Stationary measures on vector spaces

Ping Ngai (Brian) Chung

Abstract

We study the classification of stationary measures for linear actions on vector spaces. A large part is a self-contained write-up of a known result by Bougerol [Bou].

1 Introduction

Let \( \mu \) be a Borel probability measure on \( G = GL(V) \), and let \( \Gamma_\mu := (\text{supp } \mu) \subset G \) be the (topological) closure of the semigroup generated by the support of \( \mu \).

In this note, we are interested in studying the \( \mu \)-stationary measures on the vector space \( V \) with respect to the \( \Gamma_\mu \)-action on \( V \) by left multiplication.

**Definition.** We say that a Borel probability measure \( \nu \) on \( V \) is \( \mu \)-stationary if \( \mu \ast \nu = \nu \), i.e.

\[
\nu = \int_{GL(V)} g_* \nu \, d\mu(g).
\]

Clearly if \( \nu \) is \( \Gamma_\mu \)-invariant then it is \( \mu \)-stationary. Also note that since \( \text{supp } \mu \) acts linearly on \( V \), the origin of \( V \) is a fixed point, so the delta mass \( \delta_0 \) at the origin of \( V \) is always a \( \mu \)-stationary probability measure on \( V \).

We would like to understand when there are other \( \mu \)-stationary probability measures on \( V \), and if so whether we can classify all of them. In the rest of this note, we say that a \( \mu \)-stationary measure \( \nu \) on \( V \) is nontrivial if \( \nu \neq \delta_0 \).

In order to state our main classification result, we need the following two notions.

**Definition.** A Borel probability measure \( \mu \) on \( GL(V) \) has finite first moment if

\[
\int_{GL(V)} \log \max(\|g\|, \|g^{-1}\|) \, d\mu(g) < \infty.
\]

Here \( \|\cdot\| := \|\cdot\|_{GL(V)} \) is the operator norm on \( GL(V) \) with respect to a fixed norm on \( V \).

**Definition.** We define the top Lyapunov exponent of \( \mu \) on a \( \Gamma_\mu \)-invariant subspace \( W \subset V \) as

\[
\lambda_{1,W} = \lambda_{1,W}(\mu) := \lim_{n \to \infty} \frac{1}{n} \int_{GL(V)} \log \|g\|_{GL(W)} \, d\mu^{(n)}(g),
\]

where \( \mu^{(n)} := \mu \ast \mu \ast \cdots \ast \mu \) is the \( n \)-th convolution power of \( \mu \), and for \( g \in GL(V) \), \( \|g\|_{GL(W)} \) denotes the operator norm of the restriction \( g|_W \) in \( GL(W) \).

The following result gives a necessary and sufficient condition for the existence of a nontrivial \( \mu \)-stationary measure on \( V \).

**Theorem 1.1.** Let \( \mu \) be a Borel probability measure on \( GL(V) \) with finite first moment. Then there exists a nontrivial \( \mu \)-stationary probability measure \( \nu \) on \( V \) if and only if there exist \( \Gamma_\mu \)-invariant subspaces \( W' \subsetneq W \subset V \) such that

(i) \( \Gamma_\mu \) acts compactly on \( W/W' \), i.e. the image of \( \rho_{W/W'} : \Gamma_\mu \to GL(W/W') \) is compact,

(ii) either \( W' = 0 \), or the top Lyapunov exponent of \( \mu \) on \( W' \) is negative,

(iii) the support of every \( \mu \)-stationary probability measure on \( V \) is in \( W \).
Theorem 1.6. For each compact $\nu$ the measure exists and does not depend on the choice of the section $C$.

Theorem 1.3. For any compact $\Gamma$-stationary measure on $V$ and let $W' \subseteq W \subset V$ be the $\Gamma^\nu$-invariant subspaces from Theorem 1.1. Then the map $\nu \mapsto \supp \pi_* \nu$ gives a one-to-one correspondence between

\[
\{\text{ergodic $\mu$-stationary measure on $V$}\} \leftrightarrow \{\text{compact $\Gamma^\mu$-orbit in $W/W'$}\},
\]

where $\pi : W \to W/W'$ is the quotient map.

We can describe the inverse map in a more explicit way in terms of the asymptotic behavior in law of the random walk on $V$ induced by $\mu$.

Theorem 1.2. Suppose there is a nontrivial $\mu$-stationary measure on $V$ and let $m_C$ be the Haar (probability) measure supported on $C$. Let $s : W/W' \to W$ be a linear section, i.e. a linear map such that $\pi \circ s = \id$. Then the weak-\* limit

\[
\nu_C := \lim_{n \to \infty} \mu^{(n)} * (s_\ast m_C)
\]

exists and does not depend on the choice of the section $s$. Moreover, the map $C \mapsto \nu_C$ is the inverse map of the bijection in Theorem 1.2.

Using the classification of stationary measures, we can obtain the following equidistribution result.

Theorem 1.4. For all $x \in W$, let $C$ is the compact $\Gamma^\nu$-orbit of $x + W'$ in $W/W'$. Then

1. we have the weak-\* convergence

\[
\frac{1}{n} \sum_{i=0}^{n-1} \mu^{(i)} * \delta_x \to \nu_C.
\]

2. For $\mu^N$-almost every word $b = (b_1, b_2, \ldots) \in GL(V)^N$, we have the convergence of the empirical measures

\[
\frac{1}{n} \sum_{i=0}^{n-1} \delta_{b_i b_{i-1} \ldots b_1 x} \to \nu_C \quad \text{as } n \to \infty.
\]

The following definition is standard when considering stationary measures.

Proposition 1.5. [BL] Lem. II.2.1] Let $\mu$ be a Borel probability measure on $G = GL(V)$ and $\nu$ be a $\mu$-stationary measure on $V$. Then for $\mu^N$-almost every $b = (b_1, b_2, \ldots) \in G^N$, there exists a probability measure $\nu_b$ on $V$ such that for all $g \in \Gamma_{\mu}$,

\[
\nu_b = \lim_{n \to \infty} (b_1 b_2 \ldots b_n g)_\ast \nu.
\]

Moreover, we have

\[
\nu = \int_{G^N} \nu_b \, d\mu^N(b).
\]

The measure $\nu_b$ is sometimes called the limit measure of $\nu$ with respect to the word $b$.

We can describe the limit measures of any stationary measures on $V$.

Theorem 1.6. For each compact $\Gamma^\nu$-orbit $C$ in $W/W'$, for $\mu^N$-almost every word $b \in GL(V)^N$, the limit measure

\[
\nu_b = \lim_{n \to \infty} (b_1 b_2 \ldots b_n)_\ast \nu_C
\]

is supported on the compact subset $p_h(C) \subset W$ for some linear section $p_b : W/W' \to W$. In particular, $\nu_b$ is compactly supported on $W$.

If $\Gamma^\mu$ acts trivially on $W/W'$, then $\nu_b$ is a delta mass $\delta_{\xi(b)}$ for $\mu^N$-almost every word $b$, and thus $\nu$ is $\mu$-proximal (cf. [BQ1 Sect. 2.7]).
The note is structured as follows.

