with θ a cut-off function, for some α_1 small and all R sufficiently large, we have, for some $t_0 \in (1,2)$ that

$$E((h(-,t_0), \partial_t h(-,t_0)) < E_c - \frac{1}{2}\gamma\,\alpha_1$$

and

$$\int |\nabla h(t_0)|^2 < \int |\nabla W|^2 .$$

We then let v be the solution of (CP), with data $h(-,t_0)$ at $t = t_0$. By our properties of critical elements, we know that $||v||_{S(-\infty,+\infty)} \leq g(\frac{1}{2}\gamma\,\alpha_1)$, for all R large. But, since $||u_c||_{S(0,+\infty)} = +\infty$, we have that $||u_R||_{L^{2(N+1)/N-2}_{[0,1]}} \xrightarrow[R\to\infty]{} \infty$, by rescaling. But, by finite speed of propagation, we have that $v = z_{\alpha,R}$ on a large set, and after a change of variables to undo α_1, we reach a contradiction. The details of the argument are lengthy. □

To finish the proof of Theorem B, we are reduced to:

***Theorem* (Rigidity Theorem).** — *Assume that $E((u_0,u_1)) < E((W,0))$, $\int |\nabla u_0|^2 < \int |\nabla W|^2$, u the solution of (CP) with $I_+ = [0,T_+)$. Assume that*

a) $\int \nabla u_0\, u_1 = 0$.

b) $\exists\, x(t), \lambda(t) > 0$, $t \in [0,t_+)$ *s.t.*

$$K = \Big\{ v(x,t) = \Big(\frac{1}{\lambda(t)^{N-2/2}}\, u\Big(\frac{x-x(t)}{\lambda(t)}\, t\Big)\,,\, \frac{1}{\lambda(t)^{N/2}}\,\partial_t u\Big(\frac{x-x(t)}{\lambda(t)}\,,t\Big)\Big)\Big\}$$

has compact closure in $\dot H^1 \times L^2$.

c) *$x(t), \lambda(t)$ are continuous, if $T_+ = 1$ (scaling) $\lambda(t) \geq \frac{C_0(K)}{1-t}$, $\operatorname{supp} u$, $\partial_t u \subset B(0,1-t)$, when $T_+ = +\infty$, $x(0) = 0$, $\lambda(0) = 1$, $\lambda(t) \geq A_0 > 0$.*

Then, $T_+ = 1$ cannot happen and if $T_+ = +\infty$, $(u_0,u_1) = (0,0)$.

Clearly the rigidity theorem gives us the contradiction with establishes Theorem B, i).

Proof of the Rigidity Theorem. — **Case 1 :** $T_+ = +\infty$, $\lambda(t) \geq A_0 > 0$, $x(0) = 0$, $\lambda(0) = 1$, $x(t), \lambda(t)$ continuous. Assume $(u_0,u_1) \neq (0,0)$. By our variational estimates we have

$$\sup_{t>0} ||(\nabla u, \partial_t u)(t)||_{L^2} \leq CE\,,$$

where $E((u_0,u_1)) = E > 0$. We also have

$$\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq C_{\bar\delta} \int |\nabla u(t)|^2\,, \quad t > 0$$

$$\frac{1}{2}\int (\partial_t u(t))^2 + \frac{1}{2}\int |\nabla u(t)|^2 - |u(t)|^{2^*} \geq C_{\bar\delta} E$$

where $\bar\delta = \bar\delta(\delta_0)$, $E((u_0,u_1)) \leq (1-\delta_0)\,E((W,0))$. A change of variables, the compactness of $\bar K$ and $\lambda(t) \geq A_0 < 0$ now give: given $\varepsilon > 0$, $\exists\, R_0(\varepsilon)$ s.t. for all $0 \leq t < \infty$, we have

$$\int\limits_{|x+\frac{x(t)}{\lambda(t)}|\geq R_0(\varepsilon)} |\partial_t u|^2 + |\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*} \leq \varepsilon\, E\,.$$

