RESEARCH STATEMENT

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I am broadly interested in geometric representation theory and number theory. My research in recent years has focused on the geometric Langlands program, especially those aspects related to parabolic induction: geometric Eisenstein series and the geometry of semi-infinite flag varieties.

1. Proof of the GLC

This year, I was part of a nine-person team, led by Dennis Gaitsgory and Sam Raskin, that announced a proof of the geometric Langlands conjecture (henceforth abbreviated as the GLC). The proof is spread out over the preprints [GLC1], [GLC2], [GLC3], [GLC4], and [GLC5].

1.1. *Statement.* Let X be a compact Riemann surface and G a complex reductive group.

Theorem 1.2 (GLC)**.** *There is a canonical equivalence of categories*

$$\mathbb{L}_G : \text{D-mod}(\text{Bun}_G) \xrightarrow{\sim} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}}).$$

As discussed below, the equivalence $\mathbb{L}_G$ is subject to several desiderata in the form of compatibilities with simpler equivalences, which in particular characterize it uniquely.

1.3. *What are all these symbols?* I will now briefly define the notations which appear in the GLC. The perplexed reader is encouraged to consult §5 for some more discussion and heuristics.

- Bun_G denotes the moduli space of holomorphic principal G-bundles on X.
- $\text{D-mod}(\text{Bun}_G)$ denotes the category of algebraic D-modules on Bun_G (cf. §5.5).
- $\check{G}$ denotes the Langlands dual group of G (cf. §5.2).
- $\text{LS}_{\check{G}}$ denotes the moduli space of $\check{G}$-local systems on X in the de Rham sense, i.e., principal $\check{G}$-bundles equipped with a flat connection.
- $\text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})$ denotes the category of ind-coherent sheaves on $\text{LS}_{\check{G}}$ with singular support in the global nilpotent cone (a slight enlargement of the category of quasi-coherent sheaves on $\text{LS}_{\check{G}}$; cf. §5.6).

1.4. *The Eisenstein part.* I contributed to the proof the GLC as a coauthor to [GLC3] (as well as the appendices to [GLC2]), jointly with Chen, Gaitsgory, and Raskin. I will now state our main result and explain its role in the proof of the GLC.

Following a general strategy in representation theory, one decomposes both categories appearing in the GLC into two pieces: the part which can be obtained from Levi subgroups of G by a parabolic induction procedure, and the complementary "cuspidal" part. The work [GLC3] concerns the first of these pieces, and the cuspidal parts are identified in [GLC4] and [GLC5].

We take as an input the existence of the Langlands functor $\mathbb{L}_G$, which is constructed and characterized uniquely in [GLC1].

Theorem 1.5 ([GLC3] Theorem 17.1.2)**.** *Assume that $\mathbb{L}_M$ is an equivalence for any proper Levi subgroup $M \subset G$. Then $\mathbb{L}_G$ restricts to an equivalence*

$$\text{D-mod}(\text{Bun}_G)_{\text{Eis}} \xrightarrow{\sim} \text{IndCoh}_{\text{Nilp}}(\text{LS}_{\check{G}})_{\text{red}}.$$

The subcategories which appear in the theorem are defined as follows.

- D-mod$(\mathrm{Bun}_G)_{\mathrm{Eis}} \subset$ D-mod(Bun_G) is generated by the images of the *geometric Eisenstein series* (alias: parabolic induction) functors

$$\mathrm{Eis}_! : \text{D-mod}(\mathrm{Bun}_M) \longrightarrow \text{D-mod}(\mathrm{Bun}_G)$$

  as $M \subset G$ varies over proper Levi subgroups.
- $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})_{\mathrm{red}} \subset \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}})$ consists of those sheaves whose support is contained in the locus of *reducible* $\check{G}$-local systems, i.e., those which admit a reduction to a proper parabolic subgroup of $\check{G}$.

Our proof of Theorem 1.5 involves writing down the spectral counterpart to $\mathrm{Eis}_!$ and proving that the Langlands functor intertwines them. This constitutes one of the aforementioned desiderata for $\mathbb{L}_G$: compatibility with parabolic induction.

