BUN G SEMINAR 6 NOVEMBER 2024

ANONYMOUS SCRIBE

1. Recollections on the BRST complex

1.1. We recall that the BRST complex

$$\mathcal{C}^{\frac{\infty}{2},\bullet}(\mathcal{L}\mathfrak{n},\chi;\mathbf{V}_\kappa) = \big(\mathbf{V}_\kappa \otimes \Lambda^\bullet(\mathcal{L}\mathfrak{n}/\mathcal{L}^+\mathfrak{n}) \otimes \Lambda^\bullet(\mathcal{L}\mathfrak{n}^\vee/\mathcal{L}^+\mathfrak{n}^\vee dt), d\big)$$

is a chain complex of **Z**-graded vertex algebras. Let us review the meaning of the terms appearing in this expression.

1.2. The first term $\mathbf{V}_\kappa$ denotes the vacumn module at level κ. It is naturally a vertex algebra generated by the fields[1]

$$J^\alpha(z) = \Sigma\, J^\alpha_n z^{-n-1} \qquad J^\alpha_n = J^\alpha \otimes t^n.$$

1.3. The second term is an instance of a more general construction applied to the finite-dimensional Lie algebra $\mathfrak{n}$.

First recall that one associates a Clifford algebra to any quadratic space. In particular, taking $\mathfrak{n} \oplus \mathfrak{n}^\vee$ with its quadratic form

$$q(\nu \oplus \xi) = 2\xi(\nu),$$

we obtain a Clifford algebra $\mathcal{C}(\mathfrak{n} \oplus \mathfrak{n}^\vee)$. Note that this only uses the vector space structure on $\mathfrak{n}$. However, when $\mathfrak{n}$ is a Lie algebra, the map

$$\rho : \mathfrak{n} \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{n}) \simeq \mathfrak{n} \otimes \mathfrak{n}^\vee \xrightarrow{\mathrm{mult}} \mathcal{C}(\mathfrak{n} \oplus \mathfrak{n}^\vee)$$

is a Lie algebra homomorphism.

Now consider the affine version of the Clifford algebra

$$\mathcal{C}(\mathcal{L}\mathfrak{n} \oplus \mathcal{L}\mathfrak{n}^\vee dt).$$

It admits a Fock module

$$\Lambda^\bullet(\mathcal{L}\mathfrak{n}/\mathcal{L}^+\mathfrak{n}) \otimes \Lambda^\bullet(\mathcal{L}\mathfrak{n}^\vee/\mathcal{L}^+\mathfrak{n}^\vee dt) \tag{1}$$

where the generators

$$\psi_{\alpha,n} = e_\alpha \otimes t^n \qquad \psi^\star_{\alpha,n} = e^\vee_\alpha \otimes t^{n-1}dt$$

[1] Since $\mathbf{V}_\kappa$ is representation of the affine Lie algebra, we may view $J^\alpha \otimes t^n$ as defining an endomorphism of $\mathbf{V}_\kappa$. It is via this interpretation that it makes sense to interpret the expression for $J^\alpha(z)$ as a field.

act by wedge product. The Fock module (1) is naturally a vertex algebra generated by the fields

$$\psi_\alpha(z) = \sum_n e_{\alpha,n} z^{-n-1} \qquad \psi^\star_\alpha(z) = \sum_n e^\vee_{\alpha,n} z^{-n-1}$$

Note that the vectors

$$\psi_{\alpha,n}|0\rangle \qquad \psi^\star_{\alpha,n}|0\rangle$$

are nonzero only for $n < 0$ and $n \leqslant 0$, respectively.

Let us record for future use that the map ρ also has a chiral analogue. Namely, for every simple root α, we obtain a field

$$\rho(\psi_\alpha(z))$$

1.4. The tensor product

$$\mathbf{V}_\kappa \otimes \Lambda^\bullet(\mathcal{L}\mathfrak{n}/\mathcal{L}^+\mathfrak{n}) \otimes \Lambda^\bullet(\mathcal{L}\mathfrak{n}^\vee/\mathcal{L}^+\mathfrak{n}^\vee dt) \tag{2}$$

acquires a vertex algebra structure from its two factors.