We next need some algebraic identities:

Lemma. — *Let $r(R) = r(t,R) = \int\limits_{|x|\geq R} \{|\nabla u|^2 + |\partial_t u|^2 + |u|^{2^*} + \frac{|u|^2}{|x|^2}\}\,dx$. We have, if $\phi \in C_0^\infty(B_2)$, $\phi \equiv 1$ on $|x| \leq 1$, $\phi_R(x) = \phi(\frac{x}{R})$, $\psi_R(x) = x\,\phi(\frac{x}{R})$:*

i) $\partial_t\Big(\displaystyle\int \psi_R\,\nabla u\cdot\partial_t u\Big) = -\frac{N}{2}\int(\partial_t u)^2 + \frac{N-2}{2}\Big[\int |\nabla u|^2 - |u|^{2^*}\Big] + O(r(R))$

ii) $\partial_t\Big(\displaystyle\int \phi_R\,u\,\partial_t u\Big) = \int(\partial_t u)^2 - \int |\nabla u|^2 + \int |u|^{2^*} + O(r(R))$

iii) $\partial_t\Big(\displaystyle\int \psi_R\Big\{\frac{1}{2}|\nabla u|^2 + \frac{1}{2}(\partial_t u)^2 - \frac{1}{2^*}|u|^{2^*}\Big\}\Big) = -\int \nabla u\,\partial_t u + O(r(R)).$

The proof of the case $T_+ = +\infty$ is based on 2 Lemmas.

Lemma 1. — *$\exists\,\varepsilon_1 > 0$, $C > 0$ s.t. if $0 < \varepsilon < \varepsilon_1$, $\exists\,R_0(\varepsilon)$ s.t. if $R > 2R_0(\varepsilon)$, $\exists\,t_0 = t_0(R,\varepsilon)$ with $0 < t_0 \le C\,R$ s.t. $\forall\,0 < t < t_0$ we have $|\frac{x(t)}{\lambda(t)}| < R - R_0(\varepsilon)$ and $|\frac{x(t_0)}{\lambda(t_0)}| = R - R_0(\varepsilon)$.*

Remark. — *In the radial case, $x(t) \equiv 0$, so a contradiction follows directly from Lemma 10. This is the analogue of the virial identity proof for NLS. In the non-radial case we also need:*

Lemma 2. — *$\exists\,\varepsilon_2 > 0$, $R_1(\varepsilon) > 0$, $C_0 > 0$ s.t. if $R > R_1(\varepsilon)$, for $0 < \varepsilon < \varepsilon_2$, we have $t_0(R,\varepsilon) \ge \frac{C_0 R}{\varepsilon}$.*

From Lemma 1 and Lemma 2 we have: for $0 < \varepsilon < \varepsilon_1$, $R > 2R_0(\varepsilon)$, $t_0(R,\varepsilon) \le C\,R$, while for $0 < \varepsilon < \varepsilon_2$, $R > R_1(\varepsilon)$, $t_0(R,\varepsilon) \ge \frac{C_0 R}{\varepsilon}$. This is a contradiction for ε small.

Proof of Lemma 1. — If not, since $x(0) = 0$, $\lambda(0) = 1$, both $x(t), \lambda(t)$ continuous, we have $\forall\,0 < t < C\,R$ (C large) that $|\frac{x(t)}{\lambda(t)}| < R - R_0(\varepsilon)$. Let $z_R(t) = \int \psi_R\,\nabla\cdot\partial_t u + (\frac{N}{2} - \alpha)\int \phi_R u\,\partial_t u$. Then, $z_R'(t) = -\alpha\int(\partial_t u)^2 - (1-\alpha)[\int |\nabla u|^2 - |u|^{2^*}] + 0(r(R))$. But, for $|x| > R$, $0 < t < C\,R$ we have $|x + \frac{x(t)}{\lambda(t)}| \ge R_0(\varepsilon)$, so that $|r(R)| \le \tilde{C}\,\varepsilon\,E$ and so, for ε small, $\alpha = \frac{1}{2}$, $z_R'(t) \le -\frac{\tilde{\tilde{C}}\,E}{2}$. Also, $|z_R(t)| \le C_1 RE$. Integrating in t, we obtain: $\frac{C\,R\,\tilde{\tilde{C}}\,E}{2} \le 2C_1\,R\,E$, a contradiction for C large. □