2. After the proof of the GLC

One might reasonably ask what is left to do in the geometric Langlands program.

2.1. *Further directions for the global theory.* As detailed in the introduction to [GLC5], the proof of the GLC leaves open many questions in the global geometric Langlands program. In addition to their intrinsic interest, answers to some of these questions could facilitate a better conceptual understanding of the GLC itself by enabling a new proof, or by replacing parts of the existing proof. For an example of a result in this direction, see Theorem 3.2 below, part of my joint work with Andreas Hayash [CaH].

There is also the so-called quantum geometric Langlands program, a deformation of the "classical" program discussed above. Some progress has been made in this area (including my joint work with Dhillon and Raskin [CDR]), but the main conjecture analogous to Theorem 1.2 remains open. The quantum theory opens up new connections to other parts of mathematics, notably including the theory of quantum groups via the Kazhdan-Lusztig equivalence [KL].

But the elephant in the room is that Theorem 1.2 is actually the *global unramified* version of the GLC. In order to study the phenomenon of ramification, where we allow $\check{G}$-local systems to become singular at finitely many points in X, it is natural to shift our attention to the local situation near a point in X.

2.2. *The local geometric Langlands program.* Compared with the global geometric Langlands program, the local theory is still in its infancy.

One reason for this is the phenomenon of irregular singularities, called wild ramification by analogy with number theory. The tamely ramified story is by now fairly well-understood, and includes the geometric Satake equivalence, the derived geometric Satake equivalence and its parabolic variant due to Bezrukavnikov and his collaborators, and the factorizable derived Satake equivalence and its parabolic variant established in my joint work with Raskin [CR]. By contrast, many fundamental difficulties in the wildly ramified case have yet to be overcome.

Another challenge in the local theory is that it takes place one categorical level higher than global geometric Langlands. In particular, the local geometric Langlands conjecture predicts an equivalence of *2-categories*. This categorification is in accordance with principles from topological quantum field theory: the global theory takes as input the two-dimensional compact Riemann surface X, while the local theory is attached to a punctured disk, which is homotopy equivalent to a one-dimensional circle.

This additional homotopical complexity greatly enriches the local story. A key role is played by *factorization structure* (cf. §5.8), which is closely related to the notions of vertex algebra in conformal field theory and $\mathbb{E}_2$-algebra in topology. This additional structure is essential for many global applications, including the proof of the GLC, which use the theory of chiral homology (analogous to conformal blocks in CFT and factorization homology in topology). My work with Hayash [CaH] makes use of factorization structures in an essential way: see §3.6.

Yet another source of difficulty for the local geometric Langlands program is that many of the geometric objects which naturally arise are infinite-dimensional, sometimes quite badly so. A very interesting example of such a space is the "semi-infinite flag variety," discussed below in §3.5. This infinite-dimensionality is a technical obstruction when working with sheaves on such spaces, although recent advances have made this much easier. Again the theory is enriched by this complexity, which draws in some beautiful ideas like the "semi-infinite linear algebra" introduced by Tate and developed by many others (cf. [Ra]).

2.3. *Upcoming projects.* In §4 below, I describe in some detail forthcoming joint work with Gurbir Dhillon. Here are brief descriptions of some other planned projects.

Gaitsgory's central functor via degeneration of Whittaker sheaves. In my thesis defense, I conjectured that the dualizing complex on the derived Springer fiber over $0 \in \check{\mathfrak{g}}$ admits a certain Langlands dual description under the equivalence of [ABG]. Namely, it should correspond to the nearby cycles of a one-parameter family of degenerating Whittaker sheaves on the affine flag variety. In joint work with Jize Yu, we will prove this conjecture and use it to give a new interpretation of Gaitsgory's central functor [Ga1], or rather its enhanced version introduced in [AB]. A key ingredient is the spectral decomposition of bi-Whittaker sheaves on the loop group recently established by Chen and Dhillon [ChD].