1. In section 2, we recall a few preliminary facts about stationary measures and top exponents.

2. In section 3, we recall the situation when the action is irreducible, which will form the building blocks of the general case.

3. In section 4, we list a few properties of $\Gamma_\mu$-actions that satisfy (i) and (ii) of Theorem 1.1. In particular most of Theorem 1.2, 1.3, 1.4 and 1.6 will be proved in this section.

4. In section 5, we study properties of the action on the span of the support of any given stationary measure on $V$. The main result in this section is Proposition 5.5, when we show that the action on this span satisfies (i) and (ii) of Theorem 1.1.

5. In section 6, we conclude by proving Theorem 1.1 using results from the previous sections.

2 Preliminary facts

We first recall that, in the case of a compact action, we have the standard fact that any stationary measure is invariant.

**Proposition 2.1.** [BQ3, Lem. 8.4] Let $\mu$ be a Borel probability measure on $G = GL(V)$ and $\nu$ be a $\mu$-stationary measure on $V$. If $\Gamma_\mu$ acts compactly on $V$, then $\nu$ is $\Gamma_\mu$-invariant. Moreover, if $\nu$ is ergodic, then the support of $\nu$ is a single compact $\Gamma_\mu$-orbit in $V$, and $\nu$ is the unique $\mu$-stationary measure supported on this orbit.

We recall the following general theorem by Furstenberg and Kesten, which follows from Kingman’s subadditive ergodic theorem and the ergodicity of the Bernoulli shift.

**Theorem 2.2.** [FK, Thm. 2], see also [BQ1, Lem. 4.27]. Let $\mu$ be a Borel probability measure on $GL(V)$ with finite first moment. For $\mu^N$-a.e. $b = (b_1, b_2, \ldots) \in G^N$, one has

$$
\lim_{n \to \infty} \frac{1}{n} \log \|b_n \cdots b_1\| = \lim_{n \to \infty} \frac{1}{n} \log \|b_1 \cdots b_n\| = \lambda_{1,V}(\mu).
$$

In particular, if $\lambda_{1,V} < 0$, then $\|b_1 \cdots b_n\| \to 0$ for $\mu^N$-almost every word $b$.

To simplify notation, given a vector space $V'$ with a homomorphism $\rho_{V'} : \Gamma_\mu \to GL(V')$, we say that $\mu$ has **negative top exponent on $V'$** if the top Lyapunov exponent $\lambda_{1,V'}$ of $\rho_{V'}$ with respect to $\mu$ is negative.

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We need the following two lemmas that allow us to carry certain properties to invariant subspaces and quotients.

**Lemma 2.3.** Let $\mu$ be a Borel probability measure on $GL(V)$ with finite first moment. Let $W \subset V$ be a $\Gamma_\mu$-invariant subspace of $V$. Then the following are equivalent:

(i) $\mu$ has negative top exponent on $V$.

(ii) $\mu$ has negative top exponent on $W$ and $V/W$.

**Proof.** In fact the top exponent on $V$ is the maximum of the top exponents on $W$ and $V/W$. This is standard. See, for instance, [FKi, Lem. 3.6].

We also need the following elementary result about boundedness.

**Lemma 2.4.** Let $\mu$ be a Borel probability measure on $GL(V)$. Let $W \subset V$ be a $\Gamma_\mu$-invariant subspace of $V$. Given a subset $B \subset \Gamma_\mu$, if $B$ is bounded from above in $GL(V)$, then $B$ is bounded from above in $GL(W)$ and $GL(V/W)$.
3 The irreducible case

We first recall the classification of stationary measures for irreducible $\Gamma_\mu$-actions, i.e. the only $\Gamma_\mu$-invariant subspaces of $V$ are 0 and $V$.

**Proposition 3.1.** Let $\mu$ be a Borel probability measure on $GL(V)$. Suppose that $\Gamma_\mu$ acts irreducibly on $V$. Then there exists a nontrivial $\mu$-stationary probability measure $\nu$ on $V$ if and only if $\Gamma_\mu$ is compact in $GL(V)$.

**Proof.** If $\Gamma_\mu$ is compact in $GL(V)$ then clearly there is a nontrivial $\Gamma_\mu$-invariant measure on $V$ (by averaging via the finite Haar measure on $\Gamma_\mu$), hence in particular $\mu$-stationary.

The opposite direction was proved in [BL Prop. V.8.1].

We will also need another proposition that shows for irreducible actions, assuming a boundedness condition, the only two options are negative top exponent and compact action.

**Proposition 3.2.** Let $\mu$ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that $\Gamma_\mu$ is irreducible. If for $\mu^N$-almost every $b = (b_1, b_2, \ldots) \in G^N$, the sequence

$$\{b_nb_{n-1}\ldots b_1 \mid n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either $\mu$ has negative top exponent on $V$, or $\Gamma_\mu$ is compact in $GL(V)$.

**Proof.** The assumption implies that the top exponent is nonpositive by Theorem 2.2. Hence it suffices to consider the case when $\lambda_{1,V} = 0$.

Let $C : G^N \to \mathbb{R}_+ \cup \{\infty\}$ be a measurable function such that

$$\|b_nb_{n-1}\ldots b_1\| \leq C(b) \quad \text{for all } n.$$

Then by assumption, we can take $C$ to be finite $\mu^N$-almost surely. If we take $C'$ large enough, there is a subset $\mathcal{B} \subset G^N$ with $\mu^N(\mathcal{B}) > 1/2$ such that $C(b) < C'$ for all $b \in \mathcal{B}$. Now fix a $\mu$-stationary measure $\nu_\mathcal{B}$ on $\mathcal{P}(V)$, and consider the dynamical system on $G^N \times \mathcal{P}(V)$ with the map

$$T(b,v) := \left(\sigma(b), \log \frac{\|b_1v\|}{\|v\|}\right),$$

where $\sigma : G^N \to G^N$ is the left shift map. Note that $\mu^N \times \nu_\mathcal{B}$ is a $T$-invariant probability measure on $G^N \times \mathcal{P}(V)$.

By the proof of the Atkinson’s lemma ([At], [Ke], see e.g. [BQ1 Lem. 3.18]), for $\mu^N \times \nu_\mathcal{B}$-almost every $(b,v) \in G^N \times \mathcal{P}(V)$, there is an infinite sequence $\{n_k\}_k$ such that

$$\lim_{k \to \infty} \log \frac{\|b_{n_k}\ldots b_1v\|}{\|v\|} = 1. \quad (1)$$

Fix a nonzero $v \in V$ such that (1) holds for $\mu^N$-almost every $b \in G^N$. For each such word $b \in G^N$, for each $n \geq 1$, take $k$ large enough so that $n_k > n$. Then

$$\log \frac{\|b_n\ldots b_1v\|}{\|v\|} = \log \frac{\|b_{n_k}\ldots b_1v\|}{\|v\|} - \log \frac{\|b_{n_k}\ldots b_1v\|}{\|b_n\ldots b_1v\|}.$$

