Semi Infinite Cohomology

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1 Idea

For a semi-simple Lie algebra $\mathfrak{g}$, the affine Lie algebra $\hat{\mathfrak{g}}_\kappa$ is a central extension of its loop space. Its vacuum representation $\mathbb{V}_{\mathfrak{g},\kappa}$ has a structure of a vertex algebra.

Now in the semi-simple case, we saw a definition of the corresponding finite W-algebra

$$W(\mathfrak{g}) = (U(\mathfrak{g})/U(\mathfrak{g})(x - \chi(x)))^{\mathfrak{n}} = C^*_{\mathrm{Lie}}\left(\mathfrak{n}, C^{\mathrm{Lie}}_*(\mathfrak{n} \otimes -\chi) \otimes \chi\right)$$

It is an associative algebra with a filtration who's associated graded is the Poisson algebra of functions on $f + \mathfrak{b}/N$.

This is an associative algebra describing "$\mathfrak{g}$-representations for which the $\mathfrak{n}$ action is given by a character χ.

We want an affine analogue. The main idea to have in mind is that our semi-classical (or Poisson) objects are jet spaces, and their quantizations are loop spaces. We view the polar part as acting by derivations using the duality between $k((t))/k[[t]]$ and $k[[t]]dt$ given by the residue pairing. In particular, they are non-commutative. As we saw last week, a convenient tool for working with objects like that is the language of vertex / chiral / factorization algebras.

In particular, the Kac-Moody vertex algebra $\mathbb{V}_{\mathfrak{g},\kappa}$ has as an underlying vector space the level κ vacuum representation of $\hat{\mathfrak{g}}$, and there's a canonical isomorphism

$$\mathbb{V}_{\mathfrak{g},\kappa}\text{-mod} \simeq \hat{\mathfrak{g}}\text{-mod}_\kappa$$

In comclusion, the object we're interested in is of the form $\mathcal{W}_{\mathfrak{g},\kappa} \sim C^*_{\mathrm{Lie}}(\mathfrak{n}((t)), \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi)$. We will then want to prove this complex is concentrated in degree 0 and the result inherits a structure of a vertex algebra. We'll then want a filtration on $\mathcal{W}_{\mathfrak{g},\kappa}$ with an associated graded being a Poisson vertex algebra who's underlying commutative algebra is that of functions on $f + \mathfrak{b}[[t]]/N[[t]]$.

2 Tate Lie Algebras

The problem here is that we need to make sense of the expression $C^*_{\mathrm{Lie}}(\mathfrak{n}((t)), -)$: $\mathfrak{n}((t))$ is a colimit of a limit of finite dimensional Lie algebras. We can extend cohomology to a continuous functor out of ind-finite Lie algebras, and homology for pro-finite. Here we have a Tate vector space: an ind-pro-finite Lie algebra.

The formalism of semi-infinite cohomology is supposed to deal with those problems. It should be thought of as a combination of cohomology for the colimit direction $\mathfrak{n}[[t]]$ and homology for the limit direction $\mathfrak{n}((t))/\mathfrak{n}[[t]]$.

First, we need to make sense of the input category itself $\mathfrak{n}((t))$. There is a more general definition that works for $\mathfrak{g}((t))$, but here there's a shorter one: Let $\mathfrak{n}_i = \mathrm{Ad}_{t^{-i\check{\rho}}}\, \mathfrak{n}[[t]]$. Then $\mathfrak{n}_i = \lim_j \mathrm{Ad}_{t^{-i\check{\rho}}}\, \mathfrak{n}[[t]]/t^j\mathfrak{n}[[t]] =: \lim_j \mathfrak{n}_i^j$ is a profinite Lie algebra, and we define

$$\mathfrak{n}_i\text{-mod} = \mathrm{colim}_j\, \mathfrak{n}_i^j\text{-mod}$$

$\mathfrak{n}((t)) = \bigcup_i \mathfrak{n}_i$ is a union of open profinite Lie algebras, and we define

$$\mathfrak{n}((t))\text{-mod} = \lim_i \mathfrak{n}_i\text{-mod}$$

By passing to left adjoints $\mathrm{Ind}_{\mathfrak{n}_{i_1}}^{\mathfrak{n}_{i_2}}$ we can also write

$$\mathfrak{n}((t))\text{-mod} = \mathrm{colim}_i\, \mathfrak{n}_i\text{-mod}$$

Now for the definition of semi-infinite cohomology, we have functors

$$C^*_{\mathrm{Lie}}(\mathfrak{n}_i; (-) \otimes \det(\mathfrak{n}_i/\mathfrak{n}_0)) : \mathfrak{n}_i\text{-mod} \to \mathrm{Vect}$$