Semi-infinite flags and the Gelfand-Graev action. In joint work with Kevin Lin, we plan to categorify the Schwartz space on the basic affine space of a local field introduced by Braverman and Kazhdan [BK], and to identify it with sheaves on the semi-infinite flag variety. One possible definition of the latter category is suggested by Gaitsgory in [Ga3] and further studied in my work with Hayash [CaH]; it admits an obvious action of the (finite) Weyl group. The idea is that this Weyl group action should categorify the Gelfand-Graev action on the Schwartz space defined by Braverman-Kazhdan. At least when $G = \mathrm{SL}_2$, we have a good idea how to do this: in that case the Schwartz space is categorified by sheaves on the space of loops into $\mathbb{A}^2$, and the Gelfand-Graev action is given by the symplectic Fourier-Laumon transform.

The tangent complex as a solid module. In my joint work with Hayash [CaH], we encounter many technical difficulties in the deformation theory of spaces of infinite type. One basic issue is that the tangent complex, as defined in [GR], is only well-defined in finite type. In infinite type, only the cotangent complex is available, but one can "formally" dualize it to produce a topologized version of the tangent complex. In favorable cases, the fibers of this object will be Tate vector spaces. The formalism of solid modules introduced by Clausen and Scholze is well-adapted to working with these sorts of topological modules, and in further joint work we hope to streamline and extend some aspects of deformation theory in infinite type using this language.

3. Symmetries of geometric Eisenstein series

This section describes my joint work with Hayash [CaH].

3.1. *Action on compactified Eisenstein series.* The functor $\mathrm{Eis}_!$ admits a natural variant, the "compactified" Eisenstein series functor

$$\mathrm{Eis}_{!*} : \mathrm{D}(\mathrm{Bun}_M) \longrightarrow \mathrm{D}(\mathrm{Bun}_G)$$

introduced in [BG1]. Its key property is that for any *Hecke eigensheaf* $\mathcal{M} \in \mathrm{D}(\mathrm{Bun}_M)$ with eigenvalue $E_{\check{M}}$ (cf. §5.7), the object $\mathrm{Eis}_{!*}(\mathcal{M}) \in \mathrm{D}(\mathrm{Bun}_G)$ is a Hecke eigensheaf whose eigenvalue is the $\check{G}$-local system $(E_{\check{M}})_{\check{G}}$ attached to $E_{\check{M}}$.

Using the adjoint action of $\check{M}$ on $\check{\mathfrak{g}}$, we can attach to $E_{\check{M}}$ the local system of Lie algebras $\check{\mathfrak{g}}_{E_{\check{M}}}$. The associated complex of de Rham cochains $\mathrm{C}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{M}}})$ inherits a structure of DG Lie algebra.

Theorem 3.2 ([CaH] Theorem 1.1.6)**.** *For any eigensheaf* $\mathcal{M} \in \mathrm{D}(\mathrm{Bun}_M)$ *with eigenvalue* $E_{\check{M}}$*, the eigensheaf* $\mathrm{Eis}_{!*}(\mathcal{M})$ *carries a canonical action of the Lie algebra* $\mathrm{C}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{M}}})$.

Example 3.3. If $P = B$ is a Borel subgroup with Cartan quotient $M = T$, then the constant sheaf $\mathbb{C}_{\mathrm{Bun}_T} \in \mathrm{D}(\mathrm{Bun}_T)$ is a Hecke eigensheaf with eigenvalue $E_{\check{T}}^{\mathrm{triv}}$, the trivial $\check{T}$-local system.