Now on the right hand side, the first term is at least $-1$ by (1), and the second term is at least $-\log C(\sigma ^n(b))$ by definition of $C$. Therefore

$$\log \frac{\|b_{n_k}\ldots b_1v\|}{\|v\|} \geq -1 - \log C(\sigma ^n(b)).$$

However note that $C(\sigma ^n(b))$ does not depend on $b_1, b_2, \ldots, b_n$. Therefore we can replace $b$ by one of the words that starts with $b_1, b_2, \ldots, b_n$ and satisfies $\sigma ^n(b) \in \mathcal{B}$ so that $C(\sigma ^n(b)) < C'$ for the uniform constant $C'$ chosen above. Thus for $\mu^N$-almost every word $b$, for all $n \geq 1$,

$$\log \frac{\|b_n\ldots b_1v\|}{\|v\|} \geq -1 - \log C'.$$
Now consider the sequence of measures on $V$

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{b_n b_{n-1} \ldots b_1 v}.$$ 

Then any weak-$\ast$ limit $\nu$ is a $\mu$-stationary measure on $V$ by Breiman's Law of Large Numbers ([Br], also see e.g. [BQ1] Cor. 3.4]), and is a probability measure since there is a uniform bound from above on the sequence $\{b_n b_{n-1} \ldots b_1 | n \geq 1\}$ by assumption. Since

$$\frac{\|b_n \ldots b_1 v\|}{\|v\|} \geq C''$$

for some uniform $C''$, $\nu$ is not $\delta_0$, so it is a nontrivial $\mu$-stationary probability measure on $V$. By Proposition 3.1, $\Gamma_\mu$ is compact in $GL(V)$.

The same is true if the order of the matrix product $b_1 b_2 \ldots b_n$ is reversed.

**Corollary 3.3.** Let $\mu$ be a Borel probability measure on $G = GL(V)$ with finite first moment. Assume that $\Gamma_\mu$ is irreducible. If for $\mu^N$-almost every $b = (b_1, b_2, \ldots) \in G^N$, the sequence

$$\{b_1 b_2 \ldots b_n | n \geq 1\}$$

is bounded from above (with respect to the operator norm on $GL(V)$), then either $\mu$ has negative top exponent on $V$, or $\Gamma_\mu$ is compact in $GL(V)$.

**Proof.** Apply Proposition 3.2 to the pushforward $\mu^T$ of $\mu$ via the adjoint map $GL(V) \to GL(V^*)$ defined by $g \mapsto g^T$ (i.e. the matrix transpose). Note that $\|g\|_{GL(V)} = \|g^T\|_{GL(V^*)}$, so the first moments of $\mu$ and $\mu^T$ are the same. Similarly the top exponents of $\mu$ and $\mu^T$ are the same. Finally $\Gamma_\mu$ is irreducible if and only if $\Gamma_\mu^T$ is, and $\Gamma_\mu$ is compact if and only if $\Gamma_\mu^T$ is.

4 Properties of a contracting-by-compact action

In this section, we list a few properties of subspaces with a contracting-by-compact action by $\mu$, i.e. there is a proper subspace (possibly zero) with negative top exponent with respect to $\mu$ and $\Gamma_\mu$ acts compactly on the quotient.

The following proposition shows that for such action, almost every word is bounded from above with respect to the operator norm (though this bound may depend on the word).

**Proposition 4.1.** Let $\mu$ be a Borel probability measure on $GL(W)$ with finite first moment. Moreover there exists a proper $\Gamma_\mu$-invariant subspace $W' \subseteq W$ such that

(i) $\Gamma_\mu$ acts compactly on $W/W'$, and

(ii) if $W' \neq 0$, $\mu$ has negative top exponent on $W'$.

Then there exists a measurable map $C : G^N \to \mathbb{R}_+$ such that for $\mu^N$-almost every word $b = (b_1, b_2, \ldots)$,

$$\|b_1 b_2 \ldots b_n\| < C(b)$$

for all $n \geq 1$.

**Proof.** By choosing suitable basis, we can write each $b_i \in \text{supp } \mu$ as

$$\begin{bmatrix} x_i & y_i \\ 0 & z_i \end{bmatrix},$$

where $x_i \in GL(W')$, $z_i \in GL(W/W')$ and $y_i \in \text{Hom}(W/W', W')$.

Now we expand $b_1 b_2 \ldots b_n$ in terms of $x_i, y_i, z_i$,

$$b_1 b_2 \ldots b_n = \begin{bmatrix} X_n & Y_n \\ 0 & Z_n \end{bmatrix},$$
where
\[ X_n = x_1x_2 \ldots x_n, \quad Y_n = \sum_{k=1}^{n} x_1 \ldots x_{k-1}y_kz_{k+1} \ldots z_n, \quad Z_n = z_1z_2 \ldots z_n. \]

Since \( \mu \) has negative top exponent on \( W' \), \( x_1x_2 \ldots x_n \to 0 \) for \( \mu^N \)-almost every word \( b \) by Theorem 2.2. Since \( \Gamma_\mu \) acts compactly on \( W/W' \), \( Z_n \) is uniformly bounded by some constant \( C' \). Hence it remains to find a bound on \( Y_n \) that is independent of \( n \) (but may depend on the word \( b \)).

If \( W' = 0 \), we are done. If \( W' \neq 0 \), let \( \lambda_{1,W'} < 0 \) be the top exponent of \( \mu \) on \( W' \). Then for \( \mu^N \)-almost every word \( b \),
\[ \lim_{k \to \infty} \frac{1}{k} \log \| x_1x_2 \ldots x_k \| = \lambda_{1,W'} < 0. \]

Since \( \mu \) has finite first moment in \( GL(W) \), in particular, we have
\[ \int_G \log^+ \|y\| \, d\mu < \infty, \]
where \( \log^+ (x) := \max(\log(x), 0) \). This implies that (since \( \|b_k\| \geq \|y_k\| \))
\[ \sum_{k=1}^{\infty} \mu \left( \log^+ \|y_k\| - \frac{k\lambda_{1,W'}}{2} \right) \leq \sum_{k=1}^{\infty} \mu \left( \log^+ \|b_k\| - \frac{k\lambda_{1,W'}}{2} \right) < \infty. \]

By Borel-Cantelli Lemma, for \( \mu^N \)-almost every word \( b \),
\[ \limsup_{k} \frac{1}{k} \log^+ \|y_k\| \leq -\frac{\lambda_{1,W'}}{2}. \]

This implies that
\[ \limsup_{k} \frac{1}{k} \log \|x_1 \ldots x_{k-1}y_k\| \leq \limsup_{k} \frac{1}{k} \log \|x_1 \ldots x_{k-1}\| \|y_k\| \leq \frac{\lambda_{1,W'}}{2}. \]

Since \( \lambda_{1,W'} < 0 \), and \( z_i \) is in a compact subgroup of \( GL(W/W') \) with a uniform upper bound \( C' \), there exist \( n_0 = n_0(b) \) and \( C'' = C''(b) \) such that for all large enough \( n \),
\[ \|Y_n\| \leq \sum_{k=1}^{n} \|x_1 \ldots x_{k-1}y_kz_{k+1} \ldots z_n\| \leq C'' + C' \sum_{k=n_0}^{n} e^{k\lambda_{1,W'}/2} \leq C'' + \frac{C'}{1 - e^{\lambda_{1,W'/2}}} < \infty, \]

as desired. \( \Box \)

The following proposition shows that there is at least one nontrivial stationary measure in the subspace \( W \).