Define

$$C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], -) = \mathrm{colim}\, C^*_{\mathrm{Lie}}(\mathfrak{n}_i; (-) \otimes \det(\mathfrak{n}_i/\mathfrak{n}_0))$$

3 Vertex Algebra Structure

Assuming $M \in \mathfrak{n}((t))\text{-mod}^{\heartsuit}$, we can compute semi-infinite cohomology using the usual resolution. That is, $C^\bullet(\mathfrak{n}[[t]], M) \simeq M \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^*$. Then for each i, we have

$$\begin{aligned} M \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^* \otimes \det(\mathfrak{n}_i/\mathfrak{n}_0) &\simeq M \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^* \otimes \bigwedge^\bullet (\mathfrak{n}_i/\mathfrak{n}_0)^* \otimes \det(\mathfrak{n}_i/\mathfrak{n}_0) \\ &\simeq M \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^* \otimes \bigwedge^\bullet (\mathfrak{n}_i/\mathfrak{n}_0) \end{aligned}$$

Taking colimit, we get the usual complex computing semi-infinite cohomology

$$C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], M) \simeq M \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^* \otimes \bigwedge^\bullet \mathfrak{n}((t))/\mathfrak{n}[[t]]$$

Let χ be the character

$$\mathfrak{n}((t)) \to \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]((t)) \xrightarrow{\Sigma} \mathbb{C}((t)) \xrightarrow{\mathrm{Res}} \mathbb{C}$$

In particular, in our case we get the complex

$$C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi) = \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi \otimes \bigwedge^\bullet \mathfrak{n}[[t]]^* \otimes \bigwedge^\bullet \mathfrak{n}((t))/\mathfrak{n}[[t]]$$

Theorem 1. $C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi)$ *is concentrated in cohomological degree zero.*

Denote

$$\mathcal{W}_{\mathfrak{g},\kappa} \coloneqq H^0 C^{\frac{\infty}{2}} \left(\mathfrak{n}\left((t)\right), \mathfrak{n}[[t]], \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi\right)$$

Theorem 2. $\mathcal{W}_{\mathfrak{g},\kappa}$ *has a structure of a vertex algebra*

The proof idea is similar to the finite case: We give the semi-infinite complex the structure of a vertex algebra, then show that the differential is given by the action of a specific element, hence respects the vertex algebra structure. In particular, the cohomology is a vertex algebra.

We already have a vertex algebra structure on $\mathbb{V}_{\mathfrak{g},\kappa}$. It remains to give a vertex algebra structure for the other component, and then take the tensor vertex algebra.

Choose a basis of root vectors $\{e_\alpha\}_{\alpha\in\Delta_+}$. Then $\bigwedge^\bullet \mathfrak{n}[[t]]^*$ is generated by elements of the form $\psi^*_{\alpha,n} = e^*_\alpha \otimes t^n$ for $n \geq 0$, and $\bigwedge \mathfrak{n}\left((t)\right)/\mathfrak{n}[[t]]$ by elements of the form $\psi_{\alpha,n} = e_\alpha \otimes t^{-n}$ for $n > 0$.

Write $\bigwedge_{\mathfrak{n}} = \bigwedge \mathfrak{n}[[t]]^* \otimes \bigwedge \mathfrak{n}\left((t)\right)/[[t]]$.