In that case, we have

$$\mathrm{Eis}_{!*}(\mathbb{C}_{\mathrm{Bun}_T})\big|_{\mathcal{P}_{\check{G}}^{\mathrm{triv}}} \cong \mathrm{C}^\bullet(\mathrm{QMap}(X, G/B), \mathrm{IC}),$$

the intersection cohomology of Drinfeld's space of quasimaps from X into the flag variety G/B. The relevant Lie algebra is

$$\mathrm{C}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{T}}^{\mathrm{triv}}}) \cong \check{\mathfrak{g}} \otimes \mathrm{C}^\bullet(X).$$

Passing to cohomology in degree 0, we recover by different means the action

$$\check{\mathfrak{g}} \circlearrowright \mathrm{H}^\bullet(\mathrm{QMap}(X, G/B), \mathrm{IC})$$

constructed (in the case $X = \mathbb{P}^1$) in the preprint [FFKM].

Remark 3.4. As related in the introduction to [BG2], Drinfeld observed about twenty years ago that it should be possible to deduce Theorem 3.2 from Theorem 1.2 using general principles of deformation theory. We carry out this deduction in [CaH], but our main proof of Theorem 3.2 is a much more direct local-to-global construction in the spirit of [FFKM].

3.5. *Semi-infinite flags.* The main construction of [FFKM] is quite remarkable: the action of $\check{\mathfrak{g}}$ is specified by explicit cohomological correspondences, and the actions of the positive and negative parts are adjoint with respect to Poincaré duality. Our work was motivated by a desire to better understand this action and its place within the geometric Langlands program.

The philosophy expressed in *loc. cit.* is that the action of $\check{\mathfrak{g}}$ has a local origin in the geometry of the semi-infinite flag variety, a rather mysterious object proposed by Feigin and Frenkel [FF]. This space is highly infinite-dimensional and hence hard to define as an algebro-geometric object. The approach of [FFKM] is to work with certain finite-dimensional models, the so-called Zastava spaces.

But modern techniques allow us to access semi-infinite flags more directly. Let $\mathfrak{L}G$ and $\mathfrak{L}^+G$ denote the algebraic loop group of G and its subgroup of formal arcs. The quotient $\mathrm{Gr}_G := \mathfrak{L}G/\mathfrak{L}^+G$ is the *affine Grassmannian* of G; it is a filtered union of projective varieties and can be though of as a partial flag variety for $\mathfrak{L}G$.

The groups $\mathfrak{L}G$ and $\mathfrak{L}^+G$ admit natural factorization structures relative to X (cf. §5.8); the affine Grassmannian with its factorization structure is sometimes called the *Beilinson-Drinfeld Grassmannian.*

We now consider the subgroup $\mathfrak{L}N_P \cdot \mathfrak{L}^+M \subset \mathfrak{L}G$, where $N_P \subset P$ denotes the unipotent radical. It is an affine analogue of the parabolic subgroup P, distinct from the so-called parahoric subgroup. The orbits of $\mathfrak{L}N_P \cdot \mathfrak{L}^+M$ on Gr_G discretely parameterized, countably infinite in number, and of infinite codimension. If $P \neq G$, then these orbits are also infinite-dimensional, hence the term "semi-infinite."

3.6. *Action on the semi-infinite IC sheaf.* Gaitsgory [Ga3] has introduced an object

$$\mathrm{IC}^{\frac{\infty}{2}} \in \mathrm{D}\text{-}\mathrm{mod}(\mathrm{Gr}_G)^{\mathfrak{L}N_P \cdot \mathfrak{L}^+M}$$

called the *semi-infinite IC sheaf*, which can be thought of as the IC sheaf on the closure of the neutral $\mathfrak{L}N_P \cdot \mathfrak{L}^+M$-orbit (because the orbits are generally infinite-dimensional, this does not have a clear meaning *a priori*). It admits a natural factorization structure relative to that of the Beilinson-Drinfeld Grassmannian.

In our proof of Theorem 3.2, the object $\mathrm{IC}^{\frac{\infty}{2}}$ is the local analogue of the functor $\mathrm{Eis}_{!*}$. The local actor is a certain $\check{G}$-equivariant associative algebra $\mathcal{U}(\check{\mathfrak{g}})$ with compatible factorization structure. It is an analogue of the $\mathbb{E}_3$-enveloping algebra of $\check{\mathfrak{g}}$ from topology, whose construction in the de Rham setting was suggested to us by S. Raskin. In particular, it admits a PBW-type filtration whose associated graded is the commutative algebra $\mathrm{Sym}(\check{\mathfrak{g}}[-2])$.