**Proposition 4.2.** Let \( \mu \) be a Borel probability measure on \( G = GL(W) \) with finite first moment. Suppose there exists a proper \( \Gamma_\mu \)-invariant subspace \( W' \subset W \) such that

(i) \( \Gamma_\mu \) acts compactly on \( W/W' \), and

(ii) if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \).

Then for all \( x \in W \setminus W' \), any weak-* limit point of the sequence of probability measures
\[ \nu_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \mu^{(i)} \ast \delta_x \]
is a nontrivial \( \mu \)-stationary probability measure on \( W \).

**Proof.** Let \( \hat{W} := W \cup \{\infty\} \) be the one-point compactification of \( W \). Then the space of probability measures \( \mathcal{M}(\hat{W}) \) is compact, hence there exists a subsequence \( \{n_k\} \) such that \( \nu_{x,n_k} \) converges to a probability measure \( \nu \in \mathcal{M}(\hat{W}) \). Moreover,
\[ \mu * \nu_{x,n_k} - \nu_{x,n_k} = \frac{1}{n_k} (\mu^{(n_k)} \ast \delta_x - \delta_x) \to 0. \]
Hence $\nu$ is $\mu$-stationary. Since $\infty$ is a fixed point, we may consider $\nu$ as a $\mu$-stationary measure on $W$ (a priori may not be a probability measure). It remains to show that $\nu(W \setminus \{0\}) = 1$. Let $\pi : W \to W/W'$ be the quotient map.

First of all since $\Gamma_\mu$ acts compactly on $W/W'$ and $x \in W \setminus W'$, $\Gamma_\mu \pi(x) \subset W/W'$ is compact and does not contain the origin in $W/W'$. Therefore there exists a compact subset $C_x \subset W \setminus W'$ such that $\Gamma_\mu x \subset C_x + W'$. Note that $0 \notin C_x + W'$. Now clearly the support of $\nu_{x,n}$ is contained in $\Gamma_\mu x \subset C_x + W'$ for all $n$ and hence the support of $\nu$ is also contained in the closed set $C_x + W'$. In particular $\nu(\{0\}) = 0$.

It remains to show that for all $\varepsilon > 0$, there exists $C'' = C''(\varepsilon, x) > 0$ such that

$$
\nu(\{w \in W \mid \|w\| < C''\}) > 1 - \varepsilon.
$$

Since $\nu_{x,n} \to \nu$, applying this convergence to the indicator function $1_{\{w \in W \mid \|w\| < C''\}}$, we have

$$
\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu'(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_j x\| < C''\}) = \nu(\{w \in W \mid \|w\| < C''\}).
$$

But the left hand side can be bounded from below using Fatou’s lemma:

$$
\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mu'(\{(b_1, b_2, \ldots, b_i) \in G^i \mid \|b_1 b_2 \ldots b_j x\| < C''\}) \\
= \lim_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int_{\|b_1 b_2 \ldots b_j x\| < C''} 1 \, d\mu^{n}(b) \\
\geq \liminf_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \int_{\|b_1 b_2 \ldots b_j x\| < C''} 1 \, d\mu^{n}(b)
$$

Moreover, by Proposition 4.1 there exists a measurable function $C : G^n \to \mathbb{R}_+$ such that, for $\mu^n$-almost every word $b = (b_1, b_2, \ldots),

$$
\|b_1 b_2 \ldots b_n\| < C(b).
$$

Now take a subset $B_\varepsilon \subset G^n$ and large enough $C' > 0$ such that $\mu^n(B_\varepsilon) > 1 - \varepsilon$ and $C(b) < C'$ for all $b \in B_\varepsilon$. Let $C'' = C''(\varepsilon, x) := C'(\|x\|)$. Then for all $b \in B_\varepsilon,

$$
\liminf_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} 1_{\|b_1 b_2 \ldots b_i x\| < C'\varepsilon}(b) = 1.
$$

Thus

$$
\nu(\{w \in W \mid \|w\| < C''\}) \geq \liminf_{k \to \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} 1_{\|b_1 b_2 \ldots b_i x\| < C'' \varepsilon}(b) \, d\mu^{n}(b) \geq \mu^n(B_\varepsilon) > 1 - \varepsilon.
$$

The following proposition shows that any stationary measure in such subspace $W$ is uniquely determined by its pushforward on the quotient $W/W'$.

**Proposition 4.3.** Let $\mu$ be a Borel probability measure on $G = GL(W)$ with finite first moment. Let $W' \subseteq W$ be a $\Gamma_\mu$-invariant flag. Suppose

(i) $\Gamma_\mu$ acts compactly on $W/W'$, and

(ii) if $W' \neq 0$, $\mu$ has negative top exponent on $W'$.

Suppose that we have two $\mu$-stationary measures $\nu$ and $\nu'$ on $W$ that satisfy $\pi_* \nu = \pi_* \nu'$ for the quotient map $\pi : W \to W/W'$, then $\nu = \nu'$. 

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**Proof.** By Proposition 4.1, there exists a measurable map \( C : G^N \to \mathbb{R}_+ \) such that for \( \mu^N \)-almost every word \( b = (b_1, b_2, \ldots) \in G^N \), we have
\[
\|b_1 b_2 \ldots b_n\|_{GL(W)} < C(b).
\]
Also for almost every word \( b \), we have the limit measure
\[
\nu_b = \lim_{n \to \infty} (b_1 b_2 \ldots b_n)_* \nu.
\]
Therefore we can take a limit point \( b_\infty \) of the sequence \( \{b_1 b_2 \ldots b_n \ | n \geq 1\} \) in \( \text{End}(W) \), and
\[
\nu_b = (b_\infty)_* \nu.
\]
Similarly, we have, for almost every word \( b \),
\[
\nu'_b := \lim_{n \to \infty} (b_1 b_2 \ldots b_n)_* \nu' = (b_\infty)_* \nu'.
\]
Now since \( \mu \) has negative top exponent on \( W' \), for almost every word \( b \),
\[
\lim_{n \to \infty} b_1 b_2 \ldots b_n v = 0 \quad \text{for every vector} \ v \in W'.
\]
Therefore \( W' \subset \ker b_\infty \), hence the map \( b_\infty : W \to W \) factors through \( W/W' \), i.e. there exists a linear map \( b'_\infty : W/W' \to W \) such that \( b_\infty = b'_\infty \circ \pi \), where \( \pi : W \to W/W' \) is the quotient map. Since \( \pi_* \nu = \pi_* \nu' \), for \( \mu^N \)-almost every word \( b \), we have
\[
\nu_b = (b_\infty)_* \nu = (b'_\infty)_* \pi_* \nu = (b'_\infty)_* \pi_* \nu' = (b_\infty)_* \nu' = \nu'_b.
\]
Thus by Theorem 1.5,
\[
\nu = \int_{G^N} \nu_b d\mu^N(b) = \int_{G^N} \nu'_b d\mu^N(b) = \nu'.
\]

In particular the above proof shows that each limit measure \( \nu_b \) is supported on a compact subset of \( W \). We record this in the following proposition (which proves Theorem 1.6).