Define fields

$$\psi_\alpha(z) = Y\left(\psi_{\alpha,-1}, z\right) = \sum_{n\in\mathbb{Z}} \psi_{\alpha,n} z^{-n-1}, \psi^*_\alpha(z) = Y\left(\psi^*_{\alpha,-1}, z\right) = \sum_{n\in\mathbb{Z}} \psi^*_{\alpha,n} z^{-n}$$

where the $\psi_{\alpha,n}, \psi^*_{\alpha,n}$ actions are given by

$$[\psi_{\alpha,n}, \psi_{\beta,m}] = [\psi^*_{\alpha,n}, \psi^*_{\beta,m}], [\psi_{\alpha,n}, \psi^*_{\beta,m}] = \delta_{\alpha,\beta}\delta_{m,-n} \tag{3.1}$$

Define a translation operator

$$T\left|0\right\rangle = 0, [T, \psi_{n,\alpha}] = -n\psi_{-n-1,\alpha}, [T, \psi^*_{\alpha,n}] = -\left(n-1\right)\psi^*_{n-1,\alpha}$$

Finally, define a $\mathbb{Z}_+$-grading is given by $\deg \psi_{n,\alpha} = \deg \psi^*_{n,\alpha} = -n$.

The reconstruction theorem for vertex algebras says in order to describe a vertex algebra structure on a vector space, it is enough to describe the fields corresponding to a "PBW basis". More precisely, if we have a collection of vectors $v_1, \ldots, v_n$ and fields $v_i(z) = \sum v_{i,n} z^{-n-1}$ satisfying locality etc., such that the coefficients $v_{i,n}$ give a PBW basis for V, then there is a unique vertex algebra structure on V extending $v_i(z)$. For example:

$$Y\left(v_{i,-1}v_{j,-1}, z\right) =: v_i(z)\, v_j(z) :, Y\left(v^2_{i,-1}, z\right) = \partial_z v_i(z)$$

Lemma 1. $\left(\bigwedge_{\mathfrak{n}}, Y(-,z), T, \left|0\right\rangle\right)$ *is a vertex operator algebra.*

By the reconstruction theorem, we only need to verify locality for $\psi_\alpha(z), \psi^*_\beta(z)$, i.e. that the commutator of any two of those fields is supported on the diagonal. Indeed, by the commutation relations in 3.1, we have

$$[\psi_\alpha(z), \psi_\beta(z)] = [\psi^*_\alpha(z), \psi^*_\beta(z)] = 0, [\psi_\alpha(z), \psi^*_\beta(z)] = \delta_{\alpha,\beta}\delta(z-w)$$

4 Filtrations and Poisson Vertex Algebras

Recall: A commutative vertex algebra is one in which all commutators vanish. The category of commutative vertex algebras is equivalent to that of differential commutative algebras: For (R, T) a differential algebra, define

$$Y(a, z) = \exp(zT)\, a$$

Example 1. *Let R be a commutative algebra. Then JR is a commutative differential algebra, with differential given by the usual derivative of polynomials.*

Definition 1. *A Poisson vertex algebra is a commutative vertex algebra $(V, |0\rangle, T, Y)$ equipped with an additional operation*

$$\{-,-\} : V \otimes V \to z^{-1}V[[z^{-1}]]$$

satisfying axioms similar to that of vertex operations, and all Fourier coefficients of $\{v, w\} = \sum_{n\geq 0} a_n z^{-n-1}$ are derivations of the commutative product.

Example 2. *Let $(R, \{-,-\})$ be a Poisson algebra. Since JR is freely generated by R as a differential algebra, JR has a unique structure of a vertex Poisson algebra extending $\{-,-\}$.*

Definition 2. *An increasing (good) filtration on a vertex algebra V is a filtration $F^\bullet V$ on V such that*

$$F^p V_{(n)} F^q V \subset F^{p+q} V$$

for each n and

$$F^p V_{(n)} F^q V \subset F^{p+q-1} V$$

for $n \geq 0$.

Example 3. *(Li's increasing filtration) Let V be a conformal vertex algebra, so that it has a decomposition into eigenspaces of $L_0 = x\partial x$:*

$$V = \bigoplus V_\Delta$$

Let

$$F^p V = \operatorname{span}\{a^r_{(-n_r-1)} \cdots a^1_{(-n_1-1)} |0\rangle : n_i \geq 0, \sum \Delta_{a^i} \leq p\}$$

Then $F^\bullet V$ is a good filtration on V.