Theorem 3.7 ([CaH] Theorem 1.2.6)**.** *There is a canonical action of the associative algebra* $\mathcal{U}(\check{\mathfrak{g}})|_{\check{M}}$ *on* $\mathrm{IC}^{\frac{\infty}{2}}$*, compatibly with factorization, relative to the action of* $\mathrm{Rep}(\check{M})$ *on* $\mathrm{D}(\mathrm{Gr}_G)^{\mathfrak{L}N_P\mathfrak{L}^+M}$ *by Hecke functors.*

The main ingredients in the proof are the following.

- The $(\check{M}, \check{G})$-Hecke structure on $\mathrm{IC}^{\frac{\infty}{2}}$ constructed in [Ga2].
- The factorizable derived geometric Satake equivalence, established in [CR].
- A Koszul duality equivalence on the spectral side of derived geometric Satake.

Let us comment further on the third point. The original derived geometric Satake equivalence, proved in [BF], is an equivalence of monoidal categories

$$\text{D-mod}(\mathrm{Gr}_G)^{\mathfrak{L}^+G} \xrightarrow{\sim} \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}(\mathrm{Rep}(\check{G})).$$

Our Koszul duality theorem says that this equivalence can be made compatible with factorization by replacing $\mathrm{Sym}(\check{\mathfrak{g}}[-2])$ with the "chiral $\mathbb{E}_3$-enveloping algebra" $\mathcal{U}(\check{\mathfrak{g}})$.

Having proved Theorem 3.7, we deduce Theorem 3.2 by a straightforward local-to-global argument using chiral homology.

4. Localization at the critical level

The following describes forthcoming joint work with Gurbir Dhillon.

4.1. *Localization and semi-infinite flags* When Feigin-Frenkel proposed the existence of a semi-infinite flag variety, their motivation came from the study of the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$ of affine Kac-Moody representations at critical level. Whereas representations at positive or negative level can be localized onto a version of the affine flag variety, the combinatorics of representations at critical level suggest a rather exotic kind of geometry, in which Schubert cells have both infinite dimension and codimension.

Let 2ρ denote the sum of simple roots of G, which we view as a cocharacter of $\check{T}$. Choose a square root $\Omega_X^{1/2}$ of the canonical line bundle on X, from which we obtain the $\check{T}$-bundle

$$\rho(\Omega_X^1) := 2\rho(\Omega_X^{1/2}).$$

Fix a meromorphic connection ∇ on $\rho(\Omega_X^1)$. In particular, we obtain a $\check{T}$-local system

$$E_{\check{T}} := (\rho(\Omega_X^1), \nabla),$$

defined on a dense open subset $U \subset X$. Using local geometric class field theory, this determines a factorizable character D-module λ on $\mathfrak{L}T$, which is moreover unramified (i.e., trivial on $\mathfrak{L}^+T$) over U. We can then consider the category of invariants

$$\mathrm{D}(\mathfrak{L}G)^{\mathfrak{L}B,\lambda}.$$

The Miura oper construction attaches to the connection ∇ a $\check{G}$-oper on U. Using the Feigin-Frenkel isomorphism, this oper determines a central character χ for $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}$ at each point of X. Moreover, this central character is unramified over U.

Theorem 4.2 (In progress)**.** *The above data determines a localization functor*

$$\mathrm{Loc}_{\lambda}^{\frac{\infty}{2}} : \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi} \longrightarrow \mathrm{D}(\mathfrak{L}G)^{\mathfrak{L}B,\lambda},$$

which is $\mathrm{D}(\mathfrak{L}G)$*-linear and compatible with factorization.*

The idea is that the fibers of a localized module are given by weight spaces in the BRST cohomology, analogously to the fibers of Beilinson-Bernstein localization, which are computed by finite-dimensional Lie algebra homology.