**Proposition 4.4.** Let \( \mu \) be a Borel probability measure on \( G = GL(W) \) with finite first moment. Let \( W' \subset W \) be a \( \Gamma_\mu \)-invariant flag. Suppose

(i) \( \Gamma_\mu \) acts compactly on \( W/W' \), and

(ii) if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \).

Given an ergodic \( \mu \)-stationary measure \( \nu \) on \( W \), for \( \mu^N \)-almost every word \( b \), the limit measure
\[
\nu_b = \lim_{n \to \infty} (b_1 b_2 \ldots b_n)_* \nu
\]
is supported on the pushforward of a single compact \( \Gamma_\mu \)-orbit on \( W/W' \) via a linear injection \( p_b : W/W' \to W \). In particular, \( \nu_b \) is compactly supported on \( W \).

**Proof.** Take \( p_b \) to be the linear map \( b'_\infty \) defined in the proof of Proposition 4.3. Since \( \pi_* \nu \) is an ergodic \( \mu \)-stationary measure on \( W/W' \) and \( \mu \) acts compactly on \( W/W' \), \( \pi_* \nu \) is an ergodic \( \Gamma_\mu \)-invariant measure and is supported on a single compact \( \Gamma_\mu \)-orbit in \( W/W' \) by Proposition 2.1. Thus \( \nu_b = (b'_\infty)_* \pi_* \nu \) is also compactly supported on \( W \).

Using Proposition 4.3 one can refine Proposition 4.2.

**Proposition 4.5.** Let \( \mu \) be a Borel probability measure on \( G = GL(W) \) with finite first moment. Suppose there exists a proper \( \Gamma_\mu \)-invariant subspace \( W' \subset W \) such that

(i) \( \Gamma_\mu \) acts compactly on \( W/W' \), and

(ii) if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \).
For all \( x \in W \setminus W' \), let

\[
\nu_{x,n} := \frac{1}{n} \sum_{i=0}^{n-1} \mu(i) * \delta_x.
\]

Then the weak-* limit

\[
\nu_x := \lim_{n \to \infty} \nu_{x,n}
\]

exists and is a nontrivial \( \mu \)-stationary probability measure on \( W \).

**Proof.** By Proposition 4.2, we know that any limit point of the sequence \( \{ \nu_{x,n} \}_n \) is a nontrivial \( \mu \)-stationary measure on \( W \). Moreover, since the projection of \( \nu_{x,n} \) on \( W/W' \) lies in the compact \( \Gamma_\mu \)-orbit of \( x + W' \in W/W' \), any weak-* limit point projects to a \( \mu \)-stationary measure supported on the single compact orbit \( \Gamma_\mu x + W' \subset W/W' \), hence is in fact the unique invariant measure supported on the compact set \( \Gamma_\mu x + W' \). In particular, any limit point of \( \{ \nu_{x,n} \}_n \) is a \( \mu \)-stationary probability measure that projects to the same measure on \( W/W' \). By Proposition 4.3, all such limit points agree, so the sequence \( \{ \nu_{x,n} \}_n \) converges. \( \square \)

In fact, if we start with any initial measure that projects to the Haar measure supported on a compact \( \Gamma_\mu \)-orbit in \( W/W' \), then the convolution powers are not just Cesàro summable, but themselves converge.

**Proposition 4.6.** Let \( \mu \) be a Borel probability measure on \( G = GL(W) \) with finite first moment. Suppose there exists a proper \( \Gamma_\mu \)-invariant subspace \( W' \subset W \) such that

(i) \( \Gamma_\mu \) acts compactly on \( W/W' \), and

(ii) if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \).

For all \( x \in W \setminus W' \), let \( \mathcal{C}_x \) be the \( \Gamma_\mu \)-orbit of the image \( x \) in \( W/W' \), and \( m_x \) be the Haar (probability) measure on \( W/W' \) supported on \( \mathcal{C}_x \). Then for any linear section \( s : W/W' \to W \), i.e. a linear map such that \( \pi \circ s = \text{id} \), we have the following weak-* convergence

\[
\nu_x := \lim_{n \to \infty} \mu^{(n)} * (s_*m_x).
\]

Moreover, \( \nu_x \) is a nontrivial \( \mu \)-stationary probability measure on \( W \) that does not depend on the choice of the linear section \( s \). The map \( x \mapsto \nu_x \) is constant along the orbit \( \mathcal{C}_x \).

**Proof.** By Proposition 4.2, for all \( x \in W \setminus W' \), there exists a nontrivial \( \mu \)-stationary measure \( \nu_x \) on \( W \) that projects to \( m_x \) on \( W/W' \).

Similar to the proof of Proposition 4.3, there exists a measurable function \( C : G^N \to \mathbb{R}_+ \) such that for \( \mu^N \)-almost every word \( b = (b_1, b_2, \ldots) \), we have

\[
\|b_1b_2\ldots b_n\|_{GL(W)} < C(b), \quad \text{and} \quad \nu_b = \lim_{n \to \infty} (b_1 \ldots b_n)_* \nu_x,
\]

exists. Moreover, for any limit point \( b_\infty \) of \( \{b_1b_2\ldots b_n \mid n \geq 1 \} \) in \( \text{End}(W) \), there exists a linear map \( b'_\infty : W/W' \to W \) such that \( b_\infty = b'_\infty \circ \pi \). Let \( \{n_k\}_k \) be the indices of the subsequence such that

\[
\lim_{k \to \infty} b_1b_2\ldots b_{n_k} = b_\infty = b'_\infty \circ \pi.
\]

Now for any linear section \( s : W/W' \to W \), we have

\[
\lim_{k \to \infty} (b_1 \ldots b_{n_k})* (s_*m_x) = (b'_\infty)_* m_x = (b'_\infty)_* (s_*m_x)
\]

since \( \pi \circ s = \text{id} \). On the other hand since the stationary measure \( \nu_x \) projects to \( m_x \) on \( W/W' \), we also have

\[
\nu_b = \lim_{n \to \infty} (b_1 \ldots b_n)_* \nu_x = (b_\infty)_* \nu_x = (b'_\infty)_* \nu_x = (b'_\infty)_* m_x.
\]

Thus

\[
\nu_b = (b'_\infty)_* m_x = \lim_{k \to \infty} (b_1 \ldots b_{n_k})* (s_*m_x)
\]
for any convergent subsequence \( \{b_1 \ldots b_n \mid k \geq 1\} \). Since the left hand side does not depend on the subsequence, we have the convergence
\[
\nu_b = \lim_{n \to \infty} (b_1 \ldots b_n)_*(s_*m_x).
\]
Since this holds for \( \mu^N \)-almost every \( a \), we have
\[
\nu_a = \int \nu_b d\mu^N(b) = \int \lim_{n \to \infty} (b_1 \ldots b_n)_*(s_*m_x)d\mu^N(b) = \lim_{n \to \infty} \mu^N((b_1 \ldots b_n)_*(s_*m_x)).
\]

5 Properties of the span of the support of a stationary measure

In this section, we prove a few properties of the action on the span of the support of a given stationary measure. The main statement is that the span of the support of a given stationary measure must have a contracting-by-compact action by \( \mu \) (Proposition 5.5). An important auxiliary proposition leading towards this fact is Proposition 5.2.