We place $\mathcal{L}\mathfrak{n}/\mathcal{L}^+\mathfrak{n}$ in cohomological degree -1 and $\mathcal{L}\mathfrak{n}^\vee/\mathcal{L}^+\mathfrak{n}^\vee dt$ in cohomological degree 1.

We give (2) its analogue of the Kac–Kazhdan grading, where

$$\deg(J^\alpha_n|0\rangle) = -n - \rho^\vee(\alpha)$$

on the first factor and

$$\deg(\psi_{\alpha,n}|0\rangle) = -n - \rho^\vee(\alpha) \qquad \deg(\psi^\star_{\alpha,n}|0\rangle) = -n + \rho^\vee(\alpha)$$

on the second factor.

1.5. It remains to describe the BRST differential, which takes the form

$$d = d_{st} + \chi.$$

Each term is in fact the residue of a vertex operator:

$$d_{st} = Q_{(0)} \qquad \chi = \chi_{(0)}$$

Namely, we define

$$Q(z) = \sum_\alpha e_\alpha(z) \otimes \psi^\star_\alpha(z) + \sum_\alpha \mathbf{1} \otimes \rho(\psi_\alpha(z)) \cdot \psi^\star_\alpha(z)$$

where the sum is taken over all positive roots. In this formula $\rho(\psi_\alpha(z))$ is interpreted as a field acting on the entire Fock module factor (1). Note that $Q_{(0)}$ indeed has cohomological degree 1. It has weight 0.

We define

$$\chi(z) = \mathbf{1} \otimes \sum_\alpha \psi^\star_\alpha(z)$$

where α ranges over the *simple* roots. Note that $\chi_{(-1)}$ indeed has cohomological degree 1. It has weight 1.

2. The subcomplex

2.0.1. In the previous talk, a computation of the $\mathcal{W}$-algebra was performed by introducing the subcomplex

$$\mathcal{C}_{\kappa}^{-,\bullet} \hookrightarrow \mathcal{C}^{\frac{\infty}{2},\bullet}(\mathcal{L}\mathfrak{n}, \chi; \mathbf{V}_{\kappa}) \tag{3}$$

generated by $\mathfrak{b}^{-}$ and $\psi_{\alpha}^{\star}$, in a certain sense.

To be more precise, $\mathcal{C}_{\kappa}^{-,\bullet}$ is the subcomplex is generated by elements of the form

$$\tilde{J}_{(i)}^{\alpha} \cdots \psi_{\beta,(j)}^{\star} \cdots |0\rangle \tag{4}$$

where

$$\tilde{J}^{\alpha}(z) = J^{\alpha}(z) \otimes \mathbf{1} + \mathbf{1} \otimes \rho(\psi_{\alpha}(z))$$

as α ranges over the non-positive roots, and β ranges over the positive roots.

The key feature of this subcomplex was:

2.0.2 Proposition. The inclusion (3) is a quasi-isomorphism.

We used this result to show that the semi-infinite cohomology

$$H^{\frac{\infty}{2},\bullet}(\mathcal{L}\mathfrak{n}, \chi; \mathbf{V}_{\kappa})$$

is concentrated in degree zero at any level κ.

We are going to describe another way of deducing this result from Proposition 2.0.2, at least at generic[2] levels. The idea is to construct a spreading out of $\mathcal{C}_{\kappa}^{-,\bullet}$ over $k[\kappa^{-1}]$ and consider the limiting case $\kappa \to \infty$.

First let

$$\mathcal{C}_{\kappa,naive}^{-,\bullet}$$

denote the $k[\kappa^{-1}]$-module freely generated by the symbols (4). The formulas defining d_{st} and χ make sense over $k[\kappa^{-1}]$, and make $\mathcal{C}_{\kappa,naive}^{-,\bullet}$ into a graded chain complex of $k[\kappa^{-1}]$-modules.

Now consider the submodule

$$\mathcal{C}_{\kappa}^{-,\bullet} \subset \mathcal{C}_{\kappa,naive}^{-,\bullet}$$

generated by the elements

$$\frac{1}{\kappa^{n}} \cdot \tilde{J}_{(i)}^{\alpha} \cdots \psi_{\beta,(j)}^{\star} \cdots |0\rangle$$

[2]The meaning of *generic* for us is cocountably many.

where n is the number of $\tilde{J}^{\alpha}$ terms appearing on the left. This submodule is stable under d_{st} and $\kappa \cdot \chi$, so we give it the differential

$$d = d_{st} + \kappa \cdot \chi.$$

The resulting complex is the spreading out we wish to study.

