Vertex algebras.

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1 Definition and basic examples of vertex algebras.

Our main reference is Frenkel's book [F]. Another reference is Frenkel—Ben–Zvi's book [FBZ]. There are also notes from MIT seminar on affine Kac-Moody algebras on critical level that followed Frenkel's book [D1, D2].

Definition 1.1. Let V be a vector space. A field on V is an element $A(z)$ of $(\operatorname{End} V)[[z, z^{-1}]]$ such that for any $v \in V$ we have $A(z)v \in V((z))$. That is, if we write $A(z) = \sum A_j z^{-j}$, $A_j v = 0$ for large enough j.

Definition 1.2. Vertex algebra is a collection of data:

1. A vector space V, called the space of states.
2. An element $|0\rangle \in V$, called vacuum vector.
3. An endomorphism $T \colon V \to V$, called translation operator.
4. A map $Y(\cdot, z) \colon V \to \operatorname{End}(V)[[z, z^{-1}]]$ that sends $A \in V$ to a field $Y(A, z) = \sum A_{(n)} z^{-n-1}$.

This data satisfies the following axioms:

1. $Y(|0\rangle, z) = \mathrm{Id}_V$.
2. $Y(A, z)|0\rangle = A + z(\cdots) \in V[[z]]$..
3. $[T, Y(A, z)] = \partial_z Y(A, z)$.
4. $T|0\rangle = 0$.

5. Locality: for any $A, B \in V$ there exists a positive integer N such that $(z-w)^N [Y(A,z), Y(B,w)] = 0$.

V is called graded vector algebra if V is a graded vector space and we have vacuum vector in degree zero, T a map of degree one and $\deg A_{(n)} = \deg A - n - 1$ for a homogeneous element A.

An example of a vertex algebra is a commutative algebra V with derivation T. We define $|0\rangle = 1$ and

$$Y(A,z) = \sum z^n mult(\frac{T^n A}{n!}) = mult(e^{Tz}A),$$

where $mult$ is the operator of multiplication. It is not hard to check that T, $|0\rangle$ and Y define a vertex algebra structure on V. Moreover, we can take $N = 0$ while checking locality, the fields just commute: $[Y(A,z), Y(B,w)] = 0$. Also note that $Y(A,z)$ is a power series with no negative terms. One can check that these conditions are equivalent:

1. $[Y(A,z), Y(B,w)] = 0$.
2. $Y(A,z) \in \operatorname{End} V[[z]]$ for all A.
3. The structure of vertex algebra on V comes from a structure of a commutative algebra on V.

Moreover, the multiplication can be written as $AB = A_{(-1)}B$.

2 Heisenberg, Kac-Moody and Virasoro vertex algebras. Reconstruction theorem.

Now we consider vertex algebras corresponding to Heisenberg and affine Kac-Moody algebras. In all these examples, our algebra $\hat{\mathfrak{a}}$ has basis $\mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$ and $\mathfrak{a}[t] \oplus \mathbb{C} \subset \hat{\mathfrak{a}}$ is a direct sum. Consider $V^\kappa(\mathfrak{a}) = U(\hat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}$, where $\mathfrak{a}[t]$ acts trivially on $\mathbb{C}$ and $\mathbf{1}$ acts as the identity. Define $|0\rangle = 1 \otimes 1$. Define grading by the rule $\deg xt^n = -n$. Let T acts by $-\partial_t$, this is a well-define operation on V^κ.

It remains to describe state-field correspondence. For $a \in \mathfrak{a}$ let $a_n = at^n$. Define $Y(a_{-1}|0\rangle) = \sum a_k z^{-k-1}$. Here by a_k we mean the action of a_k on V^κ. One can think of this formula as follows: we should have $Y(a_{-1}|0\rangle)|0\rangle =$

$a_{-1}|0\rangle + z(...)$, this motivates the constant coefficient. Then $[T, Y(a_{-1}|0\rangle, z)] = \partial_z Y(a_{-1}|0\rangle, z)$ and higher derivatives define the endomorphisms for positive powers of z, and we use the same formula $a_k z^{-k-1}$ for nonnegative k.

In commutative vertex algebras we have $Y(TA, z) = \partial_z Y(A, z)$. One can show this property in general. In Frenkel's book this is proved as a consequence of associativity in vertex algebras that we will discuss below.