**Lemma 5.1.** Let \( \mu \) be a Borel probability measure on \( GL(V) \), \( \nu \) be a \( \mu \)-stationary probability measure on \( V \). Let \( W \) be the linear span of the support of \( \nu \). Then

(i) \( W \) is \( \Gamma_{s_*} \)-invariant.

(ii) For \( \mu^N \)-almost every word \( b = (b_1, b_2, \ldots) \in G^N \), the sequence \( \{b_1b_2 \ldots b_n \mid n \geq 1\} \) is bounded from above in \( GL(W) \).

**Proof.** (i) is clear since \( \nu \) is \( \Gamma_{s_*} \)-invariant. The proof of (ii) is similar to the proof of [3] Lem. 3.3, using ideas of [4] Thm. 1.2. By considering the restriction of the action to \( W \) we may assume that \( V = W \) and thus \( G = GL(W) \) without loss of generality. For \( b \in G^N \) for which the limit measure \( \nu_b \) exists, assume the contrary that the sequence \( \{b_1b_2 \ldots b_n \mid n \geq 0\} \) is not bounded from above in \( GL(W) \). Then we can find a subsequence \( \{n_k \mid k \in \mathbb{N}\} \) and \( \nu_{b_\infty} \in \text{End}(W) \) with \( \|\nu_{b_\infty}\|_{\text{End}(W)} = 1 \) such that
\[
\lim_{n \to \infty} \|b_1b_2 \ldots b_{n_k}\|_{GL(W)} = \infty, \quad \text{and} \quad \lim_{k \to \infty} \|b_1b_2 \ldots b_{n_k}\|_{GL(W)} = b_\infty.
\]

Let \( W_b := \ker b_\infty \subset W \). For \( v \in W \setminus W_b \), we have
\[
\lim_{k \to \infty} \|b_1b_2 \ldots b_{n_k}v\|_{W} = \infty.
\]

Thus for any continuous function \( \phi : W \to \mathbb{R} \) with compact support, for all \( v \in W \setminus W_b \),
\[
\phi(b_1b_2 \ldots b_{n_k}v) \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore
\[
\int \phi(v)d\nu_b(v) = \lim_{k \to \infty} \int \phi(v)d(b_1b_2 \ldots b_{n_k})_*\nu(v) = \lim_{k \to \infty} \int \phi(b_1b_2 \ldots b_{n_k}v)d\nu(v) = \lim_{k \to \infty} \int \mathbf{1}_{W_b}(v)\phi(b_1b_2 \ldots b_{n_k}v)d\nu(v) \leq \nu(W_b) \sup_{v \in W} |\phi(v)|.
\]

Since \( \phi \) is an arbitrary continuous function on \( W \) with compact support, by taking a sequence of such \( \phi \) supported on balls of radius \( n \to \infty \) and takes value 1 within a slightly smaller open ball, we can conclude that \( \nu(W_b) = 1 \). Since \( W_b \) is closed, we have \( \text{supp } \nu \subset W_b \).

Since \( W_b \) is a subspace of \( W \) and \( \text{supp } \nu \) spans \( W \), we have \( \ker b_\infty = W_b = W \), i.e. \( b_\infty \) is the zero map. But this is a contradiction since \( \|b_\infty\|_{\text{End}(W)} = 1 \).

\[\square\]
We shall show the following important auxiliary proposition.

**Proposition 5.2.** Let $\mu$ be a Borel probability measure $\mu$ on $G = GL(V)$ with finite first moment. Suppose there exists a $\mu$-stationary measure $\nu$ on $V$ such that $V$ is the span of supp $\nu$. Suppose there exist $\Gamma_\mu$-invariant subspaces $0 \subset W' \subset W \subset V$ such that

(i) $\Gamma_\mu$ acts compactly on $W'$;

(ii) if $W' \neq W$, $\mu$ has negative top exponent on $W/W'$;

(iii) $\Gamma_\mu$ acts compactly on $V/W$.

Then there is a $\Gamma_\mu$-invariant splitting of $V$:

$$V = W' \oplus W''$$

for some $\Gamma_\mu$-invariant subspace $W'' \subset V$.

We first prove a lemma which allows us to reduce the proposition to the case when the acting group $\Gamma_\mu$ is uniformly bounded from above in $GL(V)$.

**Lemma 5.3.** Under the assumptions of Proposition 5.2, if in addition, $\Gamma_\mu$ is unbounded from above with respect to the operator norm on $GL(V)$, i.e. there exists a sequence $\{g_k\} \subset \Gamma_\mu$ such that $\|g_k\|_{GL(V)} \to \infty$, then there is a nonzero $\Gamma_\mu$-invariant subspace $W_0 \subset W$ such that

$$W' \cap W_0 = 0.$$

**Proof.** The proof is similar to that of Lemma 5.1(ii). By Lemma 5.1(ii), for $\mu^N$-almost every word $w \in G^N$, the sequence

$$\{b_1 b_2 \ldots b_n \mid n \geq 1\}$$

is bounded from above in $GL(V)$. Let $b_\infty$ be a limit point of this sequence in $End(V)$. Moreover, by Lemma 1.3 for all $g \in \Gamma_\mu$ and each positive integer $k$, we have

$$\nu_b = \lim_{n \to \infty} (b_1 b_2 \ldots b_n g g_k)_* \nu = (b_\infty g g_k)_* \nu.$$

Let $g_\infty$ be a limit point of the sequence $\{g_k/\|g_k\|\}_k$ in $End(V)$. Then by the same argument as the proof of Lemma 5.1(ii), using the fact that $\|g_k\| \to \infty$, one can conclude that

$$b_\infty g g_\infty = 0,$$

the zero map on $V$. Hence for all $g \in \Gamma_\mu$,

$$gg_\infty V \subset ker b_\infty.$$

Let $W_0$ be the span of $\{gg_\infty V \mid g \in \Gamma_\mu\}$. Then $W_0 \subset ker b_\infty$. Since $\|g_\infty\| = 1$, $g_\infty V$ is nonzero, so $W_0$ is a nonzero $\Gamma_\mu$-invariant subspace of $V$. Moreover, since $\Gamma_\mu$ acts compactly on $W'$ and $b_\infty$ is in the closure of $\Gamma_\mu$ in $End(V)$, $W' \cap ker b_\infty = 0$.

On the other hand, we claim that $ker b_\infty \subset W$. In fact, for $v \notin W$, since $b_\infty \in \Gamma_\mu$ acts compactly on $V/W$, we have $b_\infty v \notin W$, in particular $b_\infty v \neq 0$, so $v \notin ker b_\infty$.

Now since $W_0 \subset ker b_\infty$, we have that $W_0 \subset W$ and $W' \cap W_0 = 0$, as desired. \qed

We also need an algebraic fact about compact subsemigroups of $End(V)$.

**Lemma 5.4.** [HM A.1.22] Let $S \subset End(V)$ be a nonempty compact subsemigroup. Then there exists $h \in S$ such that

(a) $h$ is idempotent, i.e. $h^2 = h$;

(b) $hSh := \{bgh \mid g \in S\}$ has the structure of a compact group with identity element $h$;

(c) there is a group action by $hSh$ on $hV$.