2.1. Consider the specialization of $\mathcal{C}_{\kappa}^{-,\bullet}$ to $\kappa^{-1} = 0$. The $\psi^{\star}$ terms form an exterior algebra at any level. However, at $\kappa^{-1} = 0$, the $\tilde{J}$ terms commute with each other. Therefore, we obtain a canonical identification

$$\mathcal{C}_{\infty}^{-,\bullet} \simeq \left(\mathrm{Sym}(\mathcal{L}\mathfrak{b}^{-}/\mathcal{L}^{+}\mathfrak{b}^{-}) \otimes \Lambda^{\bullet}(\mathcal{L}^{+}\mathfrak{n})^{\vee}, d_{\infty}\right) \simeq \left(\mathcal{O}_{\mathcal{L}^{+}\mathfrak{b}\,dt} \otimes \Lambda^{\bullet}(\mathcal{L}^{+}\mathfrak{n})^{\vee}, d_{\infty}\right).$$

Here we have used the Killing form to identify

$$(\mathcal{L}\mathfrak{b}^{-}/\mathcal{L}^{+}\mathfrak{b}^{-})^{\vee} \simeq \mathcal{L}^{+}(\mathfrak{b}^{-,\vee})\,dt \simeq \mathcal{L}^{+}\mathfrak{b}\,dt.$$

Let us explain the relation of this complex to the moduli space of opers.

2.2. In what follows we will assume that G is semisimple and adjoint. Recall that a G-oper on the formal disc is a G-local system along with a reduction of the underlying bundle to B satisfying a certain transversality condition.

To state the oper condition, suppose that the underlying B-bundle has been trivialized so that the structure of G-local system is given by a connection one-form

$$\nabla = d + A\,dt \qquad A \in \mathcal{L}^{+}\mathfrak{g}.$$

Consider the principal grading on $\mathfrak{g}$ and form the projection

$$\mathfrak{g}_{\geqslant -1} \to \mathfrak{g}_{-1}$$

from the subspace of elements of degree $\geqslant -1$ to the space of degree -1 elements. The oper condition says that

$$A(t) \in \mathcal{L}^{+}\mathfrak{g}^{\circ}_{\geqslant -1},$$

where

$$\mathfrak{g}^{\circ}_{\geqslant -1} \subset \mathfrak{g}_{\geqslant -1}$$

is the open subset of elements whose projection in $\mathfrak{g}_{-1}$ lands in the open T-orbit[3]

$$\mathfrak{g}^{\circ}_{-1} \subset \mathfrak{g}_{-1}.$$

Therefore, we obtain

$$\mathrm{Op}_{G}(D) \simeq (\mathcal{L}^{+}\mathfrak{g}^{\circ}_{\geqslant -1}\,dt)/\mathcal{L}^{+}B.$$

[3] More explicitly, $\mathfrak{g}_{-1}$ is a sum of negative root spaces indexed by simple roots; in these terms, $\mathfrak{g}^{\circ}_{-1}$ is the subspace of elements that are nonzero on each component.

Now consider the space of connections of the form

$$\nabla = d + f_{-1} + A\, dt \qquad A \in \mathcal{L}^+\mathfrak{b}.$$

The stabilizer of any such connection is $\mathcal{L}^+N \subset \mathcal{L}^+B$ and $\mathcal{L}^+T$ acts transitively on $\mathcal{L}^+\mathfrak{g}^\circ_{-1}$, so we obtain an isomorphism

$$(d + f_{-1} + \mathcal{L}^+\mathfrak{b}\, dt)/\mathcal{L}^+N \to (d + \mathcal{L}^+\mathfrak{g}^\circ_{\geqslant -1}\, dt)/\mathcal{L}^+B.$$

In particular,

$$\mathcal{O}_{\mathrm{Op}_G(D)} \simeq (\mathcal{O}_{\mathcal{L}^+\mathfrak{b}dt} \otimes \Lambda^\bullet(\mathcal{L}^+\mathfrak{n})^\vee, d_{\mathrm{Chev}}).$$

2.2.1 Proposition. We have

$$d_\infty = d_{\mathrm{Chev}}.$$

2.3. This gives the Feigin–Frenkel isomorphism at $\kappa = \infty$.

Since $C^{-,\bullet}_\kappa$ is graded and each graded piece is finite-dimensional, the same result holds for generic κ. (As there are only countably many cohomology groups, this produces a Feigin–Frenkel isomorphism after specialization at all but countably many values of κ.)