Theorem 3. *For a filtered vertex algebra $F^\bullet V$, the associated graded $\operatorname{gr}^\bullet V$ is naturally a vertex Poisson algebra.*

Example 4. *In the case of $\mathbb{V}_{\mathfrak{g},\kappa}$, Li's filtration agrees with the usual PBW filtration, and* $\operatorname{gr}^\bullet \mathbb{V}_{\mathfrak{g},\kappa} \simeq \mathbb{C}[J\mathfrak{g}^*]$.

5 BRST reduction

Classical BRST reduction realizes the construction

$$W_{\mathfrak{g}}^{\text{fin}} = C^* \left(\mathfrak{n}, C_* \left(\mathfrak{n}, U(\mathfrak{g}) \otimes -\chi\right) \otimes \chi\right)$$

as the cohomology of a single complex, who's differential is given by the adjoint action of a certain element. Explicitly, we start with the double resolution $C(\mathfrak{g}) = U(\mathfrak{g}) \otimes \bigwedge \mathfrak{n}^* \otimes \bigwedge \mathfrak{n}$. We can describe the $\mathfrak{n}$ module structure on $\bigwedge \mathfrak{n}^* \otimes \bigwedge \mathfrak{n}$ through identifying the last with the underlying vector space for the Clifford algebra $\mathrm{Cl}(\mathfrak{n} \oplus \mathfrak{n}^*)$ associated with the evaluation pairing. We then have a Lie algebra homomorphism $\mathfrak{n} \to \mathrm{Cl}(\mathfrak{n} \oplus \mathfrak{n}^*)$ given by

$$\rho : x_\alpha \mapsto \sum [x_\alpha, x_\beta] x_\beta^*$$

which gives the representation. The $\mathfrak{n}$-algebra structure on the complex $C(\mathfrak{g})$ is then given by

$$\theta(x) = (\mu^*(x) - \chi(x)) \otimes 1 + 1 \otimes \rho(x)$$

The last can be written as the adjoint action of the element

$$Q = \sum_\alpha \theta(x_\alpha) \otimes x_\alpha^*$$

Using the fact that Q is odd, we get $\operatorname{ad} Q^2 = 0$. In particular, we get $(C(\mathfrak{g}), \operatorname{ad} Q)$ is a complex computing finite W-algebras. Finally, with respect to the Kazhdan grading $\operatorname{ad} Q$ will have degree zero, and so the associated graded of this complex will compute the Poisson algebra $\mathbb{C}[\mathcal{S}]$. In particular, since we know that the associated graded is concentrated in degree zero, we get the same result for the quantized complex (using a spectral sequence argument).

To generalize that to the affine case, we simply take the corresponding fields: Let

$$Q(z) = \sum_\alpha \theta(\psi_\alpha(z)) \otimes \psi_\alpha^*(z) = \sum_n Q_{(n)} z^{-n-1}$$

Its residue is $Q_{(0)}$, and the complex $\left(\mathbb{V}_{\mathfrak{g},\kappa} \otimes \bigwedge_{\mathfrak{n}}, \operatorname{ad} Q_{(0)}\right)$ computes semi-infinite cohomology. Introduce the Kazhdan grading:

$$\deg^{KK} x_{\alpha,n} = \deg^{KK} \psi_{\alpha,n} = \alpha(\check{\rho}) - n, \deg^{KK} \psi_{\alpha,n}^* = -\alpha(\check{\rho}) - n$$

We get another filtration on the vertex algebra $\mathbb{V}_{\mathfrak{g},\kappa}$, and with respect to this filtration the operator $Q_{(0)}$ has degree zero. However, there is a price: The new filtration is not bounded below.

6 Computation of the Semi-infinite Complex

Consider first semi-infinite cohomology applied to the associated graded.

Proposition 1. $C^{\frac{\infty}{2}}(\mathfrak{n}((t)), \mathfrak{n}[[t]], \operatorname{gr} \mathbb{V}_{\mathfrak{g},\kappa} \otimes \chi) \simeq \mathbb{C}[J\mathcal{S}]$.