Since

$$a_{-k}|0\rangle = \frac{T^{k-1}}{(k-1)!} a_{-1}|0\rangle,$$

we have

$$Y(a_{-k}|0\rangle, z) = \frac{\partial_z^{k-1}}{(k-1)!} Y(a_{-1}|0\rangle, z) = \sum \binom{-l-1}{k} a_l z^{-k-l-1}.$$

What about all the other elements of the basis, $a_{-n_1} a_{-n_2} \cdots a_{-n_k}|0\rangle$? In commutative case, we would just take the product of fields. We cannot do that now, the product of fields is not a well-defined element of $\operatorname{End} V[[z, z^{-1}]]$. But we can take normally ordered product:

$$: A(z)B(z) := A(z)_+ B(z) + B(z) A(z)_-,$$

where $A(z)_+$ has nonnegative powers of z and $A(z)_-$ has negative powers of z. Normal ordering is non-associative operation, for several arguments we just close all brackets on the right: $: A(z)B(z)C(z) :=: A(z) : B(z)C(z) ::$.

Choose basis $J^1, \ldots, J^d$ of the algebra $\mathfrak{a}$ and denote $J^a(z) = \sum J_i^a z^{-i-1}$. So the final formula is

$$Y(J_{n_1}^{a_1} J_{n_2}^{a_2} \cdots J_{n_k}^{a_k}|0\rangle, z) =: \frac{\partial_z^{-n_1-1}}{(-n_1-1)!} J^{a_1}(z) \frac{\partial_z^{-n_2-1}}{(-n_2-1)!} J_{a_2}(z) \cdots \frac{\partial_z^{-n_k-1}}{(-n_k-1)!} J_{a_k}(z),$$

where we take $n_1, \ldots, n_k$ strictly negative.

We gave the data of the vertex algebra on V^κ. Checking the axioms of vertex algebra is relatively straightforward for everything except locality.

Recall that the Lie bracket is $[a_n, b_m] = [a, b]_{n+m} + n\delta_{n+m,0}\kappa(a, b)$, where κ is an invariant bilinear form. It follows that

$$[J^a(z), J^b(z)] = [J^a, J^b](w)\delta(z-w) + \kappa(J^a, J^b)\delta'(z-w)$$

All other fields are constructed from $J^0(z), \ldots, J^d(z)$ using normally ordered products. Now locality axiom follows from Dong's lemma (Lemma

2.2.3 in Frenkel's book): if A, B, C are mutually local, then $:AB:, C$ are also local with respect to each other.

The same reasoning proves the weak reconstruction theorem (Theorem 2.2.4 in Frenkel's book): if we have fields $a^\alpha(z) = \sum \alpha_{(n)} z^{-n-1}$ such that $[T, a^\alpha] = \partial_z a^\alpha(z)$, $T|0\rangle = 0$, $a^\alpha(z)|0\rangle = a^\alpha + z\cdots$, they are mutually local, and the elements $\alpha_{(n_1)} \cdots \alpha_{(n_m)}|0\rangle$ form a basis of V, then there is a vertex algebra structure on V such that $Y(a^\alpha, z) = a^\alpha(z)$ and $T = T$, $|0\rangle = |0\rangle$.

In strong reconstruction theorem we require the elements above to be just spanning set, not necessarily the generators. Moreover, this is unique structure of a vertex algebra on V such that $Y(a^\alpha, z) = a^\alpha(z)$ and $T = T$, $|0\rangle = |0\rangle$.

Virasoro algebra is a Lie algebra with

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}\mathbf{1}.$$

Here L_n corresponds to $-t^{n+1}\partial_t$ in Witt algebra. The elements $\mathbf{1}, L_{-1}, L_0, \cdots$, form a subalgebra of Vir, which has representation $\mathbb{C}_c$: $\mathbf{1}$ acts by c, central charge, L_i act by zero. Let Vir_c be the induced representation. Let $\deg L_n = -n$, $T = L_{-1} = -\partial_t$ and $Y(L_{-2}|0\rangle, z) = \sum L_n z^{-n-2}$, a field of degree -2, as required by axioms of graded vertex algebras.

Using weak reconstruction it is enough to compute

$$[T(z), T(w)] = \frac{c}{12}\delta'''(z-w) + 2T(w)\delta'(z-w) + 2T'(w)\delta(z-w).$$

A *conformal* vertex algebra (also called vertex operator algebra) is a graded vertex algebra with a conformal vector: $Y(\omega, z) = \sum L_n^V z^{-n-2}$, L_n^V satisfy relations of Virasoro, $L_{-1}^V = T$, L_0^V is a grading operator. This gives a vertex algebra homomorphism from Vir to V, we will not define this notion.