Proof of Lemma 2. — For $0 \le t \le t_0$, set

$$y_R(t) = \int \psi_R\Big\{\frac{1}{2}\,(\partial_t u)^2 + \frac{1}{2}\,|\nabla u|^2 - \frac{1}{2^*}\,|u|^{2^*}\Big\}.$$

We have for $|x| > R$, $|x + \frac{x(t)}{\lambda(t)}| > R_0(\varepsilon)$ so that, since $\int \nabla u_0\,u_1 = 0$, $y_R'(t) = O(r(R))$ and hence

$$|y_R(t_0) - y_R(0)| \le \tilde{C}\varepsilon E t_0.$$

But,

$$|y_R(0)| \le \tilde{C}\,R_0(\varepsilon)\,E + O(Rr(R_0(\varepsilon))) \le \tilde{C}E\{R_0(\varepsilon) + \varepsilon\,R\}$$

and

$$|y_R(t_0)| \ge \Big|\int_{|x+\frac{x(t_0)}{\lambda(t_0)}| \le R_0(\varepsilon)}\Big| - \Big|\int_{|x+\frac{x(t_0)}{\lambda(t_0)}| > R_0(\varepsilon)}\Big|.$$

In the first integral, $|x| \le R$, so that $\psi_R(x) = x$. The second integral is bounded by $MR\varepsilon E$ so that

$$|y_R(t_0| \ge \Big|\int_{|x+\frac{x(t_0)}{\lambda(t_0)}| \le R_0(\varepsilon)} x\Big[\frac{1}{2}\,(\partial_t u)^2 + \frac{1}{2}\,|\nabla u|^2 - \frac{1}{2^*}\,|u|^{2^*}\Big]\Big| - MR\varepsilon E.$$

The integral equals

$$\frac{-x(t_0)}{\lambda(t_0)} \int\limits_{|x+\frac{x(t_0)}{\lambda(t_0)}|\le R_0(\varepsilon)} \frac{1}{2}(\partial_t u)^2(t_0) + \frac{1}{2}|\nabla u|^2(t_0) - \frac{1}{2^*}|u|^{2^*}(t_0) + \int\limits_{|x+\frac{x(t_0)}{\lambda(t_0)}|\le R_0(\varepsilon)} \Big(x + \frac{x(t_0)}{\lambda(t_0)}\Big)\Big\{ \ \Big\}$$

$$= \frac{-x(t_0)}{\lambda(t_0)} \int \Big\{\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2^*}|u|^{2^*}\Big\} + \frac{x(t_0)}{\lambda(t_0)} \int_{|x+\frac{x(t_0)}{\lambda(t_0)}|\le R_0(\varepsilon)} \Big\{ \ \Big\}$$

$$+ \int_{|x+\frac{x(t_0)}{\lambda(t_0)}|\le R_0(\varepsilon)} \Big(x + \frac{x(t_0)}{\lambda(t_0)}\Big)\Big\{ \ \Big\}.$$