Proof. Just as in the finite case, we first take homology in the $\mathfrak{n}((t))/\mathfrak{n}[[t]]$-direction. We have a moment map $J\mu : J\mathfrak{g}^* \to J\mathfrak{n}^*$ and Lie algebra homology here is the restriction to $J\mu^{-1}(\chi)$. Since it is defined by a regular sequence, it is concentrated in cohomological degree 0. Then we have a decomposition

$$J\mu^{-1}(\chi) \simeq JN \times J\mathcal{S}$$

from which we deduce that the cohomology

$$C^*\left(\mathfrak{n}[[t]], \mathbb{C}[J\mu^{-1}(\chi)]\right) \simeq C^*(\mathfrak{n}[[t]], N[[t]]) \otimes \mathbb{C}[J\mathcal{S}] \simeq \mathbb{C}[J\mathcal{S}]$$

is concentrated in cohomological degree 0. □

We would then want to use that to compute the semi-infinite cohomology of $\mathbb{V}_{\mathfrak{g},\kappa}$ itself. However, the Kazhdan filtration is not bounded below, and so the corresponding spectral sequence may not converge. The standard solution to that is to find a quasi-isomorphic subcomplex which is bounded below.

Here's the general idea: We can extend the Lie algebra map $\theta : \mathfrak{n} \to C(\mathfrak{g})$ to a map $\tilde{\theta} : \mathfrak{g} \to C(\mathfrak{g})$ by the same formula, and its restriction to $\mathfrak{b}^-$ is a Lie algebra homomorphism. We can do the same in the affine case, and the result gives morphisms of vertex algebras

$$\mathbb{V}_{\mathfrak{b}^-,\kappa_\mathfrak{b}} \otimes \mathbb{V}_\mathfrak{n} \to \mathbb{V}_{\mathfrak{g},\kappa} \otimes \bigwedge\nolimits_\mathfrak{n}$$

We decompose $\mathbb{V}_{\mathfrak{g},\kappa} \otimes \bigwedge_\mathfrak{n}$ into two complexes, one generated by the image of $\mathfrak{n}$ and $\psi_\alpha(z)$ and the other by $\mathfrak{b}^-$ and ψ^*_α. One then shows that the first complex has cohomology $\mathbb{C}$. Elements of the second complex only have positive KK degree, and so the spectral sequence converges and we're done.

7 Zhu's Algebra

Finally, one can recover the finite case from the affine one using Zhu's algebra:

Proposition 2. $\mathrm{Zhu}(\mathcal{W}_{\mathfrak{g},\kappa}) = \mathcal{W}^{\mathrm{fin}}_\mathfrak{g}$.

Proof. We already know $\mathrm{Zhu}(\mathbb{V}_{\mathfrak{g},\kappa}) \simeq U(\mathfrak{g})$ and $\mathrm{Zhu}(\bigwedge_\mathfrak{n}) \simeq \mathrm{Cl}(\mathfrak{n} \oplus \mathfrak{n}^*)$, and so the Zhu algebra of the vertex algebra computing semi-infinite cohomology is the Poisson algebra computing finite W-algebras. Furthermore the operator $Q_{(0)}$ is compatible with the operation defining Zhu's algebra, and so commutes with taking cohomology. □

To summarize, we have the following diagrams:

$$\begin{array}{ccc} \mathcal{W}_{\mathfrak{g},\kappa} & \xrightarrow{\mathrm{gr}} & \mathbb{C}[J\mathcal{S}] \\ \downarrow{\scriptstyle \mathrm{Zhu}} & & \downarrow{\scriptstyle \mathrm{Zhu}} \\ \mathcal{W}^{\mathrm{fin}}_{\mathfrak{g},\kappa} & \xrightarrow{\mathrm{gr}} & \mathbb{C}[\mathcal{S}] \end{array}$$

analogous to the diagram

$$\begin{array}{ccc} \mathbb{V}_{\mathfrak{g},\kappa} & \xrightarrow{\text{gr}} & \mathbb{C}[J\mathfrak{g}^*] \\ \downarrow{\scriptstyle \text{Zhu}} & & \downarrow{\scriptstyle \text{Zhu}} \\ U(\mathfrak{g}) & \xrightarrow{\text{gr}} & \mathbb{C}[\mathfrak{g}^*] \end{array}$$