For completeness we include a sketch of the proof here.
Sketch of Proof. Let $r$ be the smallest rank among elements in $S$, and let $S_0 := \{ g \in S \mid \text{rank}(g) = r \}$. Then $S_0$ is itself a compact subsemigroup of $\text{End}(V)$ since the rank cannot increase when taking products and limits. By Ellis-Numakura lemma ([HM A.1.16]), any nonempty compact semigroup has an idempotent element, so there exists $h \in S_0$ with $h^2 = h$. Then $hSh$ is a compact semigroup with $h$ acting as the identity element.

We claim that $h$ is the only idempotent element in $hSh$. In fact let $h'$ be another idempotent element in $hSh$. Then the image of $h'$ is contained in the image of $h$. But $h$ has minimal rank in $S$ and $hSh$ is contained in $S$, so the images of $h$ and $h'$ are the same. Moreover, since $h$ and $h'$ are idempotents in $\text{End}(V)$, we have the decompositions

$$V = \text{im} \, h \oplus \ker h = \text{im} \, h' \oplus \ker h'.$$

Since $h' \in hSh$, $\ker h \subset \ker h'$. But since $\text{im} \, h = \text{im} \, h'$, the dimensions of $\ker h$ and $\ker h'$ agree, so $\ker h = \ker h'$. Any idempotent in $\text{End}(V)$ is determined by its image and kernel, so $h = h'$.

On the other hand, one can check that if a compact semigroup $K$ with identity has no other idempotent, then it is a compact group. In fact, for any $t \in K$, $tK$ and $Kt$ are nonempty compact subsemigroups of $K$, so they also have idempotent elements. But by assumption, this idempotent element must be the identity, so $t$ has left and right inverses for all $t \in K$, as desired.

Thus we have shown that $K = hSh$ is a compact group with identity $h$. $hSh$ acts on $hV$ since the identity element $h$ acts trivially on $hV$.

Now we are ready to prove Proposition 5.2.

Proof of Proposition 5.2. We prove the statement by induction on $\dim V$.

Base case: $\dim V = 1$.
Since $W$ is a proper subspace of $V$, we have $W' = W = 0$. Therefore we can take $W'' = V$.

Induction step.
If $\Gamma_\mu$ is unbounded from above in $\text{GL}(V)$, by Lemma 5.3 there exists a nonzero $\Gamma_\mu$-invariant subspace $W_0 \subset W$ with $W' \cap W_0 = 0$. Now consider the $\Gamma_\mu$-invariant flag

$$0 \subsetneq W' \subset W/W_0 \subsetneq V/W_0.$$

Since $W_0$ is nonzero, $\dim V/W_0 < \dim V$, so by the induction hypothesis, there exists a $\Gamma_\mu$-invariant subspace $W_2 \subset V$ with $W_0 \subset W_2$ such that there is the $\Gamma_\mu$-invariant splitting

$$V/W_0 = W' \oplus W_2/W_0.$$ 

Thus we can take $W'' = W_2$.

Hence in the remaining part of the proof we assume also that there exists $C > 0$ such that $\|g\| \leq C$ for all $g \in \Gamma_\mu$. Let $\Gamma_\mu$ be the (topological) closure of $\Gamma_\mu$ in $\text{End}(V)$, then $\Gamma_\mu$ is a compact semigroup in $\text{End}(V)$. By Lemma 5.4 there exists an idempotent $h \in \Gamma_\mu$ (i.e. $h^2 = h$) such that

$$K := h\Gamma_\mu h$$

is a compact group with identity $h$. Moreover $K$ acts on $hV$, and preserves $W'$ (note that $hW' = W'$ since $\Gamma_\mu$ acts compactly on $W'$). Since $K$ is compact, there exists a $K$-invariant complementary subspace $W_1 \subset hV$ of $W'$, i.e.

$$hV = W' \oplus W_1.$$

Note that $hW_1 = W_1$ since $h \in K$. Now let $W''$ be the span of $\{ ghW_1 \mid g \in \Gamma_\mu \}$. Then $W''$ is $\Gamma_\mu$-invariant.

Let $v \in W'' \cap W'$. On one hand, $hv \in hW' = W'$, on the other hand,

$$hv \in \text{span}(\{ ghW_1 \mid g \in \Gamma_\mu \}) = W_1$$

since $hgh \in K$ for $g \in \Gamma_\mu$ and $W_1$ is $K$-invariant. Thus $hv \in W' \cap W_1 = 0$, i.e. $v \in \ker h$. Now since $\Gamma_\mu$ acts compactly on $W'$, $\ker h \cap W' = 0$. But $v \in \ker h \cap W'$, so $v = 0$. Therefore $W'' \cap W' = 0$.

Hence we have found a $\Gamma_\mu$-invariant subspace $W''$ with trivial intersection with $W'$. It remains to show that $W'' + W' = V$. 

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We first observe that \( \ker h \subset W \). In fact, consider \( v \notin W \). Since \( h \) acts compactly on \( V/W \), \( hv \neq 0 \) in \( V/W \); so \( hv \neq 0 \) in \( V \), thus \( v \notin \ker h \).

Since \( h \) is idempotent, we have that

\[
V = \im h \oplus \ker h = W' \oplus W_1 \oplus \ker h.
\]

Since \( W_1 \subset W'' \) and \( W' \oplus \ker h \subset W \), we have

\[
V = W'' + W.
\]

If \( W' = W \), we are done. If \( W' \neq W \), by assumption, \( \mu \) has negative top exponent on \( W/W' \). Now

\[
V/W'' = (W'' + W)/W'' = W/((W'' \cap W)
\]

Since \( W' \) is \( \Gamma \mu \)-invariant, \( W' \subset W \) and \( W' \cap W'' = 0 \), we have the following \( \Gamma \mu \)-equivariant identification

\[
V/((W'' \oplus W') = W/((W'' \cap W') = (W/W')(W'' \cap (W/W')).
\]

Since \( \mu \) has negative top exponent on \( W/W' \), it also has negative top exponent on \( (W/W')(W'' \cap (W/W')) \), thus on \( V/((W'' \oplus W') \). Therefore the only \( \mu \)-stationary measure on \( V/((W'' \oplus W') \) is \( \delta_0 \). On the other hand, since \( \nu \) is a \( \mu \)-stationary measure on \( V \) with \( \text{span}(\text{supp} \ \nu) = V \), the pushforward of \( \nu \) on \( V/((W'' \oplus W') \) also spans. But this pushforward is \( \mu \)-stationary on \( V/((W'' \oplus W') \), so it equals \( \delta_0 \). Therefore \( V = W'' \oplus W' \), as desired.

Now we are ready to prove that the \( \mu \)-action on the span of the support of a stationary measure is contracting-by-compact.

**Proposition 5.5.**[Bon] Thm. 5.1 necessity direction] Let \( \mu \) be a Borel probability measure \( \mu \) on \( G = GL(V) \) with finite first moment, and \( \nu \) be a nontrivial \( \mu \)-stationary measure on \( V \). Let \( W \) be the linear span of \( \text{supp} \ \nu \). Then there exists a \( \Gamma \mu \)-invariant proper subspace \( W' \subset W \) such that

(i) \( \Gamma \mu \) acts compactly on \( W/W' \), and

(ii) if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \).