The absolute value of the first term equals $(R - R_0(\varepsilon))\,E$. The last two are bounded by $\tilde{C}(R - R_0(\varepsilon))\,\varepsilon\,E + \tilde{C}\,R_0(\varepsilon)\,E$. Thus,

$$|y(t_0| \ge (R - R_0(\varepsilon))\,E(1 - \tilde{C}\,\varepsilon) - M\,R\,\varepsilon\,E - \tilde{C}\,R_0(\varepsilon)\,E \ge \frac{E\,R}{4}$$

for ε small, R large. Thus,

$$\tilde{C}\,\varepsilon\,E\,t_0 \ge \frac{E\,R}{4} - \tilde{C}\,E(R_0(\varepsilon) + \varepsilon\,R)$$

which yields the lemma for ε small, R large. □

We next turn to:

Case 2 : $T_+ = 1$, $\lambda(t) \ge \frac{C_0(K)}{1-t}$, $\operatorname{supp} u,\ \partial_t u \subset B(0, 1-t)$. In this case we cannot use the conservation of the L^2 norm as in the case of (NLS) and a new approach is needed.

The first step is:

Lemma 3. — $\lambda(t) \le \frac{C_1(K)}{1-t}$.

Proof. — If not, $\exists\, t_n \uparrow 1$ s.t. $\lambda(t_n)(1-t_n) \uparrow +\infty$. Let $z(t) = \int x\cdot\nabla u\cdot\partial_t u + (\frac{N}{2} - \alpha)\int u\,\partial_t u$, $0 < \alpha < 1$. This is defined for $0 < t < 1$ and

$$z'(t) = -\alpha\int(\partial_t u)^2 - (1-\alpha)\int|\nabla u|^2 - |u|^{2^*}.$$

By our variational estimates $E((u_0, u_1)) = E > 0$ and $\sup\limits_{0<t<1} ||(\nabla u, \partial_t u)(t)|| \le C\,E$. Also, $z'(t) \le -C_\alpha E$, $0 < t < 1$. From the support properties, we easily see that $\lim\limits_{t\uparrow 1} z(t) = 0$, so that, integrating in t, $z(t) \ge C_\alpha E(1-t)$. We will show that $\frac{z(t_n)}{1-t_n} \to 0$, yielding a contradiction. We know that $\int\nabla u\,\partial_t u = 0$, $0 < t < 1$. Hence,

$$\frac{z(t_n)}{(1-t_n)} = \frac{\int(x + \frac{x(t_n)}{\lambda(t_n)})\nabla u\cdot\partial_t u}{(1-t_n)} + \Big(\frac{N}{2} - \alpha\Big)\int\frac{u\,\partial_t u}{(1-t_{n)}}.$$

Note that, for $\varepsilon > 0$ given, we have

$$\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\le\varepsilon(1-t_n)} \Big|x + \frac{x(t_n)}{\lambda(t_n)}\Big|\,|\nabla u(t_n)|\,|\partial_t u(t_n)| \le C\,\varepsilon\,E(1-t_n)$$

and similarly for $\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\leq\varepsilon(1-t_n)} |u(t_n)|\,|\partial_t u(t_n)|$. Next we show that $|x+\frac{x(t_n)}{\lambda(t_n)}|\leq 2(1-t_n)$. If not, $B(\frac{-x(t_n)}{\lambda(t_n)},(1-t_n))\cap B(0,1-t_n)=\emptyset$, so that $\int\limits_{B(\frac{-x(t_n)}{\lambda(t_n)},(1-t_n))} |\nabla u(t_n)|^2=0$, while

$$\int\limits_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq(1-t_n)} |\nabla u(x,t_n|^2\,dx=\int\limits_{|y|\geq\lambda(t_n)(1-t_n)} \Big|\nabla u\Big(\frac{y-x(t_n)}{\lambda(t_n)},t_n\Big)\Big|^2\frac{dy}{\lambda(t_n)^N}\xrightarrow[n\to\infty]{}0$$

by $\lambda(t_n)(1-t_n)\to+\infty$, compactness of $\bar K$. Arguing similarly for $\partial_t u(t_n)$, we obtain that $E((u(t_n),\partial_t u(t_n)))\to 0$, a contradiction. But,