3 Basis properties of vertex algebras.

Skew symmetry: for all $A, B \in V$ we have $Y(A, z)B = e^{zT}Y(B, -z)A$.

Associativity: expressions

$$Y(A, z)Y(B, w)C, \qquad Y(B, w)Y(A, z)C, \qquad Y(Y(A, z-w)B, w)C$$

are expansions in $V((z))((w))$, $V((w))((z))$ and $V((w))((z-w))$ of the same element of $V[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$.

Skew-symmetry and associativity are proved in Frenkel's book. In the case of commutative vertex algebra we get $Y(Y(A,z-w)B,w)C = \sum(z-w)^i Y(\frac{T^iA}{i!}B,w)C = \sum(z-w)^i w^j \frac{T^j}{j!}(\frac{T^iA}{i!}B)C$. We have

$$T^j(T^iA\cdot B) = \sum T^{i+k}AT^{j-k}B\frac{j!}{k!(j-k)!}.$$

For fixed $i+k=a$ and $j-k=b$ we get

$$\sum_{j=b}^{a+b}(z-w)^{a+b-j}w^j\frac{1}{(a+b-j)!b!(j-b)!}.$$

Writing $w^j = w^b w^{j-b}$ we use binomial formula to get

$$\frac{z^a w^b}{a!b!}T^aAT^bBC,$$

as we should.

Using associativity we get $Y(A,z)Y(B,w) = Y(Y(A,z-w)B,w) = \sum Y(A_{(n)}B,w)(z-w)^{-n-1}$. This expression is called operator product expansion, they are used a lot in physics.

One of the consequence of associativity is the following formula (formula (2.3-8) from Frenkel's book):

$$[A_{(m)},B_{(k)}] = \sum_{n\geq 0}\binom{m}{n}(A_{(n)}B)_{m+k-n}. \tag{1}$$

We will also need Borcherds identity:

$$(A_{(m)}B)_{(n)}C = \sum_{i\geq 0}(-1)^i\binom{m}{i}\Big(A_{(m-i)}(B_{(n+i)}(C)) - (-1)^m B_{(m+n-i)}(A_{(i)}(C))\Big). \tag{2}$$

C_2-algebra of a vertex algebra V is defined as

$$V/\operatorname{Span}\{a_{(-2)}b \mid a,b\in V\}$$

with operation $\overline{a}\cdot\overline{b} = \overline{a_{(-1)}b}$, compare with the formula for commutative vertex algebra.

This operation is well-defined:

$$a_{(-1)}b_{(-2)}c \sim [a_{(-1)}, b_{-2}]c = \sum_{n\geq 0} \binom{-1}{n} (A_{(n)}B)_{-3-n}C \sim 0,$$

here we used (1). Using (2) we get

$$(a_{-2}b)_{-1}c = \sum_{i\geq 0} (\cdots A_{-2-i} \cdots B_{-3-i} \cdots) \sim 0.$$

Using Borcherds identity again for $m = n = -1$ we see that the only term not equivalent to zero is $A_{-1}(B_{-1}C)$, this gives

$$(A_{-1}B)_{-1}C = A_{-1}(B_{-1}C).$$

Using (1) we see that this operation is commutative.

It can also be checked that $a_{(0)}b$ gives a Poisson bracket on C_2 algebra.

Zhu algebra of a graded vertex algebra V is defined as

$$Zhu(V) = V/(V \circ V).$$

Here

$$a \circ b = \sum_{i\geq 0} \binom{\deg a}{i} a_{(i-2)}b$$

for homogeneous elements. We define operation on $Zhu(V)$ using

$$a * b = \sum_{i\geq 0} \binom{\deg a}{i} a_{(i-1)}b.$$

Let $o(a) = a_{(\deg a - 1)}$. Then $o(a)$ preserves graded components of any graded V-module (for a V-module M by $a_{(m)}$ we mean the corresponding field on M.) The $\overline{a} \mapsto o(a)$ gives a representation of $Zhu(V)$ on M_{top}, the lowest degree component of M. This provides an equivalence between irreducible positive energy representations of V and simple $Zhu(V)$-modules.

V is called *chiralization* of A if $Zhu(V) = A$.

Positive grading on V gives filtration on V, hence filtration on $Zhu(V)$. It can be checked that C_2-algebra surjects onto $\operatorname{gr}\mathrm{Zhu}(V)$. Moreover, comparing formulas we see that this is a map of Poisson algebras.

Example: one can show that $V^{\kappa}(\mathfrak{a})$ is a chiralization of $U(\mathfrak{a})$.

References

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