**Proof.** We prove this by induction on \( \dim W \).

**Base case:** \( \dim W = 1 \).

In this case, \( \Gamma \mu \) acts irreducibly on \( W \). By Proposition 3.1, \( \Gamma \mu \) acts compactly on \( W \) and we can take \( W' = 0 \).

**Induction step.**

If \( \Gamma \mu \) acts irreducibly on \( W \), then again by Proposition 3.1, \( \Gamma \mu \) acts compactly on \( W \) and we can take \( W' = 0 \).

If \( \Gamma \mu \) does not act irreducibly on \( W \), take a minimal nonzero \( \Gamma \mu \)-invariant proper subspace \( 0 \subsetneq W_0 \subsetneq W \). The pushforward of \( \nu \) under the map \( W \to W/W_0 \) is a stationary measure on \( W/W_0 \) whose support spans \( W/W_0 \). Since \( \dim W/W_0 < \dim W \), by the induction hypothesis, we know that there exists a \( \Gamma \mu \)-invariant proper subspace \( W_1 \subsetneq W \) such that

(i) \( 0 \subsetneq W_0 \subsetneq W_1 \subsetneq W \),

(ii) \( \Gamma \mu \) acts compactly on \( W/W_1 \), and

(iii) either \( W_1 = W_0 \) or \( \mu \) has negative top exponent on \( W_1/W_0 \).

By minimality of \( W_0 \), we know that \( \Gamma \mu \) acts irreducibly on \( W_0 \). Since \( W \) is the linear span of \( \text{supp} \ \nu \), by Lemma 5.1 for \( \mu^N \)-almost every word \( b \in G^N \), the sequence \( \{ b_1 b_2 \ldots b_n \mid n \geq 1 \} \) is bounded from above in \( GL(W) \). By Lemma 2.3 \( \{ b_1 b_2 \ldots b_n \mid n \geq 1 \} \) is also bounded from above in \( GL(W_0) \). Thus by Corollary 3.3 either \( \mu \) has negative top exponent on \( W_0 \) or \( \Gamma \mu \) acts compactly on \( W_0 \).

**Case 1: \( \mu \) has negative top exponent on \( W_0 \).**

We claim that in this case, \( \mu \) has negative top exponent on \( W_1 \). The claim is clear if \( W_1 = W_0 \). If \( W_0 \subsetneq W_1 \),
since \( \mu \) has negative top exponent on \( W_1/W_0 \), by Lemma 2.3 \( \mu \) also has negative top exponent on \( W_1 \). Thus we can take \( W' = W_1 \).

**Case 2: \( \mu \) acts compactly on \( W_0 \).**

In this case, by Proposition 5.2, there exists a proper \( \Gamma_\mu \)-invariant subspace \( W_2 \subset W \) such that

\[
W = W_0 \oplus W_2.
\]

Let \( W_2 := W_1 \cap W_2 \). Then we can \( \Gamma_\mu \)-equivariantly identify \( W_2 \) and \( W_1/W_0 \). Thus either \( W_2 = 0 \) or \( \mu \) has negative top exponent on \( W_2 \), and \( \Gamma_\mu \) acts compactly \( W_2/W' \). Moreover, since

\[
W/W_2 = W_0 \oplus W_2/W',
\]

and \( \Gamma_\mu \) acts compactly on \( W_0 \) and \( W_2/W' \), we have that \( \Gamma_\mu \) acts compactly on \( W/W' \). Therefore we can take \( W' = W_2 \).

\[\square\]

6 Proofs of the main theorems

Using properties proved in the previous two sections, we can now prove the main theorems.

**Proof of Theorem 1.1.** Let \( W \subset V \) be the \( \Gamma_\mu \)-invariant subspace of maximal dimension such that \( W = \text{span}(\text{supp} \, \nu_0) \) for some \( \mu \)-stationary measure \( \nu_0 \) on \( V \).

We now claim that every \( \mu \)-stationary measure \( \nu \) satisfies \( \text{supp} \, \nu \subset W \). In fact, assume that there is some stationary measure \( \nu' \) such that \( \text{supp} \, \nu' \nsubseteq W \). Let \( U = \text{span}(\text{supp} \, \nu') \). Now let \( \nu'' = \frac{1}{2} \nu + \frac{1}{2} \nu' \). Then \( W + U = \text{span}(\text{supp} \, \nu'') \). Since \( W + U \) has strictly larger dimension than \( W \), this contradicts the maximality of \( \text{dim} \, W \), hence condition (i) in the theorem holds.

By Proposition 5.5, there exists a \( \Gamma_\mu \)-invariant proper subspace \( W' \subset W \) such that \( \Gamma_\mu \) acts compactly on \( W/W' \), and if \( W' \neq 0 \), \( \mu \) has negative top exponent on \( W' \). Thus (ii) and (iii) in the theorem hold.

**Proof of Theorem 1.2.** Let \( \pi : W \to W/W' \) be the quotient map. By Theorem 1.1 and Proposition 2.1, the map

\[
\Phi : \{\text{ergodic } \mu \text{-stationary measure on } V\} \to \{\text{compact } \Gamma_\mu \text{-orbit in } W/W'\},
\]

defined by \( \Phi(\nu) := \text{supp} \, \pi_* \nu \) is well-defined.

- **\( \Phi \) is injective.**
  This follows from Proposition 4.3 and the uniqueness of the \( \Gamma_\mu \)-invariant measure supported on a single compact \( \Gamma_\mu \)-orbit.

- **\( \Phi \) is surjective and determine \( \Phi^{-1} \)**
  The origin 0 of \( W/W' \) is a compact invariant subset of \( W/W' \), and is the image of the invariant measure \( \delta_0 \) on \( V \). Now given a compact \( \Gamma_\mu \)-invariant subset \( C \neq \{0\} \) in \( W/W' \), let \( x \in \pi^{-1}(C) \subset W \setminus W' \). By Proposition 4.6, \( \nu_x = \lim_{n \to \infty} \mu^{(n)} \ast (s_* m_x) \) is a \( \mu \)-stationary probability measure on \( V \) such that \( \text{supp} \, \pi_* \nu_x \) is \( C \), where as we recall, \( s : W/W' \to W \) is any linear section and \( m_x \) is the unique \( \Gamma_\mu \)-invariant measure supported on \( C \). Thus \( C \mapsto \nu_x \) is the inverse of \( \Phi \).

**Proof of Theorem 1.3.** The first claim was proved in Proposition 4.6. The second claim was shown in the proof of Theorem 1.2.

**Proof of Theorem 1.4.** The convergence of the limit in the first claim was shown in Proposition 4.5. That the limiting measure is \( \nu_C \) follows from the injectivity of \( \Phi \) in Theorem 1.2. The second claim is true since by Breiman’s law of large number [15], for \( \mu \)-almost every word \( b \in G^n \), every weak-* limit point of the empirical measures is a \( \mu \)-stationary probability measure. Now the rest follows from the same argument as Proposition 4.5 and the injectivity of \( \Phi \) in Theorem 1.2.

**Proof of Theorem 1.6.** This follows from Proposition 4.4.
References