$$\begin{aligned}\frac{1}{(1-t_n)}&\int_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq\varepsilon(1-t_n)}\Big|x+\frac{x(t_n)}{\lambda(t_n)}\Big|\,|\nabla u(x,t_n)|\,|\partial_t u(x,t_n)|\\ &\leq 3\int_{|x+\frac{x(t_n)}{\lambda(t_n)}|\geq\varepsilon(1-t_n)}|\nabla u(x,t_n)|\,|\partial_t u(x,t_n)|\\ &=\frac{3}{\lambda(t_n)^N}\int\limits_{|y|\geq\varepsilon(1-t_n)\lambda(t_n)}\Big|\nabla u\Big(\frac{y-x(t_n)}{\lambda(t_n)},t_n\Big)\Big|\,\Big|\partial_t u\,\Big(\frac{y-x(t_n)}{\lambda(t_n)}\,t_n\Big)\Big|\,dy\xrightarrow[n\to\infty]{}0,\end{aligned}$$

by compactness of $\bar K$, $(\lambda(t_n)(1-t_n))\to 0$. Arguing similarly (using Hardy) for $\int\frac{u\cdot\partial_t u}{(1-t_n)}$, we conclude the proof. □

Proposition. — *In this case we have* $(T_+=1)$ $\operatorname{supp} u,\partial_t u\subset B(0,1-t)$ *and* $K=((1-t)^{N/2}(\nabla u((1-t)x,t),\partial_t u((1-t)x,t))$ *is compact in* $L^2(\mathbb{R}^N)^{N+1}$.

Proof. — $\big\{\vec v(x,t)=(1-t)^{N/2}(\nabla u((1-t)(x-x(t)),t),\partial_t u((1-t)(x-x(t)),t)\big\}$ has compact closure since $C_0(K)\leq(1-t)\,\lambda(t)\leq C_1(K)$ and if $\bar K$ is compact, $K_1=\{\lambda^{N/2}\,\vec v(\lambda x):\vec v\in K,\,C_0\leq\lambda\leq C_1\}$ has $\bar K_1$ compact. Let now $\tilde v(x,t)=(1-t)^{N/2}(\nabla u((1-t)\,x,t),\partial_t u((1-t)\,x,t))$, so that $\tilde v(x,t)=\vec v(x+x(t),t)$. Since $\operatorname{supp}\vec v(-,t)\subset\{x:|x-x(t)|\leq 1\}$ and $E>0$, the fact that $\{\vec v(-,t)\}$ is compact $\Rightarrow|x(t)|\leq C$. But, if $K_2=\{\vec v(x+x_0,t):|x_0|\leq C\}$, then $\bar K_2$ is compact, giving the Proposition. □

At this point, because of the lack of the L^2 conservation law, we cannot go further and a new idea is needed. Following Giga-Kohn [**8**] in the parabolic case and Merle-Zaag [**23**] in the hyperbolic case $\big((\partial_t^2-\Delta)\,u-|u|^{p-1}\,u=0,1<p<\frac{4}{N-1}+1\big)$, we introduce self-similar variables. We set: $y=\frac{x}{1-t}$, $s=\log\frac{1}{1-t}$, $0<t<1$ and define

$$w(y,s;0)=(1-t)^{N-2/2}\,u(x,t)=e^{-s(N-2)/2}\,u(e^{-s}\,y,1-e^{-s})\,,$$

which is defined for $0\leq s<\infty$, $\operatorname{supp} w(-,s;0)\subset\{|y|\leq 1\}$. We also consider, for $\delta>0$, $u_\delta(x,t)=u(x,t+\delta)$ and the corresponding w. In other words, we set $y=\frac{x}{1+\delta-t}$, $s=\log\frac{1}{1+\delta-t}$ and

$$w(y,s;\delta)=(1+\delta-t)^{N-2/2}\,u(x,t)=e^{-s(N-2)/2}\,u(e^{-s}\,y,1+\delta-e^{-s})$$