4.3. *A localization equivalence* The functor $\mathrm{Loc}_{\lambda}^{\frac{\infty}{2}}$ has no chance of being an equivalence of categories since, as pointed out by Gaitsgory in [Ga3], the respective categories of $\mathfrak{L}^+G$-invariants are inequivalent. Following his suggestion, we propose the following conjecture.

Let Vac denote the vacuum representation at critical level, which is the unit for the factorization structure on $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-mod. We can form the central quotient Vac_{χ}, which is the unit for the factorization structure on $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi}$ over U. The semi-infinite IC sheaf admits a twisted variant

$$\mathrm{IC}_{\lambda}^{\frac{\infty}{2}} \in \mathrm{D}(\mathfrak{L}G)^{\mathfrak{L}B,\lambda}.$$

Theorem 4.4 (In progress)**.** *There is a canonical isomorphism*

$$\mathrm{Loc}_{\lambda}^{\frac{\infty}{2}}(\mathrm{Vac}_{\chi}) \xrightarrow{\sim} \mathrm{IC}_{\lambda}^{\frac{\infty}{2}}$$

of factorization algebras over U *in* $\mathrm{D}(\mathfrak{L}G)^{\mathfrak{L}B,\lambda}$. *The induced functor*

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi} \longrightarrow \mathrm{IC}^{\frac{\infty}{2}}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}G)^{\mathfrak{L}B,\lambda})$$

is an equivalence, equivariant for $\mathrm{D}(\mathfrak{L}G)$ *and compatible with factorization structures.*

We remark that for $x \in U$, the central character χ is unramified, and hence the category $\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi,x}$ is a fairly well-studied object in representation theory.

On the other hand, the localization equivalence

$$\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\chi,x} \longrightarrow \mathrm{IC}^{\frac{\infty}{2}}\text{-mod}^{\mathrm{fact}}(\mathrm{D}(\mathfrak{L}G)_{\mathfrak{L}B,\lambda})_x$$

at a pole $x \in X \setminus U$ should yield interesting information about representation theory at ramified central characters. We remark that this is an equivalence of factorization *module* categories for the factorization categories over U which appear in Theorem 4.4.

5. Some background and heuristics

This section is intended as a glossary of sorts.

5.1. *Fourier transform.* One can think of the geometric Langlands equivalence $\mathbb{L}_G$ as a kind of categorical nonabelian Fourier transform. In this analogy, Bun_G is the fundamental domain and $\mathrm{LS}_{\check{G}}$ the corresponding space of spectral parameters, while D-mod and $\mathrm{IndCoh}_{\mathrm{Nilp}}$ are sheaf theories that categorify the spaces of distributions appearing in standard Fourier analysis.

5.2. *Langlands dual group.* Complex reductive groups, or equivalently compact Lie groups, are classified by discrete combinatorial objects called root data. The collection of all root data admits a natural involutive symmetry (exchange weights/roots with coweights/coroots). If we apply this involution to the root datum of a given group, the resulting root datum corresponds to the so-called Langlands dual group.

For example, the group $G = \mathrm{GL}_n$ is Langlands self-dual: $\check{G} = \mathrm{GL}_n$. More interestingly, odd orthogonal groups are Langlands dual to symplectic groups: if $G = SO_{2n+1}$, then $\check{G} = \mathrm{Sp}_{2n}$.

5.3. *Analogy with number theory.* To pass from the original arithmetic Langlands program to its geometric version, one consults the dictionary known as "Weil's Rosetta Stone." The number field in the arithmetic theory is replaced by a Riemann surface X. In this analogy, points of X are analogous to primes in a number field, the category $\text{D-mod}(\mathrm{Bun}_G)$ corresponds to the space of unramified automorphic forms, and local systems on X are analogous to everywhere unramified Galois representations.