2.4. A nice comment from Justin. In this seminar, we are more familiar with the Feigin–Frenkel isomorphism at critical level, which identifies

$$\mathcal{W}_{\kappa_{\mathrm{crit}}}(\mathfrak{g}) \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}(D)}.$$

So it may be concerning that G-opers have appeared instead of $\check{G}$-opers.

The resolution to this conundrum is that sending $\kappa \to \infty$ for $\mathfrak{g}$ corresponds to taking the critical level for $\check{\mathfrak{g}}$, i.e.

$$\mathcal{W}_\infty(\mathfrak{g}) \simeq \mathcal{W}_{\check{\kappa}_{\mathrm{crit}}}(\check{\mathfrak{g}}).$$

The critical level Feigin–Frenkel for $\check{\mathfrak{g}}$ takes us back to opers for G.

3. Screening operators

3.1. Now let us consider the spectral sequence computing the cohomology of $C^{-,\bullet}_\kappa$ induced by the Kac–Kazhdan grading. Its E_1 page takes the form

$$\left(H^\bullet\left(\mathcal{O}_{\mathcal{L}^+\mathfrak{b}dt} \otimes \Lambda^\bullet(\mathcal{L}^+\mathfrak{n})^\vee, d_{st,\infty}\right), \chi\right) \tag{5}$$

3.2. Consider the space of connections

$$\nabla = d + A\,dt \qquad A \in \mathcal{L}^+\mathfrak{b}$$

with its gauge action of $\mathcal{L}^+N$. As before, one computes that $d_{st,\infty}$ matches the Chevalley differential arising from this group action.

Let $\mathcal{L}^{++}N$ denote the first congruence subgroup. One computes that the map

$$(d + \mathcal{L}^+\mathfrak{t}\,dt) \to (d + \mathcal{L}^+\mathfrak{b}\,dt)/\mathcal{L}^{++}N \tag{6}$$

is an isomorphism, so (5) is a complex

$$\mathcal{O}_{\mathcal{L}^+\mathfrak{t}dt} \to \bigoplus \mathcal{O}_{\mathcal{L}^+\mathfrak{t}dt} \otimes k_{-\alpha} \to \cdots \tag{7}$$

where the second term is a sum over simple roots.

3.3. Remark. Here is a sanity check on the assertion that (6) is an isomorphism. This assertion implies that the map

$$\begin{aligned}(d + \mathcal{L}^+\mathfrak{t}\,dt)/\mathcal{L}^+T \xrightarrow{\sim} (d + \mathcal{L}^+\mathfrak{b}\,dt)/\mathcal{L}^{++}N \cdot \mathcal{L}^+T \\ \to (d + \mathcal{L}^+\mathfrak{b}\,dt)/\mathcal{L}^+B\end{aligned} \tag{8}$$

is an N-fibration. This is true because (8) is the map

$$LS_T(D) \to LS_B(D).$$

3.4. Note that (using the identification $\mathfrak{t}^\vee \simeq \mathfrak{t}$ induced by the Killing form) we have an isomorphism

$$\mathcal{O}_{\mathcal{L}^+\mathfrak{t}dt} \simeq \mathrm{Sym}(\mathcal{L}\mathfrak{t}/\mathcal{L}^+\mathfrak{t})$$

with the Heisenberg vertex algebra for $\mathfrak{t}$. Each of the terms in the complex (7) is a Fock module for this algebra. For this reason, (7) is called the free field realization of the $\mathcal{W}$-algebra and the differentials are called screening operators.

Furthermore, one can compute that the complex (7) for the dual level of $\mathfrak{g}^L$ identifies with (7) itself. This proves Feigin–Frenkel duality for generic levels.