which is defined for $0 \leq s \leq -\log \delta$, with $\operatorname{supp} w(-,s,\delta) \subset \{|y| \leq \frac{e^{-s}-\delta}{e^{-s}} = \frac{1-t}{1+\delta-t} \leq 1-\delta\}$. The w solve, in their domain,

$$\partial_s^2 w = \frac{1}{\rho} \operatorname{div}(\rho \nabla w - \rho(y\cdot\nabla w)\, y) - \frac{N(N-2)}{4} w + |w|^{4/N-2} w - 2y\cdot\nabla\,\partial_s w - (N-1)\,\partial_s w\,,$$

where $\rho(y) = (1-|y|^2)^{-1/2}$.

The elliptic part of this operator degenerates. In fact, $\frac{1}{\rho}\operatorname{div}(\rho\,\nabla w - \rho(y\cdot\nabla w)y) = \frac{1}{\rho}\operatorname{div}(\rho(I - y\otimes y)\nabla w)$, which is elliptic for $|y| < 1$ and degenerates when $|y| = 1$. This new equation gives us a new set of formulas. The reason for introducing $\delta > 0$ is that, on $\operatorname{supp} w(-,s,\delta)$, $(1-|y|^2) \geq \delta$, so we stay away from the degeneracy. Bounds on w (obvious): $\int_{B_1} |w|^{2^*} + |\nabla w|^2 + |\partial_s w|^2 \leq C$, $w \in H_0^1(B_1)$ and hence $\int_{B_1} \frac{|w|^2}{(1-|y|^2)^2} \leq C$. All these bounds are uniform in δ, s.

We introduce an energy, which will provide a Liapunov function for v :

$$\begin{aligned}\tilde{E}(w(s)) = \int_{B_1} \left\{ \frac{1}{2}(\partial_s w)^2 + |\nabla w|^2 - (y\cdot\nabla w)^2 \right\} \frac{dy}{(1-|y|^2)^{1/2}} \\ + \int_{B_1} \left\{ \frac{N(N-2)}{8} w^2 - \frac{(N-2)}{2N} |w|^{2^*} \right\} \frac{dy}{(1-|y|^2)^{1/2}}\end{aligned}$$

which is finite for $\delta > 0$. Our new formulas are $(0 < s_1 < s_2 < \log 1/\delta)$

i) $\tilde{E}(w(s_2)) - \tilde{E}(w(s_1)) = \int_{s_1}^{s_2}\int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}}\, dy\, ds \quad (\tilde{E}\uparrow)$.

ii)

$$\begin{aligned}\frac{1}{2}\int_{B_1} \left[\partial_s w \cdot w) - \frac{(1+N)}{2} w^2\right] \Big| \frac{dy}{(1-|y|^2)^{1/2}} \Big|_{s_1}^{s_2} = \\ -\int_{s_1}^{s_2} \tilde{E}(w(s))\, ds + \frac{1}{N}\int_{s_1}^{s_2}\int_{B_1} \frac{|w|^{2^*}}{(1-|y|^2)^{1/2}} \\ + \int_{s_1}^{s_2}\int_{B_1} \left\{ (\partial_s w)^2 + \partial_s w\cdot y\cdot\nabla w + \frac{\partial_s w\cdot w|y|^2}{(1-|y|^2)} \right\} \frac{dy}{(1-|y|^2)^{1/2}}.\end{aligned}$$

iiii) $\lim_{s\to\log\frac{1}{\delta}} \tilde{E}(w(s)) = E = E(u_0,u_1)$, so that $\tilde{E}(w(s)) \leq E$, for $0 \leq s < \log\frac{1}{\delta}$.