5.4. *Categorification.* A key feature of the analogy with number theory is that the passage from arithmetic to geometry necessarily involves "categorification." One instance of this principle is that vector spaces (e.g., the space of automorphic forms) are replaced by categories (e.g., $\text{D-mod}(\text{Bun}_G)$). At a more basic level, numbers are replaced by vector spaces, e.g., the geometric analogue of the Whittaker coefficient of an automorphic form is a vector space. As a consequence, functions are replaced by *sheaves*: the former are mappings from points of a geometric object to numbers, while the latter can be thought of as mappings from points to vector spaces.

5.5. *D-modules.* The casual reader can replace all instances of the word "D-module" by "sheaf," since it is a particular flavor of sheaf theory. But at a second glance, the theory of D-modules has some special features. An algebraic D-module is essentially a system of linear partial differential equations with polynomial coefficients. An interesting feature of such equations is that their solutions need not be algebraic: consider $f' = f$.

The passage from D-modules to their sheaves of local solutions can be made into an equivalence after restricting to the so-called *regular holonomic* D-modules: this is a generalization of the classical Riemann-Hilbert correspondence. Not all D-modules are regular (e.g., the exponential D-module whose equation is written above is irregular) or holonomic (e.g., the sheaf of differential operators, corresponding to the empty set of PDEs, is not holonomic). Thus the category of D-modules on a complex variety contains, but is strictly larger than, the category of constructible sheaves.

5.6. *Ind-coherent sheaves.* Ind-coherent sheaves are a modification of quasicoherent sheaves obtained by slightly enlarging the subcategory of compact objects. They only differ from quasicoherent sheaves when we consider unbounded complexes; that is, the difference lies "in cohomological degree $-\infty$."

In fact, one can single out many categories intermediate between quasicoherent and ind-coherent sheaves using so-called singular support conditions (cf. [AG]). These conditions derive their name from the more well-known singular support theory for D-modules.

5.7. *Hecke eigensheaves.* Hecke eigensheaves are the "pure tones" in the Fourier transform analogy. Analogously to the space of automorphic forms, the category $\text{D-mod}(\text{Bun}_G)$ carries a large family of commuting *Hecke operators.* Namely, for any point $x \in X$ and any representation $V \in \text{Rep}(\check{G})$, the so-called geometric Satake transform defines an operator $\text{Sat}_{G,x}(V)$ on $\text{D-mod}(\text{Bun}_G)$.

Fix a $\check{G}$-local system $E_{\check{G}}$ on X. A *Hecke eigensheaf* on Bun_G with eigenvalue $E_{\check{G}}$ is an object $\mathcal{M} \in \text{D-mod}(\text{Bun}_G)$ which transforms in the following way under Hecke operators:

$$\text{Sat}_{G,x}(V) \star \mathcal{M} \cong (V_{E_{\check{G}}})\big|_x \otimes \mathcal{M} \quad (x \in X,\ V \in \text{Rep}(\check{G})).$$

5.8. *Factorization structure.* Roughly speaking, a *factorization space* Y over X assigns to any finite subset $\underline{x} \subset X$ a space $Y_{\underline{x}}$, compatibly in families as $\underline{x}$ varies, together with isomorphisms

$$Y_{\underline{x}} \cong \prod_{x \in \underline{x}} Y_{\underline{x}}.$$

Although it may seem that such a datum is already determined by the spaces Y_x for single points $x \in X$, there is nontrivial information encoded by families in which points collide. One can view Y as fibered over the so-called Ran space, which parameterizes all finite subsets of X.

The linear analogue of a factorization space is a *factorization algebra* A, which assigns to any to any finite subset $\underline{x} \subset X$ a vector space $A_{\underline{x}}$ (compatibly in families), together with isomorphisms

$$A_{\underline{x}} \cong \bigotimes_{x \in \underline{x}} A_x.$$

More precisely, one can view A as a sheaf on the Ran space of X equipped with the additional structure of factorization isomorphisms. Given a sheaf $\mathcal{M}$ on a factorization space Y, one can also speak of a factorization algebra structure on $\mathcal{M}$ relative to the factorization structure on Y.

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