Our first improvement is $(\delta > 0)$:

Lemma. —

$$\int_0^1\int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)}\, dy\, ds \leq C\, \log\frac{1}{\delta}.$$

Proof. — We notice that

$$\begin{aligned}-2\int \frac{(\partial_s w)^2}{(1-|y|^2)} =& \frac{d}{ds}\Big\{ \int \Big[\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}\left(|\nabla w|^2 - (y\cdot w)^2\right) + \frac{(N-2)N}{8} w^2 \\ &- \frac{(N-2)}{2N}\,|w|^{2^*}\Big](-\log(1-|y|^2)\,dy\Big\} \\ &+ \int \big[\log(1-|y|^2)+2\big]\, y\cdot\nabla w\,\partial_s w - \int \log(1-|y|^2)(\partial_s w)^2 - 2\int (\partial_s w)^2 .\end{aligned}$$

We integrate between 0 and 1 and drop the next to last term by sign. One finishes by (C-S), support of $w(-,s,\delta)$. □

Corollary. —

$$\int_0^1 \int_{B_1} \frac{|w|^{2^*}}{(1-|y|^2)^{1/2}} \le C\Big(\log\frac{1}{\delta}\Big)^{1/2}$$

$$\tilde{E}(w(1)) \ge -C\,\log\Big(\frac{1}{\delta}\Big)^{1/2}.$$

Proof. — The first estimate follows from ii), iii) above, C-S and the Lemma. Note that (CS) give the $\frac{1}{2}$ power. The second estimate follows from i) and the fact that $\int_0^1 \tilde{E}(w(s))\,ds \ge -C\big(\log\frac{1}{\delta}\big)^{1/2}$, which follows from the definition of $\tilde{E}$ and the first bound. □

Our second improvement is:

Lemma. —

$$\int_1^{(\log\frac{1}{\delta})^{3/4}} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \le C\Big(\log\frac{1}{\delta}\Big)^{1/2}.$$

Proof. — Use i), iii) and the second bound in Corollary. Note that the upper limit of integration is not important in the bound. It is chosen for the subsequent applications. □

Corollary. — $\exists \bar{s}_\delta \in (1,(\log\frac{1}{\delta})^{3/4})$ *s.t.*

$$\int_{\bar{s}_\delta}^{\bar{s}_\delta+(\log\frac{1}{\delta})^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1-|y|^2)^{3/2}} \le \frac{C}{(\log\frac{1}{\delta})^{1/8}}.$$

Proof. — Split $(1,(\log\frac{1}{\delta})^{3/4})$ into disjoint intervals of length $(\log\frac{1}{\delta})^{1/8}$. Their number is $(\log\frac{1}{\delta})^{5/8}$ and $\frac{5}{8}-\frac{1}{8}=\frac{1}{2}$. Note the length $\to +\infty$, the bound $\to 0$. □

Now it is not hard to see that, since $\bar{s}_\delta \in (1,(\log\frac{1}{\delta})^{3/4})$, if $\bar{s}_\delta = -\log(1+\delta-\bar{t}_\delta)$, $|\frac{(1-\bar{t}_\delta)}{(1+\delta-\bar{t}_\delta)} - 1| \le C\,\delta^{1/4} \to 0$, which is the point of our choice of $(\log\frac{1}{\delta})^{3/4}$. From this and the compactness of $\bar{K}$, one can find $w^*(y,s)$ which solves our self-similar equation in $s\in[0,S]$, which is a limit of $w(y,\bar{s}_{\delta_j}+s,\delta_j)$ as $\delta_j \to 0$, in $C([0,S];\dot{H}^1_0\times L^2)$. The estimate in the corollary shows that w^* is independent of s. Moreover, the coercivity of u shows that $w^* \not\equiv 0$. Thus, $w^*\in H^1_0(B_1)$, solves the (degenerate) elliptic equation: $(\rho(y)=(1-|y|^2)^{-1/2})$, $\frac{1}{\rho}\operatorname{div}(\rho\,\nabla\, w^* -$

$\rho(y\cdot\nabla w^*)y) - \frac{N(N-2)}{4}\, w^* + |w^*|^{4/N-2}\, w^* = 0$. We next show that w^* satisfies the additional (crucial) estimates:

$$\int_{B_1} \frac{|w^*|^{2^*}}{(1-|y|^2)^{1/2}} + \int_{B_1} \frac{[|\nabla w^*|^2 - (y\cdot\nabla w^*)^2]}{(1-|y|^2)^{1/2}} < \infty\,.$$

Indeed, for the first estimate, it is enough to show that

$$\int_{\bar{s}_{\delta_j}}^{\bar{s}_{\delta_j}+S} \int_{B_1} \frac{|w(y,s;\delta_j)|^{2^*}}{(1-|y|^2)^{1/2}}\, dy\, ds \le C \quad \text{for } j \text{ large}\,.$$

But this follows from ii) above once more, together with the choice of $\bar{s}_{\delta_j}$ (Corollary) and (C-S). The proof of the second estimate is similar, using the first one, iii) and the formula for $\tilde{E}$.

The conclusion of the proof is obtained by showing that a w^* in $H_0^1(B_1)$, solving the degenerate elliptic equation, with the additional bounds, must be 0. To do this, we will use unique continuation. Recall that for $|y| \le 1-\eta_0$, $\eta_0 > 0$, the linear operator is uniformly elliptic, with smooth coefficient and the non-linearity is critical. An argument going back to Trudinger [**31**] shows that w^* is bounded on $|y| \le 1-\eta_0$, for each $\eta_0 > 0$. Hence, if we show that $w^* \equiv 0$ near $|y| = 1$, the standard unique continuation principle [**13**] will show that $w^* \equiv 0$. Near $|y| = 1$, our equation is modelled by (in variables $z \in \mathbb{R}^{N-1}$, $r \in \mathbb{R}$, $r > 0$ near $r = 0$)

$$r^{1/2}\,\partial_r(r^{1/2}\,\partial_r w^*) + \Delta_z\, w^* + |w^*|^{4/N-2}\, w = 0\,.$$

In these variables, our information is $w^* \in H_0^1((0,1]\times(|z|<1))$ and the additional estimates are:

$$\int_0^1 \int_{|z|<1} |w^*(r,z)|^{2^*}\, \frac{dr}{r^{1/2}}\, dz < \infty,$$

$$\int_0^1 \int_{|z|<1} |\nabla_z\, w^*(r,z)|^2\, \frac{dr}{r^{1/2}}\, dz < \infty\,.$$

We now take advantage of the degeneracy of the equation. We "desingularize" the problem by writting $r = a^2$, setting $v(a,z) = w^*(a^2,z)$, so that $\partial_a\, v(a,z) = 2a\,\partial_r w^*(r,z) = 2r^{1/2}\,\partial_r w^*(r,z)$. Our equation becomes $\partial_a^2 v + \Delta_z v + |v|^{4/N-2}\, v = 0$, $0 < a < 1$, $|z| < 1$ and our bounds are:

$$\int_0^1 \int_{|z|<1} |\nabla_z v(a,z)|^2\, da\, dz = \int_0^1 \int_{|z|<1} |\nabla_z w^*(r,z)|^2\, \frac{dr}{r^{1/2}}\, dz < \infty$$

and

$$\int_0^1 \int_{|z|<1} |\partial_a v(a,z)|^2\, \frac{da}{a}\, dz = \int_0^1 \int_{|z|<1} |\partial_r w^*(r,z)|^2\, dr\, dz < \infty\,,$$

and $v \in H_0^1((0,1]\times B_1)$. But, from the additional bound we see that "$\partial_a v(a,z)|_{a=0} = 0$". One then extends v by 0 to $a < 0$ and checks that the extension is an H^1 solution to the same equation. By Trudinger's argument, it is bounded. But, since it vanishes for $a < 0$, by unique continuation [**13**], $v \equiv 0$. Hence $w^* \equiv 0$, reaching our contradiction.

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